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Generalized (σ, τ)-Derivations in Prime Rings

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Abstract: *Let R be a prime ring, I be a non-zero ideal and α any mapping on R, and σ, τ be a automorphisms of R. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation g and d on R respectively and $\tau(I) \neq 0$. In this paper, we studied the following identities in prime rings: (i) $G(xy) + F(x)F(y) \pm [\alpha(x), y]_{\sigma,\tau} = 0$; (ii) $G(xy) - F(x)F(y) \pm [\alpha(x), y]_{\sigma,\tau} = 0$; (iii) $G(xy) + F(x)F(y) - \sigma(yx) = 0$; (iv) $G(xy) \pm F(x)F(y) \pm [x, y]_{\sigma,\tau} = 0$; (v) $G(xy) \pm F(x)F(y) \pm (xoy)_{\sigma,\tau} = 0$; (vi) $G(xy) \pm F(x)F(y) = 0$; (vii) $G(xy) \pm F(x)F(y) \pm \tau(x)\sigma(y) = 0$; (viii) $G(xy) \pm F(y)F(x) = 0$; (ix) $F(xoy) \pm (d(x)oy)_{\sigma,\tau} = 0$; (x) $F[x, y] \pm [d(x), y]_{\sigma,\tau} = 0$; (xi) $G(xy) \pm [F(x), y]_{\sigma,\tau} \pm \tau(yx) = 0$; (xii) $G(xy) \pm [F(x), y]_{\sigma,\tau} \pm [x, y]_{\sigma,\tau} = 0$; (xiii) $G(xy) \pm [F(x), y]_{\sigma,\tau} \pm (xoy)_{\sigma,\tau} = 0$; (xiv) $G(xy) \pm [F(x), y]_{\sigma,\tau} = 0$; (xv) $G(xy) \pm [F(x), y]_{\sigma,\tau} \pm \sigma(xy) = 0$; (xvi) $G(xy) \pm [F(y), x]_{\sigma,\tau} \pm \sigma(yx) = 0$; (xvii) $G(xy) \pm [F(y), x]_{\sigma,\tau} \pm [x, y]_{\sigma,\tau} = 0$; (xviii) $G(xy) \pm [F(y), x]_{\sigma,\tau} \pm (xoy)_{\sigma,\tau} = 0$; (xix) $G(xy) \pm [F(y), x]_{\sigma,\tau} \pm \sigma(xy) = 0$; (xx) $G(xy) \pm [F(y), x]_{\sigma,\tau} = 0$; (xxi) $G(xy) \pm (F(x)oy)_{\sigma,\tau} \pm \tau(yx) = 0$; (xxii) $G(xy) \pm (F(x)oy)_{\sigma,\tau} \pm [x, y]_{\sigma,\tau} = 0$; (xxiii) $G(xy) \pm (F(x)oy)_{\sigma,\tau} \pm (xoy)_{\sigma,\tau} = 0$; (xxiv) $G(xy) \pm (F(x)oy)_{\sigma,\tau} \pm \sigma(xy) = 0$; (xxv) $G(xy) \pm (F(x)oy)_{\sigma,\tau} = 0$; (xxvi) $G(xy) \pm (F(y)ox)_{\sigma,\tau} \pm \sigma(yx) = 0$; (xxvii) $G(xy) \pm (F(y)ox)_{\sigma,\tau} \pm \sigma(xy) = 0$; (xxviii) $G(xy) \pm (F(y)ox)_{\sigma,\tau} = 0$; for all x, y in some suitable sub sets of R.*

Keywords: *Prime ring, Derivation, Generalized derivation, (σ,τ)-derivation, Generalized (σ,τ)-derivation.*

I. Introduction

Bresar in [2], first time introduced the notion of generalized derivation. In 1992, Daif et al. in [4], proved a result which is given as let R be a semiprime ring, I be a non zero ideal of R and d be a derivation on R such that $d([x, y]) = [x, y]$, for all $x, y \in I$, then $I \subseteq Z(R)$. In 2002, Ashraf and Rehman [1] extended the result of Daif et al. [4] by replacing ideal to lie ideal. In 2003, Quadri et al. in [7] extended the result of Ashraf et al. [1] on generalized derivation given as let R be a prime ring with characteristic different from two, I be a nonzero ideal of R and F be a generalized derivation on R associated with a derivation d on R such that $F([x, y]) = [x, y]$, for all $x, y \in I$, then R is commutative. Golbasi et al. in [5] extended the result of Quadri et al. [7] by replacing ideal to lie ideal. Recently, S.K. Tiwari et al. in [8] studied Multiplicative (generalized)-derivation in semiprime rings. Further ChiragGarg et al. in [3] studied on generalized (α,β)-derivations in prime rings. In this paper we inspire ofS.K.Tiwari et al. [8], we proved some results on generalized (σ,τ)-derivations in prime rings.

II. Preliminaries

Throughout this paper R denote an associative ring with center Z. Recall that a ring R is prime if $xRy = \{0\}$ implies $x = 0$or $y = 0$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol (xoy) denotes the anticommutator $xy + yx$. Let σ, τ be any two automorphisms of R. For any $x, y \in R$, we set $[x, y]_{\sigma,\tau} = x\sigma(y) - \tau(y)x$ and $(xoy)_{\sigma,\tau} = x\sigma(y) + \tau(y)x$. An additive mapping $d: R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $d: R \to R$ is called a (σ,τ)-derivation if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ holds for all $x, y \in R$. An additive mapping $F: R \to R$ is called a generalized derivation, if there exists a derivation $d: R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $F: R \to R$ is said to be a generalized (σ,τ)-derivation of R, if there exists a (σ,τ)-derivation $d: R \to R$ such that $F(xy) = F(x)\sigma(y) + \tau(x)d(y)$ holds for all $x, y \in R$, whereσ and τ be a automorphisms of R.

Throughout this paper, we shall make use of the basic commutator identities:

$[x, yz] = y[x, z] + [x, y]z$,

$[xy, z] = [x, z]y + x[y, z]$,

$$(xo(yz)) = (xoy)z - y[x, z] = y(xoz) + [x, y]z,$$

$[xy, z]_{\sigma,\tau} = x[y, z]_{\sigma,\tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma,\tau}y$,

$[x, yz]_{\sigma,\tau} = \tau(y)[x, z]_{\sigma,\tau} + [x, y]_{\sigma,\tau}\sigma(z)$,

$(xo(yz))_{\sigma,\tau} = (xoy)_{\sigma,\tau}\sigma(z) - \tau(y)[x, z]_{\sigma,\tau} = \tau(y)(xoz)_{\sigma,\tau} + [x, y]_{\sigma,\tau}\sigma(z)$,

$((xy)oz)_{\sigma,\tau} = x(yoz)_{\sigma,\tau} - [x, \tau(z)]y = (xoz)_{\sigma,\tau}y + x[y, \sigma(z)]$.

Lemma 1[6, Lemma 1.1]: Let R be a prime ring with characteristics not two and U a nonzero lie ideal of R. If d is a non zero (σ,τ)- derivation of R such that $d(U)=O$, then $U\subseteq Z$.

