IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 12, Issue 5 Ver. VI (Sep. - Oct.2016), PP 54-58
www.iosrjournals.org

Upper and Lower Bounds for Ranks of Discrete Tropical Divisors

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Abstract: *We survey the results of ranks of discrete tropical divisors on tropical curves. Moreover, we found both upper and lower bounds for ranks of a given divisor, which help us quickly find the exact value of the ranks of discrete tropical divisors.*
Keywords: *Tropical Geometry, Discrete Tropical Divisor, Tropical Rank, Tropical Riemann-Roch Theorem, Metric Graph*

I. Introduction

A tropical curve is the image of a classical algebraic curve through certain valuation map. The image is a metric graph with possibly unbounded edges. Therefore, one can define that an abstract tropical curve Γ is simply a metric graph with possibly unbounded edges. That is, Γ is a graph, such that each edge has been associated with a length (possibly of infinity.)

The rank of a divisor$D$ for a tropical curve is the tropical counterpart of the dimension of the vector space of meromorphic functions satisfying $\operatorname{div}(f) + D$ is effective. For instance, we have tropical analogous of the Riemann-Roch theorem. Baker and Norine [1] introduced a version of the Riemann-Roch theorem for graphs. Gathmann and Kerber [2], and Mikhalkin and Zharkov [3] used the result to prove the Riemann-Roch theorem for tropical curves. Finally, Amini and Caporaso [4] extended the Riemann-Roch theorem to weighted tropical curves.

We use an example to elaborate our approach. An example of tropical curve Γ is as in Figure 1. We do a “surgery” to remove the “tentacles” (unbounded edges) of the tropical curve, and get the corresponding graph G, as shown in Figure 2.

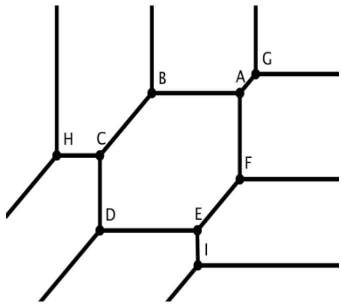

Fig. 1: a tropical curveΓ

Yoshitomi [5] has a similar consideration to the surgery we describe that he called the resulting object the “bunch” of the tropical curve Γ. However, there are some subtle differences in the process and totally different usages of the resulting objects.

We are in a position to apply *the Riemann-Roch theorem for graphs*. The theorem says that for a given divisor D on the graph G, we have

$$\operatorname{rank}(D) - \operatorname{rank}(K - D) = \deg(D) + 1 - g, \qquad (1)$$

where $\operatorname{rank}(D)$ is the quantity that is of primary interest, $\deg(D)$ is the degree of the divisor, and K is the canonical divisor.

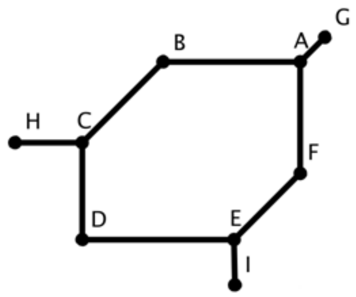

Fig. 2: metric graph corresponding to tropical curveΓ

In this paper, we apply the Riemann-Roch theorem for graphs to gain both upper and lower bounds for $\operatorname{rank}(D)$ in terms of $\deg(D)$. Our main theorem narrows down the possible values for $\operatorname{rank}(D)$, so one can quickly find the exact number of the rank.

II. Divisors On Metric Graphs

We shall discuss the definitions and main results we use in the theory of metric graphs. Let G be a metric graph without unbounded edges. Let $V(G)$ and $E(G)$ be the set of vertices and edges of G, respectively. A *divisorD* is the formal integer combination of elements of $V(G)$. That is,

$$D = \sum_{v \in V(G)} a_v \, v. \tag{2}$$

We denote by $\operatorname{Div}(G)$ the collection of all divisor on G. The degree of a divisor D is defined by

$$\deg(D) = \sum_{v \in V(G)} a_v. \tag{3}$$

A divisor is effective if $a_v \geq 0$ for all $v \in V(G)$.

Let $\mathcal{M}(G) = \operatorname{Hom}(V(G), \mathbb{Z})$, which we can regard as *analogous to the collection of meromorphic functions on a curve*. The *Laplacian operator*Δ: $\mathcal{M}(G) \to \operatorname{Div}(G)$ is defined by

$$\Delta(f) = \sum_{v \in V(G)} \Delta_v(f) v, \tag{4}$$

where

$$\Delta_v(f) = \sum_{e = wv \in E_v} \big(f(v) - f(w)\big). \tag{5}$$

We also define the subgroup $\operatorname{Prin}(G)$ of $\operatorname{Div}(G)$ to be the image of $\mathcal{M}(G)$under the Laplacian operator. That is, $Prin(G) = \Delta(\mathcal{M}(G))$. As in classical situation, we say two divisors D and D' are *linearly equivalent* if $D - D' \in \operatorname{Prin}(G)$.

For any divisor D, we define the *linear system* associate to D to be the set $|D|$ where

$$|D| = \{E \in \operatorname{Div}(G) \mid E \geq 0, E \sim D\}.$$

In this paper, we are especially interested inthe *rank* of a divisor. If $|D|$ is empty, we set $\operatorname{rank}(D) = -1$. Otherwise,

$$\operatorname{rank}(D) = \min\{s \geq 0 \mid |D - E| \neq \emptyset, \text{ for all effective divisors } E \text{ of degree } s\}$$

Finally, the *canonical divisor* on G is the divisor K given by

$$K = \sum_{v \in V(G)} (\deg(v) - 2)v. \tag{6}$$

It is easy to verify that $\deg(K) = 2g - 2$. Besides the Riemann-Roch theorem for graphs, we found there are at least three techniques that can help us to calculate the rank of a divisor.

First, Baker and Norine [1] pointed out that two divisors D, D' are linearly equivalent if and only if there is a sequence of moves taking D to D' in the *chip-firing game*. Many papers devoted the the chip-firing game, such as , for example. The initial configuration of the game assigns to each vertex v in G an integer number of dollars. Such a configuration of course can be identified with a divisor $D \in \operatorname{Div}(G)$. A *move* in the game consists of a vertex v either borrowing one dollar from each of its neighbors or giving one dollar to each of its neighbors. The chip-firing game provides a down-to-earth method (not necessary easy) to determine if two divisors are linearly equivalent.

Second, Amini and Caporaso [4] provided (kind of) explicitly structure for the principal divisor group $\operatorname{Prin}(G)$ which we will describe now. Set a binary operator on $V(G)$:

$$(v \cdot w) = \begin{cases} \text{number of edges joining } v \text{ and } w, \text{ if } v \neq w, \text{ and} \\ -\operatorname{val}(v) + 2 \cdot \operatorname{loop}(v), \text{ if } v = w, \end{cases} \tag{7}$$

where $\operatorname{val}(v)$ is the valency of v, and $\operatorname{loop}(v)$is the number of loops based at v. For a vertex $v \in V(G)$ we define $T_v \in \operatorname{Div}(G)$to be the following divisor

$$T_v = \sum_{w \in V(G)} (v \cdot w) w. \tag{8}$$

Then the principal divisors of G is generated by the divisors T_v, for all $v \in V(\mathrm{G})$.

Third, Hladký, Král, and Norine [7] proved that there exists an algorithm for computing the rank. What they founded is an exponential-time algorithm. We will really carry out the computation and since we take different approach, we hope that there is a less complicated algorithm.

III. Divisors On Tropical Curves

Let Γ be a tropical curve. We define the finite metric graph G corresponding to Γ to be the graph G removing all the unbounded edges of Γ. For any graph G, we define (discrete) *divisors* are formal sum of $\mathbb{Z}$-linear combination of the vertices. That is, a divisor D on G (Γ) is of the form:

$$D = \sum_{v \in V} a_v \cdot v, \tag{9}$$

where $a_v \in \mathbb{Z}$. We say that a divisor D on the curve Γ is exactly a divisor on the corresponding graph G. The set of all divisors on G (Γ) is denoted by $\mathrm{Div}(G)$ or $\mathrm{Div}(\Gamma)$. The *degree* of a divisor is again the sum of all coefficients.

A *meromorphic function* on G is simply a function

$$f : V \to \mathbb{Z}. \tag{10}$$

That is, $f \in \mathrm{Hom}(V, \mathbb{Z})$ and we denote the set $\mathrm{Hom}(V, \mathbb{Z})$ by $\mathcal{M}(G)$.