Theorem 1: Let R be a prime ring,I be a non-zero ideal and α any mapping on$R$. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation$g$ and$d$ on$R$ respectively and $\tau(I)\neq 0$. If $G(xy)+F(x)F(y)\pm[\alpha(x),y]_{\sigma,\tau}=0$, for all $x,y\in I$, then either$[d(x),\tau(x)]=0$,$[g(x),\tau(x)]=0$, for all $x\in I$ or$R$ is commutative.

Proof: First we consider the case$G(xy)+F(x)F(y)-[\alpha(x),y]_{\sigma,\tau}=0$, for all $x,y\in I$.(1)
We replacing y by yzin equation (1), we obtain
$G(xyz)+F(x)F(yz)-[\alpha(x),yz]_{\sigma,\tau}=0$, for all $x,y,z\in I$

$$G(xy)\sigma(z)+\tau(xy)g(z)+F(x)(F(y)\sigma(z)+\tau(y)d(z))-[\alpha(x),y]_{\sigma,\tau}\sigma(z)-\tau(y)[\alpha(x),z]_{\sigma,\tau}=0$$
$$(G(xy)+F(x)F(y)-[\alpha(x),y]_{\sigma,\tau})\sigma(z)+\tau(xy)g(z)+F(x)\tau(y)d(z)-\tau(y)[\alpha(x),z]_{\sigma,\tau}=0$$

Using equation (1), it reduces to
$\tau(xy)g(z)+F(x)\tau(y)d(z)-\tau(y)[\alpha(x),z]_{\sigma,\tau}=0$, for all $x,y,z\in I$. (2)
We replacing y by ryin equation (2), we get
$\tau(xry)g(z)+F(x)\tau(ry)d(z)-\tau(ry)[\alpha(x),z]_{\sigma,\tau}=0$, for all $x,y,z,r\in I$. (3)
We replacing x by xrin equation (2), we get

$$\tau(xry)g(z)+F(xr)\tau(y)d(z)-\tau(y)[\alpha(xr),z]_{\sigma,\tau}=0$$
$$\tau(xry)g(z)+(F(x)\sigma(r)+\tau(x)d(r))\tau(y)d(z)-\tau(y)[\alpha(xr),z]_{\sigma,\tau}=0$$

$\tau(xry)g(z)+F(x)\sigma(r)\tau(y)d(z)+\tau(x)d(r)\tau(y)d(z)-\tau(y)[\alpha(xr),z]_{\sigma,\tau}=0$,for all $x,y,z,r\in I$. (4)
We replacing $\sigma(r)$ by $\tau(r)$ in equation (4), we get
$\tau(xry)g(z)+F(x)\tau(r)\tau(y)d(z)+\tau(x)d(r)\tau(y)d(z)-\tau(y)[\alpha(xr),z]_{\sigma,\tau}=0$,for all $x,y,z,r\in I$. (5)
We subtracting equation (3) from equation (5), we get
$\tau(x)d(r)\tau(y)d(z)-\tau(y)[\alpha(xr),z]_{\sigma,\tau}+\tau(ry)[\alpha(x),z]_{\sigma,\tau}=0$, for all $x,y,z,r\in I$.(6)
We replacing y by ry in equation (6), we get
$\tau(x)d(r)\tau(ry)d(z)-\tau(ry)[\alpha(xr),z]_{\sigma,\tau}+\tau(r^2y)[\alpha(x),z]_{\sigma,\tau}=0$, for all $x,y,z,r\in I$. (7)
Left multiplying equation (6) by $\tau(r)$, we get
$\tau(r)\tau(x)d(r)\tau(y)d(z)-\tau(r)\tau(y)[\alpha(xr),z]_{\sigma,\tau}+\tau(r)\tau(ry)[\alpha(x),z]_{\sigma,\tau}=0$, for all $x,y,z\in I$,$r\in R$.(8)
We subtracting equation (8) from equation (7), we get

$$\tau(x)d(r)\tau(ry)d(z)-\tau(r)\tau(x)d(r)\tau(y)d(z)=0$$

$[\tau(x)d(r),\tau(r)]\tau(y)d(z)=0$, for all $x,y,z\in I$, $r\in R$. (9)
We replacing r by x in equation (9), we get

$$[\tau(x)d(x),\tau(x)]\tau(y)d(z)=0$$
$$\tau(x)[d(x),\tau(x)]\tau(y)d(z)+[\tau(x),\tau(x)]d(x)\tau(y)d(z)=0$$
$$\tau(x)[d(x),\tau(x)]\tau(y)d(z)=0$$

$[d(x),\tau(x)]\tau(y)d(z)=0$, for all $x,y,z\in I$.
We replacing $y$by sy, $s\in R$ in the above equation, we get

$$[d(x),\tau(x)]\tau(sy)d(z)=0$$

$[d(x),\tau(x)]R\tau(y)d(z)=0$, for all $x,y,z\in I$, $s\in R$. (10)
Since R is prime, we geteither $[d(x),\tau(x)]=0$, for all $x\in I$ or $\tau(y)d(z)=0$, for all $y,z\in I$.
Since τ is an automorphism of $R$and $\tau(I)\neq 0$, we have either $[d(x),\tau(x)]=0$, for all $x\in I$ or $d(z)=0$, for all $z\in I$.
Now let $A=\{x\in I/[d(x),\tau(x)]=0\}$and $B=\{x\in I/d(x)=0\}$.
Clearly,A and B are additive proper subgroups of I whose union is I.
Since a group cannot be the set theoretic union of two proper subgroups.
Hence either $A=I$or $B=I$.
If $B=I$, then $d(x)=0$, for all $x\in I$, by lemma 1 implies that R is commutative.
On the other hand if $A=I$, then $[d(x),\tau(x)]=0$, for all $x\in I$.
If $[d(x),\tau(x)]=0$, then $d(x)$ is said to be τ-commuting on R.(11)
Again we replacing y by yz in equation (2), we get
$\tau(xyz)g(z)+F(x)\tau(yz)d(z)-\tau(yz)[\alpha(x),z]_{\sigma,\tau}=0$, for all $x,y,z\in I$. (12)
Right multiplying equation (2) by $\tau(z)$, we get
$\tau(xy)g(z)\tau(z)+F(x)\tau(yz)d(z)\tau(z)-\tau(y)[\alpha(x),z]_{\sigma,\tau}\tau(z)=0$, for all $x,y,z\in I$. (13)
We subtracting equation (12) from equation (13), we get

$$\tau(xy)[g(z),\tau(z)]+F(x)\tau(y)[d(z),\tau(z)]-\tau(y)\big[[\alpha(x),z]_{\sigma,\tau},\tau(z)\big]=0$$