Each $f \in \mathcal{M}(G)$ is corresponding to a divisor

$$D(f) = \sum_{v \in V} \delta_v(f) \cdot v, \tag{11}$$

where

$$\delta_v(f) = \sum_{e = wv \in E_v} (f(v) - f(w)). \tag{12}$$

A divisor of this form is called a *principal divisor*. Two divisors D_1, D_2 are equivalent ($D_1 \sim D_2$) if they are differed by a principal divisor. That is, there is $f \in \mathcal{M}(G)$ such that

$$D_1 - D_2 = D(f). \tag{13}$$

An *effective divisor* E is a divisor that coefficients are all nonnegative, and we use $E \geq 0$ to indicate it is an effective divisor. For a divisor $D \in \mathrm{Div}(G)$, we define the *linear system* associated to D to be the set

$$|D| = \{E \in \mathrm{Div}(G) \mid E \geq 0, E \sim D\}. \tag{14}$$

Finally, we define the *rank* of a given divisor D. The rank of a divisor $D \in \mathrm{Div}(G)$ is defined as the following.

$$\mathrm{rank}(D) = \max\{s \,|\, |D - E| \neq \emptyset \text{ for all } E \geq 0 \text{ and } \deg E = s\} \tag{15}$$

For a graph G, we define the *canonical divisor*

$$K = \sum_{v \in V(G)} (\deg(v) - 2) \cdot v. \tag{16}$$

Baker and Norine [1] gave a version of *tropical Riemann-Roch Theorem*:

$$\mathrm{rank}(D) - \mathrm{rank}(K - D) = \deg(D) + g - 1. \tag{17}$$

The theorem will be the most important tool to help us find upper and lower bounds of the rank of a tropical divisor.

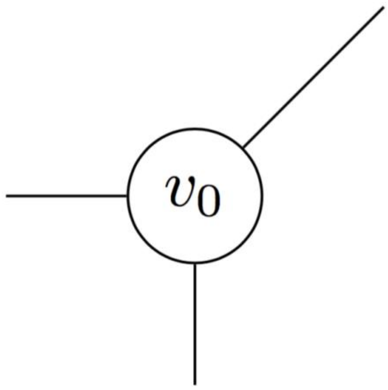

Fig. 3: a tropical line

Fig. 4: metric graph corresponding to a tropical line

DOI: 10.9790/5728-1205065458

IV. Illustrated Example

In this section, we give examples to illustrate how we calculate the rank of a divisor. Let $G = (E, V)$ be a graph where E is the collection of edges and V is the collection of vertices.Let Γ be a tropical line, we emphasize the vertex at center by making it a large point, and denoted the vertex by v_0, as shown in Figure3.By removing the unbounded edges of the tropical line, we get exactly one point (the vertex v_0,) as shown in Figure 4.

We can check the tropical Riemann-Roch theorem for the simple example. Let D be the divisor

$$D = 3 \cdot (v_0). \quad (18)$$

Since a meromorphic function f on G is simply a function from $V(G) = \{v_0\}$ to $\mathbb{Z}$, there is a $c \in \mathbb{Z}$ such that $f(v_0) = c$. Thus we can find the divisor corresponding to f:

$$(f) = 0 \cdot (v_0). \quad (19)$$

The rank of D is either

$$\max\{n \mid \text{forall} E, \deg(E) = n, E \geq 0, \text{wehave}|D - E| \neq \emptyset\}, \quad (20)$$

or

$$\min\{m|\text{thereis} E \geq 0, \text{suchthat}\deg(E) = m, |D - E| = \emptyset - 1\}. \quad (21)$$