Using equation (11) in the above equation, we get

$\tau(xy)[g(z),\tau(z)]-\tau(y)\big[[\alpha(x),z]_{\sigma,\tau},\tau(z)\big]=0$, for all $x,y,z\in I$. (14)
We replacing y by ry in equation (14), we get
$\tau(xry)[g(z),\tau(z)]-\tau(ry)\big[[\alpha(x),z]_{\sigma,\tau},\tau(z)\big]=0$, for all $x,y,z\in I$, $r\in R$. (15)
Left multiplying equation (14) by $\tau(r)$, we get
$\tau(r)\tau(xy)[g(z),\tau(z)]-\tau(r)\tau(y)\big[[\alpha(x),z]_{\sigma,\tau},\tau(z)\big]=0$, for all $x,y,z\in I$, $r\in R$. (16)
We subtracting equation (16) from equation (15), we get
$[\tau(x),\tau(r)]\tau(y)[g(z),\tau(z)]=0$, for all $x,y,z\in I$, $r\in R$. (17)
We replacing $y$by sy, $s\in R$ in equation (17), we get

$$[\tau(x),\tau(r)]\tau(sy)[g(z),\tau(z)]=0$$

$[\tau(x),\tau(r)]R\tau(y)[g(z),\tau(z)]=0$, for all $x,y,z\in I$, , $r\in R$. (18)
Using primenessof$R$, we get either $[\tau(x),\tau(r)]=0$ or $\tau(y)[g(z),\tau(z)]=0$
If $[\tau(x),\tau(r)]=0$ then R is commutative.
If $\tau(y)[g(z),\tau(z)]=0$, since $\tau(I)\neq 0$ then $[g(z),\tau(z)]=0$ implies that $g(z)$ is said to be τ-commuting on R. (19)
Using similar approach we conclude that the same results holds for $G(xy)+F(x)F(y)+[\alpha(x),y]_{\sigma,\tau}=0$, for all $x,y\in I$.
Using similar techniques with some necessary variations we can prove the following theorem:

Theorem 2: Let R be a prime ring,I be a non-zero ideal and α any mapping on$R$. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation$g$ and$d$ on$R$ respectively and $\tau(I)\neq 0$. If $G(xy)-F(x)F(y)\pm[\alpha(x),y]_{\sigma,\tau}=0$, for all $x,y\in I$, then either $[d(x),\tau(x)]=0$,$[g(x),\tau(x)]=0$, for all $x\in I$or R is a commutative.

Theorem 3: Let R be a prime ring and I be a non-zero idealon R. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation$g$ and$d$ on$R$ respectively and $\tau(I)\neq 0$. If $G(xy)+F(x)F(y)-\sigma(yx)=0$, for all $x,y\in I$, then either R is commutative or $[F(x),\tau(x)]=0$, for all $x\in I$.

Proof: We have $G(xy)+F(x)F(y)-\sigma(yx)=0$, for all $x,y\in I$. (20)
We replacing y by yz in equation (20), we get

$$G(xyz)+F(x)F(yz)-\sigma(yzx)=0$$
$$G(xy)\sigma(z)+\tau(xy)g(z)+F(x)F(y)\sigma(z)+F(x)\tau(y)d(z)-\sigma(yzx)=0$$
$$\big(G(xy)+F(x)F(y)\big)\sigma(z)+\tau(xy)g(z)+F(x)\tau(y)d(z)-\sigma(yzx)=0$$

Using equation (20), it reduces
$\sigma(yx)\sigma(z)+\tau(xy)g(z)+F(x)\tau(y)d(z)-\sigma(yzx)=0$, for all $x,y,z\in I$. (21)
We replacing x by x^2 in equation (21), we get

$$\sigma(yx^2z)+\tau(x^2y)g(z)+F(x^2)\tau(y)d(z)-\sigma(yzx^2)=0$$

$\sigma(yx^2z)+\tau(x^2y)g(z)+F(x)\sigma(x)\tau(y)d(z)+\tau(x)d(x)\tau(y)d(z)-\sigma(yzx^2)=0$, for all $x,y,z\in I$.
(22)Left multiplying equation (21) by $\tau(x)$, we get
$\tau(x)\sigma(yxz)+\tau(x)\tau(xy)g(z)+\tau(x)F(x)\tau(y)d(z)-\tau(x)\sigma(yzx)=0$, for all $x,y,z\in I$. (23)
We subtracting equation (23) from equation (22), we get
$\sigma(yx^2z)+F(x)\sigma(x)\tau(y)d(z)+\tau(x)d(x)\tau(y)d(z)-\sigma(yzx^2)-\tau(x)\big(\sigma(yxz)-\sigma(yzx)\big)-$
$\tau(x)F(x)\tau(y)d(z)=0$, for all $x,y,z\in I$. (24)
We replacing z by x in equation (24), we get

$$\sigma(yx^3)+F(x)\sigma(x)\tau(y)d(x)+\tau(x)d(x)\tau(y)d(x)-\sigma(yx^3)-\tau(x)\big(\sigma(yx^2)-\sigma(yx^2)\big)$$
$$-\tau(x)F(x)\tau(y)d(x)=0$$
$$F(x)\sigma(x)\tau(y)d(x)+\tau(x)d(x)\tau(y)d(x)-\tau(x)F(x)\tau(y)d(x)=0$$

$(F(x)\sigma(x)+\tau(x)d(x)-\tau(x)F(x))\tau(y)d(x)=0$, for all $x,y\in I$. (25)
We replacing $y$by sy, $s\in R$ in equation (25), we get

$$(F(x)\sigma(x)+\tau(x)d(x)-\tau(x)F(x))\tau(sy)d(x)=0$$

$(F(x^2)-\tau(x)F(x))R\tau(y)d(x)=0$, for all $x,y\in I$. (26)
Using primenessof$R$, we get either $F(x^2)-\tau(x)F(x)=0$ or $\tau(y)d(x)=0$, for all $x,y\in I$
Since τ is an automorphism of $R$and $\tau(I)\neq 0$, we have either $F(x^2)-\tau(x)F(x)=0$, for all $x\in I$ or $d(x)=0$, for all $x\in I$.
Now let $A=\{x\in I/F(x^2)-\tau(x)F(x)=0\}$and $B=\{x\in I/d(x)=0\}$.
Clearly, A and B are additive proper subgroups of I whose union is I.
Since a group cannot be the set theoretic union of two proper subgroups.
Hence either $A=I$or $B=I$.

If $B = I$, then $d(x) = 0$, for all $x \in I$, by lemma 1 implies that R is commutative.
On the other hand if $A = I$, then $F(x^2) - \tau(x)F(x) = 0$, for all $x \in I$.(27)
We replacing y by xx in equation (20), we get
$G(xxx) - \sigma(xxx) = -F(x)F(xx)$, for all $x \in I$. (28)
We replacing x by xx and y by x in equation (20), we get
$G(xxx) - \sigma(xxx) = -F(xx)F(x)$, for all $x \in I$. (29)
From equation (28) and equation (29), we get

$$F(x)F(x^2) = F(x^2)F(x)$$

Using equation (27), it reduces to

$$F(x)\tau(x)F(x) = \tau(x)F(x)F(x)$$
$$\big(F(x)\tau(x) - \tau(x)F(x)\big)F(x) = 0$$

We conclude that $[F(x),\tau(x)] = 0$, for all $x \in I$. (30)

Theorem 4: Let R be a prime ring and I be a non-zero idealon R. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation g and d on R respectively and $\tau(I) \neq 0$. If $G(xy) \pm F(x)F(y) \pm [x,y]_{\sigma,\tau} = 0$, for all $x,y \in I$, then either $[d(x),\tau(x)] = 0$, $[g(x),\tau(x)] = 0$, for all $x \in I$ or R is commutative.