The only divisor E such that $\deg(E) = 3$ is $E = 3 \cdot v_0$. Therefore, $D - E = 0 \cdot v_0 = (f)$. We conclude that $\text{rank}(D) \geq 3$. Moreover, the only divisor E such that $\deg(E) = 4$ is $E = 4 \cdot v_0$. Clearly, $|D - E| = \emptyset$, thus we have

$$\text{rank}(D) = 3. \quad (22)$$

It is easy to check that the canonical divisor is

$$K = -2 \cdot v_0. \quad (23)$$

Then $K - D = -5 \cdot (v_0)$, so $|K - D| = \emptyset$. That is,

$$\text{rank}(K - D) = -1 \circ \quad (24)$$

The left hand side of the tropical Riemann-Roch, which we presented in Equation(17), is

$$r(D) - r(K - D) = 3 - (-1) = 4. \quad (25)$$

Since $\deg(D) = 3$, and the genus $g = |E(G)| - |V(G)| + 1 = 0$, so the right hand side of the tropical Riemann-Roch theorem is

$$\deg(D) - g + 1 = 4. \quad (26)$$

Thus, we verify the tropical Riemann-Roch theorem for a tropical line. In general, calculating the rank is not an easy task. We would like to find some bounds for the value of the rank to narrow down possible values.

V. Rank Theorem

Let Γ be a tropical curve. We remove the unbounded edges of the tropical curve and get the corresponding finite graph G. Define

$$\text{Div}(\Gamma) := \text{Div}(G). \quad (27)$$

The graph G is called the graph corresponding to the tropical curve Γ. What we mean by a tropical divisor D is actually a divisor on the graph G.

Main Theorem. Let Γ be a tropical curve and let D be a divisor on Γ.
(a) If $\deg D < 0$ then $\text{rank}(D) = -1$.
(b) If $\deg D \geq 0$ then $\deg D - g \leq \text{rank}(D) \leq \deg D$.

Proof.
Part (a) is easy. Since $\deg D < 0$, by definition $|D - E|$ is empty for all effective divisor E on Γ. Thus, $\text{rank}(D) = -1$.
For part (b), we have either$|K - D| = \emptyset$or$|K - D| \neq \emptyset$.
If $|K - D| = \emptyset$, we get$\text{rank}(K - D) = -1$ by definition. By the tropical Riemann-Roch theorem, we obtain

$$\text{rank}(D) - (-1) = \deg D - g + 1, \quad (28)$$

thus $\text{rank}(D) = \deg D - g$. In particular,

$$\deg D - g \leq \text{rank}(D) \leq \deg D \quad (29)$$

holds.
Now, if $|K - D| \neq \emptyset$. Let E be an arbitrary effective divisor on Γ of degree $\deg D + 1$. Then

$$\deg(D - E) = \deg D - \deg E = -1. \quad (30)$$

Hence $|D - E| = \emptyset$. Therefore, $\text{rank}(D)$ is at most $\deg D$.
Note that $|K - D| \neq \emptyset$, so $\text{rank}(K - D) \geq 0$. By the tropical Riemann-Roch theorem, we have

$$\mathrm{rank}(D) \geq \deg D - g + 1. \tag{31}$$

Therefore we conclude that

$$\deg D - g \leq \mathrm{rank}(D) \leq \deg D. \tag{32}$$

□

Remark. Let $D \in \mathrm{Div}(\Gamma)$ such that $\deg D \geq 0$. In the proof of our Main Theorem, we can get an even better inequality for the cases $|K - D| \neq 0$, namely

$$\deg D - g + 1 \leq \mathrm{rank}(D) \leq \deg D. \tag{33}$$

Our main theorem from previous section gives us a range for the rank of a divisor D. Therefore, we only need to check a few possible numbers to see which one is the correct number for $\mathrm{rank}(D)$. Sometimes, we even get the exact value of the rank immediately such as the following example.

Let Γ be a tropical curve of genus 1. Let $D = 3 \cdot v_1 - 2 \cdot v_2 + 5 \cdot v_3 \in \mathrm{Div}(\Gamma)$. We have $\deg D = 3 - 2 + 5 = 6 \geq 0$. By the Remark, we have

$$\deg D - g + 1 \leq \mathrm{rank}(D) \leq \deg D, \tag{34}$$

but

$$\deg D - g + 1 = \deg D - 1 + 1 = \deg D = 6. \tag{35}$$

Therefore, we have $\mathrm{rank}(D) = 6$.

VI. Conclusion

We find bounds for the rank of given divisor on a tropical curve, which make it much easier to find the exact value of the rank. The definition of divisors on a tropical curve we use here is "discrete" version of definition. Actually, we can get similar results for "continuous" type of definition, which will be shown in our future papers.

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