Proof: First we consider the case $G(xy) + F(x)F(y) + [x,y]_{\sigma,\tau} = 0$, for all $x,y \in I$.(31)
We replacing y by yzin equation (31), we obtain
$G(xyz) + F(x)F(yz) + [x,yz]_{\sigma,\tau} = 0$, for all $x,y,z \in I$

$$G(xy)\sigma(z) + \tau(xy)g(z) + F(x)(F(y)\sigma(z) + \tau(y)d(z)) + [x,y]_{\sigma,\tau}\sigma(z) + \tau(y)[x,z]_{\sigma,\tau} = 0$$
$$(G(xy) + F(x)F(y) + [x,y]_{\sigma,\tau})\sigma(z) + \tau(xy)g(z) + F(x)\tau(y)d(z) + \tau(y)[x,z]_{\sigma,\tau} = 0$$

Using equation (31), it reduces to
$\tau(xy)g(z) + F(x)\tau(y)d(z) + \tau(y)[x,z]_{\sigma,\tau} = 0$, for all $x,y,z \in I$. (32)
We replacing y by ryin equation (32), we get
$\tau(xry)g(z) + F(x)\tau(ry)d(z) + \tau(ry)[x,z]_{\sigma,\tau} = 0$, for all $x,y,z,r \in I$. (33)
We replacing x by xrin equation (32), we get

$$\tau(xry)g(z) + F(xr)\tau(y)d(z) + \tau(y)[xr,z]_{\sigma,\tau} = 0$$
$$\tau(xry)g(z) + (F(x)\sigma(r) + \tau(x)d(r))\tau(y)d(z) + \tau(y)[xr,z]_{\sigma,\tau} = 0$$

$\tau(xry)g(z) + F(x)\sigma(r)\tau(y)d(z) + \tau(x)d(r)\tau(y)d(z) + \tau(y)[xr,z]_{\sigma,\tau} = 0$,for all $x,y,z,r \in I$. (34)
We replacing $\sigma(r)$ by $\tau(r)$ in equation (34), we get
$\tau(xry)g(z) + F(x)\tau(r)\tau(y)d(z) + \tau(x)d(r)\tau(y)d(z) + \tau(y)[xr,z]_{\sigma,\tau} = 0$,for all $x,y,z,r \in I$. (35)
We subtracting equation (33) from equation (35), we get
$\tau(x)d(r)\tau(y)d(z) + \tau(y)[xr,z]_{\sigma,\tau} - \tau(ry)[x,z]_{\sigma,\tau} = 0$, for all $x,y,z,r \in I$.(36)
We replacing y by ry in equation (36), we get
$\tau(x)d(r)\tau(ry)d(z) + \tau(ry)[xr,z]_{\sigma,\tau} - \tau(r^2y)[xr,z]_{\sigma,\tau} = 0$, for all $x,y,z,r \in I$. (37)
Left multiplying equation (36) by $\tau(r)$, we get
$\tau(r)\tau(x)d(r)\tau(y)d(z) + \tau(r)\tau(y)[xr,z]_{\sigma,\tau} - \tau(r)\tau(ry)[x,z]_{\sigma,\tau} = 0$,for all $x,y,z \in I$,$r \in R$. (38)
We subtracting equation (38) from equation (37), we get

$$\tau(x)d(r)\tau(ry)d(z) - \tau(r)\tau(x)d(r)\tau(y)d(z) = 0$$

$[\tau(x)d(r),\tau(r)]\tau(y)d(z) = 0$, for all $x,y,z \in I$, $r \in R$. (39)
The equation (39) is same as equation (9) in theorem 1. Thus, by same argument of theorem 1, we can conclude the result are $[d(x),\tau(x)] = 0$,then$d(x)$ is said to be τ-commuting on R or R is commutative.
Again we replacing y by yz in equation (32), we get
$\tau(xyz)g(z) + F(x)\tau(yz)d(z) + \tau(yz)[x,z]_{\sigma,\tau} = 0$, for all $x,y,z \in I$. (40)
Right multiplying equation (32) by $\tau(z)$, we get
$\tau(xy)g(z)\tau(z) + F(x)\tau(yz)d(z)\tau(z) + \tau(y)[x,z]_{\sigma,\tau}\tau(z) = 0$, for all $x,y,z \in I$. (41)
We subtracting equation (40) from equation (41), we get

$$\tau(xy)[g(z),\tau(z)] + F(x)\tau(y)[d(z),\tau(z)] + \tau(y)\big[[x,z]_{\sigma,\tau},\tau(z)\big] = 0$$

Using $[d(z),\tau(z)] = 0$in the above equation, we get
$\tau(xy)[g(z),\tau(z)] + \tau(y)\big[[x,z]_{\sigma,\tau},\tau(z)\big] = 0$, for all $x,y,z \in I$. (42)
We replacing y by ry in equation (42), we get
$\tau(xry)[g(z),\tau(z)] + \tau(ry)\big[[x,z]_{\sigma,\tau},\tau(z)\big] = 0$, for all $x,y,z \in I$, $r \in R$. (43)
Left multiplying equation (42) by $\tau(r)$, we get
$\tau(r)\tau(xy)[g(z),\tau(z)] + \tau(r)\tau(y)\big[[x,z]_{\sigma,\tau},\tau(z)\big] = 0$, for all $x,y,z \in I$, $r \in R$. (44)
We subtracting equation (44) from equation (43), we get

$[\tau(x),\tau(r)]\tau(y)[g(z),\tau(z)]=0$, for all $x,y,z\in I, r\in R$. (45)
The equation (45) is same as equation (17) in theorem 1. Thus, by same argument of theorem 1, we can conclude the result are $[g(x),\tau(x)]=0$, then $g(x)$ is said to be τ-commuting on R or R is commutative.
Using similar approach we conclude that the same results holds for $G(xy)+F(x)F(y)-[x,y]_{\sigma,\tau}=0$, for all $x,y\in I$.

Theorem 5: Let R be a prime ring and I be a non-zero idealon R. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation g and d on R respectively and $\tau(I)\neq 0$. If $G(xy)\pm F(x)F(y)\pm(xoy)_{\sigma,\tau}=0$, for all $x,y\in I$, then either $[d(x),\tau(x)]=0$, $[g(x),\tau(x)]=0$, for all $x\in I$ or R is commutative.

Proof: First we consider the case $G(xy)+F(x)F(y)+(xoy)_{\sigma,\tau}=0$, for all $x,y\in I$. (46)
We replacing y by yzin equation (46), we obtain
$G(xyz)+F(x)F(yz)+(xoyz)_{\sigma,\tau}=0$, for all $x,y,z\in I$

$$G(xy)\sigma(z)+\tau(xy)g(z)+F(x)(F(y)\sigma(z)+\tau(y)d(z))+(xoy)_{\sigma,\tau}\sigma(z)-\tau(y)[x,z]_{\sigma,\tau}=0$$

$$(G(xy)+F(x)F(y)+(xoy)_{\sigma,\tau})\sigma(z)+\tau(xy)g(z)+F(x)\tau(y)d(z)-\tau(y)[x,z]_{\sigma,\tau}=0$$

Using equation (46), it reduces to
$\tau(xy)g(z)+F(x)\tau(y)d(z)-\tau(y)[x,z]_{\sigma,\tau}=0$, for all $x,y,z\in I$. (47)
We replacing y by ryin equation (47), we get
$\tau(xry)g(z)+F(x)\tau(ry)d(z)-\tau(ry)[x,z]_{\sigma,\tau}=0$, for all $x,y,z,r\in I$. (48)
We replacing x by xrin equation (47), we get

$$\tau(xry)g(z)+F(xr)\tau(y)d(z)-\tau(y)[xr,z]_{\sigma,\tau}=0$$

$$\tau(xry)g(z)+\{F(x)\sigma(r)+\tau(x)d(r)\}\tau(y)d(z)-\tau(y)[xr,z]_{\sigma,\tau}=0$$

$\tau(xry)g(z)+F(x)\sigma(r)\tau(y)d(z)+\tau(x)d(r)\tau(y)d(z)-\tau(y)[xr,z]_{\sigma,\tau}=0$,for all $x,y,z,r\in I$. (49)
We replacing $\sigma(r)$ by $\tau(r)$ in equation (49), we get
$\tau(xry)g(z)+F(x)\tau(r)\tau(y)d(z)+\tau(x)d(r)\tau(y)d(z)-\tau(y)[xr,z]_{\sigma,\tau}=0$,for all $x,y,z,r\in I$. (50)
We subtracting equation (48) from equation (50), we get
$\tau(x)d(r)\tau(y)d(z)-\tau(y)[xr,z]_{\sigma,\tau}+\tau(ry)[x,z]_{\sigma,\tau}=0$, for all $x,y,z,r\in I$. (51)
We replacing y by ry in equation (51), we get
$\tau(x)d(r)\tau(ry)d(z)-\tau(ry)[xr,z]_{\sigma,\tau}+\tau(r^2y)[xr,z]_{\sigma,\tau}=0$, for all $x,y,z,r\in I$. (52)
Left multiplying equation (51) by $\tau(r)$, we get
$\tau(r)\tau(x)d(r)\tau(y)d(z)-\tau(r)\tau(y)[xr,z]_{\sigma,\tau}+\tau(r)\tau(ry)[x,z]_{\sigma,\tau}=0$,for all $x,y,z\in I$,$r\in R$. (53)
We subtracting equation (53) from equation (52), we get

$$\tau(x)d(r)\tau(ry)d(z)-\tau(r)\tau(x)d(r)\tau(y)d(z)=0$$

$[\tau(x)d(r),\tau(r)]\tau(y)d(z)=0$, for all $x,y,z\in I, r\in R$. (54)
The equation (54) is same as equation (9) in theorem 1. Thus, by same argument of theorem 1, we can conclude the result are $[d(x),\tau(x)]=0$, then $d(x)$ is said to be τ-commuting on R or R is commutative.
Again we replacing y by yz in equation (47), we get
$\tau(xyz)g(z)+F(x)\tau(yz)d(z)-\tau(yz)[x,z]_{\sigma,\tau}=0$, for all $x,y,z\in I$. (55)
Right multiplying equation (47) by $\tau(z)$, we get
$\tau(xy)g(z)\tau(z)+F(x)\tau(yz)d(z)\tau(z)-\tau(y)[x,z]_{\sigma,\tau}\tau(z)=0$, for all $x,y,z\in I$. (56)
We subtracting equation (55) from equation (56), we get

$$\tau(xy)[g(z),\tau(z)]+F(x)\tau(y)[d(z),\tau(z)]-\tau(y)\big[[x,z]_{\sigma,\tau},\tau(z)\big]=0$$

Using $[d(z),\tau(z)]=0$ in the above equation, we get
$\tau(xy)[g(z),\tau(z)]-\tau(y)\big[[x,z]_{\sigma,\tau},\tau(z)\big]=0$, for all $x,y,z\in I$. (57)
We replacing y by ry in equation (57), we get
$\tau(xry)[g(z),\tau(z)]-\tau(ry)\big[[x,z]_{\sigma,\tau},\tau(z)\big]=0$, for all $x,y,z\in I, r\in R$. (58)
Left multiplying equation (57) by $\tau(r)$, we get
$\tau(r)\tau(xy)[g(z),\tau(z)]-\tau(r)\tau(y)\big[[x,z]_{\sigma,\tau},\tau(z)\big]=0$, for all $x,y,z\in I, r\in R$. (59)
We subtracting equation (59) from equation (58), we get
$[\tau(x),\tau(r)]\tau(y)[g(z),\tau(z)]=0$, for all $x,y,z\in I, r\in R$. (60)
The equation (60) is same as equation (17) in theorem 1. Thus, by same argument of theorem 1, we can conclude the result are $[g(x),\tau(x)]=0$, then $g(x)$ is said to be τ-commuting on R or R is commutative.
Using similar approach we conclude that the same results holds for $G(xy)+F(x)F(y)-(x,y)_{\sigma,\tau}=0$, for all $x,y\in I$.

Theorem 6: Let R be a prime ring and I be a non-zero idealon R. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation g and d on R respectively and $\tau(I)\neq 0$. If $G(xy)\pm F(x)F(y)=0$, for all $x,y\in I$, then either R is commutative or $[F(x),\tau(x)]=0$ and $[G(x),\tau(x)]=0$, for all $x\in I$.

Proof: First we consider the case $G(xy)+F(x)F(y)=0$, for all $x,y\in I$. (61)
We replacing y by yz in equation (61), we obtain
$G(xyz)+F(x)F(yz)=0$, for all $x,y,z\in I$

$$G(xy)\sigma(z)+\tau(xy)g(z)+F(x)\{F(y)\sigma(z)+\tau(y)d(z)\}=0$$
$$\{G(xy)+F(x)F(y)\}\sigma(z)+\tau(xy)g(z)+F(x)\tau(y)d(z)=0$$

Using equation (61), it reduces to
$\tau(xy)g(z)+F(x)\tau(y)d(z)=0$, for all $x,y,z\in I$. (62)
We replacing y by ry in equation (62), we get
$\tau(xry)g(z)+F(x)\tau(ry)d(z)=0$, for all $x,y,z,r\in I$. (63)
We replacing x by xr in equation (62), we get

$$\tau(xry)g(z)+F(xr)\tau(y)d(z)=0$$
$$\tau(xry)g(z)+\{F(x)\sigma(r)+\tau(x)d(r)\}\tau(y)d(z)=0$$

$\tau(xry)g(z)+F(x)\sigma(r)\tau(y)d(z)+\tau(x)d(r)\tau(y)d(z)=0$, for all $x,y,z,r\in I$. (64)
We subtracting equation (63) from equation (64), we get

$$F(x)\sigma(r)\tau(y)d(z)+\tau(x)d(r)\tau(y)d(z)-F(x)\tau(ry)d(z)=0$$

$\big(F(x)\sigma(r)+\tau(x)d(r)-F(x)\tau(r)\big)\tau(y)d(z)=0$, for all $x,y,z,r\in I$. (65)
We replacing y by sy, $s\in R$ in equation (65), we get

$$\big(F(x)\sigma(r)+\tau(x)d(r)-F(x)\tau(r)\big)\tau(sy)d(z)=0$$
$$\big(F(xr)-F(x)\tau(r)\big)R\tau(y)d(z)=0$$

Using primenessof R, we get either $F(xr)-F(x)\tau(r)=0$ or $\tau(y)d(z)=0$, for all $y,z\in I$
Since τ is an automorphism of R and $\tau(I)\neq 0$, we have either $F(xr)-F(x)\tau(r)=0$, for all $x\in I$ and $r\in R$ or $d(x)=0$, for all $x\in I$.
Now let $A=\{x\in I/F(xr)-F(x)\tau(r)=0, r\in R\}$ and $B=\{x\in I/d(x)=0\}$.
Clearly, A and B are additive proper subgroups of I whose union is I.
Since a group cannot be the set theoretic union of two proper subgroups.
Hence either $A=I$ or $B=I$.
If $B=I$, then $d(x)=0$, for all $x\in I$, by lemma 1 implies that R is commutative.
On the other hand if $A=I$, then $F(xr)=F(x)\tau(r)$, for all $x,r\in I$. (66)
We replacing y by yz in equation (61), we get
$G(xyz)=-F(x)F(yz)$, for all $x,y,z\in I$. (67)
We replacing x by xy and y by z in equation (61), we get
$G(xyz)=-F(xy)F(z)$, for all $x,y,z\in I$. (68)
From equation (67) and equation (68), we get

$$F(x)F(yz)=F(xy)F(z)$$

Using equation (66), it reduces to

$$F(x)F(y)\tau(z)=F(x)\tau(y)F(z)$$

$F(x)(F(y)\tau(z)-\tau(y)F(z))=0$, for all $x,y,z\in I$. (69)
We replacing x by xw, $w\in R$ in equation (69), we get

$$F(xw)(F(y)\tau(z)-\tau(y)F(z))=0$$
$$F(x)\tau(w)(F(y)\tau(z)-\tau(y)F(z))=0$$
$$F(x)R(F(y)\tau(z)-\tau(y)F(z))=0$$

Using primenessof R, we conclude that
$F(x)\tau(y)-\tau(y)F(x)=0$, for all $x,y\in I$.
In particular $[F(x),\tau(x)]=0$, for all $x\in I$. (70)
We replacing y by yz in equation (61), we obtain

$$G(xyz)+F(x)F(yz)=0$$
$$G(xy)\sigma(z)+\tau(xy)g(z)=-F(x)F(yz)=-F(x)F(y)\tau(z)=-F(x)\tau(y)F(z)=-\tau(x)F(y)F(z)$$
$$=\tau(x)G(yz)=\tau(x)(G(y)\sigma(z)+\tau(y)g(z)$$

That is $\big(G(xy)-\tau(x)G(y)\big)\sigma(z)=0$, for all $x,y,z\in I$. (71)
Since I is non zero, so using primeness of R, we conclude that
$G(xy)-\tau(x)G(y)=0$, for all $x,y\in I$. (72)
Using equation (72), we have
$G(xyz)=\tau(xy)G(z)$, for all $x,y,z\in I$. (73)

$G(xyz) = -F(x)F(yz) = -F(x)F(y)\tau(z) = G(xy)\tau(z) = \tau(x)G(y)\tau(z)$, for all $x,y,z \in I$. (74)
From equation (73) and equation (74), we have
$\tau(x)\big(G(y)\tau(z) - \tau(y)G(z)\big) = 0$.
Hence, $G(x)\tau(y) - \tau(x)G(y) = 0$, for all $x,y \in I$.
In particular, $[G(x),\tau(x)] = 0$, for all $x \in I$. (75)
There by the proof of the theorem is completed.

Corollary 1: Let R be a prime ring and I be a non-zero idealon R. Suppose that F is a generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation d on R respectively and $\tau(I) \neq 0$ such that F acts as homomorphism on I i.e., $F(xy) \pm F(x)F(y) = 0$, for all $x,y \in I$, then either R is commutative or $[F(x),\tau(x)] = 0$, for all $x \in I$.

Proof: We replacing G by F and g by d in theorem 6, we get the required result.
Theorem 7: Let R be a prime ring and I be a non-zero idealon R. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation g and d on R respectively and $\tau(I) \neq 0$. If $G(xy) \pm F(x)F(y) \pm \tau(x)\sigma(y) = 0$, for all $x,y \in I$, then either R is commutative or $[F(x),\tau(x)] = 0$, for all $x \in I$.

Proof: First we consider the case $G(xy) + F(x)F(y) + \tau(x)\sigma(y) = 0$, for all $x,y \in I$. (76)
We replacing y by yz in equation (76), we obtain
$G(xyz) + F(x)F(yz) + \tau(x)\sigma(yz) = 0$, for all $x,y,z \in I$

$$G(xy)\sigma(z) + \tau(xy)g(z) + F(x)\{F(y)\sigma(z) + \tau(y)d(z)\} + \tau(x)\sigma(yz) = 0$$
$$\{G(xy) + F(x)F(y) + \tau(x)\sigma(y)\}\sigma(z) + \tau(xy)g(z) + F(x)\tau(y)d(z) = 0$$

Using equation (76), it reduces to
$\tau(xy)g(z) + F(x)\tau(y)d(z) = 0$, for all $x,y,z \in I$. (77)
The equation (77) is same as equation (62) in theorem 6. Thus, by same argument of theorem 6, we can conclude the result here.

Theorem 8: Let R be a prime ring and I be a non-zero idealon R. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation g and d on R respectively and $\tau(I) \neq 0$. If $G(xy) \pm F(y)F(x) = 0$, for all $x,y \in I$, then either R is commutative or $[F(x),\tau(x)] = 0$, for all $x \in I$.
Proof: We have $G(xy) + F(y)F(x) = 0$, for all $x,y \in I$. (78)
We replacing x by xz in equation (78), we get

$$G(xzy) + F(y)F(xz) = 0$$
$$G(x)\sigma(zy) + \tau(x)g(zy) + F(y)F(x)\sigma(z) + F(y)\tau(x)d(z) = 0$$

Using equation (78), it reduces to

$$G(x)\sigma(zy) + \tau(x)g(zy) - G(xy)\sigma(z) + F(y)\tau(x)d(z) = 0$$
$$G(x)\sigma(zy) + \tau(x)g(z)\sigma(y) + \tau(x)\tau(z)g(y) - G(x)\sigma(y)\sigma(z) - \tau(x)g(y)\sigma(z) + F(y)\tau(x)d(z) = 0$$

$G(x)(\sigma(zy) - \sigma(yz)) + \tau(x)g(z)\sigma(y) + \tau(x)\tau(z)g(y) - \tau(x)g(y)\sigma(z) + F(y)\tau(x)d(z) = 0$, for all $x,y,z \in I$. (79)
We replacing z by y in equation (79), we get
$\tau(x)\tau(y)g(y) + F(y)\tau(x)d(y) = 0$, for all $x,y \in I$. (80)
We replacing x by zx in equation (80), we get
$\tau(z)\tau(x)\tau(y)g(y) + F(y)\tau(z)\tau(x)d(y) = 0$, for all $x,y,z \in I$. (81)
Left multiplying equation (80) by $\tau(z)$, we get
$\tau(z)\tau(x)\tau(y)g(y) + \tau(z)F(y)\tau(x)d(y) = 0$, for all $x,y,z \in I$. (82)
We subtracting equation (82) from equation (81), we get

$$\big(F(y)\tau(z) - \tau(z)F(y)\big)\tau(x)d(y) = 0$$

$[F(y),\tau(z)]\tau(x)d(y) = 0$, for all $x,y,z \in I$. (83)
We replacing x by sx, $s \in R$ in equation (83), we get

$$[F(y),\tau(z)]\tau(sx)d(y) = 0$$
$$[F(y),\tau(z)]R\tau(x)d(y) = 0$$

Since R is prime, we get either $[F(y),\tau(z)] = 0$, for all $y,z \in I$ or $\tau(x)d(y) = 0$, for all $x,y \in I$.
Since τ is an automorphism of R and $\tau(I) \neq 0$, we have either $[F(y),\tau(z)] = 0$, for all $y,z \in I$ or $d(y) = 0$, for all $y \in I$. (84)
Now let $A = \{x \in I / [F(x),\tau(x)] = 0\}$ and $B = \{x \in I / d(x) = 0\}$.
Clearly, A and B are additive proper subgroups of I whose union is I.
Since a group cannot be the set theoretic union of two proper subgroups.
Hence either $A = I$ or $B = I$.

If $B = I$, then $d(x) = 0$, for all $x \in I$, by lemma 1 implies that R is commutative.
On the other hand if $A = I$, then $[F(x), \tau(x)] = 0$, for all $x \in I$.
We replacing z by $x(\tau^{-1}(\sigma))(z)$ in equation (84), we get

$$[F(y), \tau(x)\sigma(z)] = 0$$

$[F(y), \tau(x)]\sigma(z) + \tau(x)[F(y), \sigma(z)] = 0$, for all $x, y, z \in I$. (85)
Using equation (84) in equation (85), we get
$\tau(x)[F(y), \sigma(z)] = 0$, for all $x, y, z \in I$.
Using primeness of Rgives $[F(y), \sigma(z)] = 0$, for all $y, z \in I$. (86)
We replacing y by xy in the above equation, we get

$$[F(xy), \sigma(z)] = 0$$

$$[F(x)\sigma(y), \sigma(z)] + [\tau(x)d(y), \sigma(z)] = 0$$

$F(x)[\sigma(y), \sigma(z)] + [F(x), \sigma(z)]\sigma(y) + \tau(x)[d(y), \sigma(z)] + [\tau(x), \sigma(z)]d(y) = 0$,for all $x, y, z \in I$. (87)
We replacing z by y in equation (86) and the results used in equation (87), we get
$\tau(x)[d(y), \sigma(y)] + [\tau(x), \sigma(y)]d(y) = 0$, for all $x, y \in I$. (88)
We replacing x by zx in equation (88) and using equation (88), we get
$[\tau(z), \sigma(y)]\tau(x)d(y) = 0$, for all $x, y, z \in I$. (89)
Using primeness of R implies that either $[\tau(z), \sigma(y)] = 0$or $d(y) = 0$.
If $d(y) = 0$, then by lemma 1, R is commutative.
If $[\tau(z), \sigma(y)] = 0$, for all $y, z \in I$, which easily implies that R is commutative.

Corollary 2: Let R be a prime ring and I be a non-zero idealon R. Suppose that F is a generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation d on$R$ respectively and $\tau(I) \neq 0$ such that F acts as anti-homomorphism on I i.e., $F(xy) \pm F(y)F(x) = 0$ for all $x, y \in I$, then either R is commutative or $[F(x), \tau(x)] = 0$, for all $x \in I$.

Proof: We replacing G by F and g by d in theorem 8, we get the required result.
Theorem 9: Let R be a prime ring and I be a non-zero idealon R. Suppose that F is a generalized(σ,τ)-derivation on R associated with (σ,τ)-derivation d on$R$ such that $\tau(I)d(I) \neq 0$. If $F(xoy) \pm (d(x)oy)_{\sigma,\tau} = 0$, for all $x, y \in I$, then R contains a non zero central ideal.

Proof: First, we have $F(xoy) - (d(x)oy)_{\sigma,\tau} = 0$, for all $x, y \in I$. (90)
We replacing y by yx in equation (90), we get

$$F(xo(yx)) - (d(x)o(yx))_{\sigma,\tau} = 0$$

$$F((xoy)x - y[x,x]) - ((d(x)oy)_{\sigma,\tau}\sigma(x) - \tau(y)[x,x]_{\sigma,\tau}) = 0$$

$$F((xoy)x) - (d(x)oy)_{\sigma,\tau}\sigma(x) + \tau(y)[x,x]_{\sigma,\tau} = 0$$

$$F(xoy)\sigma(x) + \tau(xoy)d(x) - (d(x)oy)_{\sigma,\tau}\sigma(x) + \tau(y)[x,x]_{\sigma,\tau} = 0$$

$(F(xoy) - (d(x)oy)_{\sigma,\tau})\sigma(x) + \tau(xoy)d(x) + \tau(y)[x,x]_{\sigma,\tau} = 0$, for all $x, y \in I$. (91)
Using equation (90) in equation (91), we get
$\tau(xoy)d(x) + \tau(y)[x,x]_{\sigma,\tau} = 0$, for all $x, y \in I$. (92)
We replacing $y$by ry, $r \in R$ in equation (92), we get

$$\tau(xo(ry))d(x) + \tau(ry)[x,x]_{\sigma,\tau} = 0$$

$$\tau(r(xoy) + [x,r]y)d(x) + \tau(r)\tau(y)[x,x]_{\sigma,\tau} = 0$$

$\tau(r)\tau(xoy)d(x) + \tau[x,r]\tau(y)d(x) + \tau(r)\tau(y)[x,x]_{\sigma,\tau} = 0$, for all $x, y \in I$and $r \in R$. (93)
Left multiplying equation (92) by $\tau(r)$, we have
$\tau(r)\tau(xoy)d(x) + \tau(r)\tau(y)[x,x]_{\sigma,\tau} = 0$, for all $x, y \in I$and $r \in R$. (94)
We subtracting equation (94) from equation (93), we get
$\tau[x,r]\tau(y)d(x) = 0$, for all $x, y \in I$and $r \in R$. (95)
We replacing $y$by sy, $s \in R$ in the equation (95), we get
$\tau([x,r])\tau(s)\tau(y)d(x) = 0$, for all $x, y \in I$and $s, r \in R$.
$\tau([x,r])R\tau(y)d(x) = 0$, for all $x, y \in I$and $r \in R$. (96)
Since R is prime, we get either $\tau([x,r]) = 0$, for all $x \in I$, $r \in R$ or $\tau(y)d(x) = 0$, for all $x, y \in I$.
Since τ is an automorphism of $R$and $\tau(I) \neq 0$, we have either $[x,r] = 0$, for all $x \in I$ or $d(x) = 0$, for all $x \in I$.
Now let $A = \{x \in I / [x,r] = 0\}$and $B = \{x \in I / d(x) = 0\}$.
Clearly, A and B are additive proper subgroups of I whose union is I.
Since a group cannot be the set theoretic union of two proper subgroups.
Hence either $A = I$or $B = I$.
If $A = I$, then $[x,r] = 0$, for all $x \in I$ implies that R is commutative

On the other hand, If $B = I$, then $d(x) = 0$, for all $x \in I$, by lemma 1 implies that R contains a non zero central ideal.
Similarly, we can obtain the same conclusion when $F(xoy) + (d(x)oy)_{\sigma,\tau} = 0$, for all $x, y \in I$. Thus theproof is completed.
Theorem 10: Let R be a prime ring and I be a non-zero idealon R. Suppose that F is a generalized(σ,τ)-derivation on R associated with (σ,τ)-derivation d on$R$ such that $\tau(I)d(I) \neq 0$. If $F[x,y] \pm [d(x),y]_{\sigma,\tau} = 0$, for all $x, y \in I$, then R contains a non zero central ideal.
Proof: First, we have $F[x,y] - [d(x),y]_{\sigma,\tau} = 0$, for all $x, y \in I$. (97)
We replacing y by yx in equation (97), we get

$$F[x,yx] - [d(x),yx]_{\sigma,\tau} = 0$$
$$F([x,y]x + y[x,x]) - \left([d(x),y]_{\sigma,\tau}\sigma(x) + \tau(y)[x,x]_{\sigma,\tau}\right) = 0$$
$$F([x,y]x) - [d(x),y]_{\sigma,\tau}\sigma(x) - \tau(y)[x,x]_{\sigma,\tau} = 0$$
$$F[x,y]\sigma(x) + \tau[x,y]d(x) - [d(x),y]_{\sigma,\tau}\sigma(x) - \tau(y)[x,x]_{\sigma,\tau} = 0$$

$\left(F[x,y] - [d(x),y]_{\sigma,\tau}\right)\sigma(x) + \tau[x,y]d(x) - \tau(y)[x,x]_{\sigma,\tau} = 0$, for all $x, y \in I$. (98)
Using equation (97) in equation (98), we get
$\tau[x,y]d(x) - \tau(y)[x,x]_{\sigma,\tau} = 0$, for all $x, y \in I$. (99)
We replacing $y$by ry, $r \in R$ in equation (99), we get

$$\tau[x,ry]d(x) - \tau(ry)[x,x]_{\sigma,\tau} = 0$$
$$\tau(r[x,y] + [x,r]y)d(x) - \tau(r)\tau(y)[x,x]_{\sigma,\tau} = 0$$

$\tau(r)\tau[x,y]d(x) + \tau[x,r]\tau(y)d(x) - \tau(r)\tau(y)[x,x]_{\sigma,\tau} = 0$, for all $x, y \in I$and $r \in R$. (100)
Left multiplying equation (99) by $\tau(r)$, we have
$\tau(r)\tau[x,y]d(x) - \tau(r)\tau(y)[x,x]_{\sigma,\tau} = 0$, for all $x, y \in I$and $r \in R$. (101)
We subtracting equation (101) from equation (100), we get
$\tau[x,r]\tau(y)d(x) = 0$, for all $x, y \in I$and $r \in R$. (102)
The equation (102) is same as equation (95) in theorem 9. Thus, by same argument of theorem 9, we can conclude the result here.
Similarly, we can obtain the same conclusion when $F[x,y] + [d(x),y]_{\sigma,\tau} = 0$, for all $x, y \in I$. Thus theproof is completed.

Theorem 11: Let R be a prime ring and I be a non-zero idealon R. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation g and d on R respectively and $\tau(I) \neq 0$. If $G(xy) \pm [F(x),y]_{\sigma,\tau} \pm \tau(yx) = 0$, for all $x, y \in I$, then either$R$ is commutative or $[g(x),\tau(x)] = 0$, $[F(x),x]_{\sigma,\tau} \in Z(R)$ and $[d(x),x]_{\sigma,\tau} = 0$, for all $x \in I$.

Proof: First we have $G(xy) + [F(x),y]_{\sigma,\tau} + \tau(yx) = 0$, for all $x, y \in I$. (103)
We replacing y by yz in equation (103), we get

$$G(xyz) + [F(x),yz]_{\sigma,\tau} + \tau(yzx) = 0$$
$$G(xy)\sigma(z) + \tau(xy)g(z) + [F(x),y]_{\sigma,\tau}\sigma(z) + \tau(y)[F(x),z]_{\sigma,\tau} + \tau(yzx) = 0$$
$$\tau(xy)g(z) + \tau(y)[F(x),z]_{\sigma,\tau} + \tau(yzx) + (G(xy) + [F(x),y]_{\sigma,\tau})\sigma(z) = 0$$

Using equation (103), it reduces to
$\tau(xy)g(z) + \tau(y)[F(x),z]_{\sigma,\tau} + \tau(yzx) - \tau(yx)\sigma(z) = 0$, for all $x, y, z \in I$. (104)
We replacing $y$by ry, $r \in R$ in equation (104), we get
$\tau(xry)g(z) + \tau(ry)[F(x),z]_{\sigma,\tau} + \tau(ryzx) - \tau(ryx)\sigma(z) = 0$, for all $x, y, z \in I$and $r \in R$. (105)
Left multiplying equation (104) by $\tau(r)$, we get
$\tau(r)\tau(xy)g(z) + \tau(r)\tau(y)[F(x),z]_{\sigma,\tau} + \tau(r)\tau(yzx) - \tau(r)\tau(yx)\sigma(z) = 0$,for all $x, y, z \in I$and $r \in R$. (106)
We subtracting equation (106) from equation (105), we get
$\tau[x,r]\tau(y)g(z) = 0$, for all $x, y, z \in I$and $r \in R$. (107)
We replacing y by yx in the equation (107), we get
$\tau[x,r]\tau(yx)g(z) = 0$, for all $x, y, z \in I$and $r \in R$. (108)
Right multiplying equation (107) by $\tau(x)$, we get
$\tau[x,r]\tau(y)g(z)\tau(x) = 0$, for all $x, y, z \in I$and $r \in R$. (109)
We subtracting equation (108) from equation (109), we get
$\tau[x,r]\tau(y)[g(z),\tau(x)] = 0$, for all $x, y, z \in I$and $r \in R$.
We replacing $y$by sy, $s \in R$ and z by x in the above equation, we get

$$\tau([x,r])\tau(s)\tau(y)[g(x),\tau(x)] = 0$$

$\tau([x,r])R\tau(y)[g(x),\tau(x)] = 0$, for all $x, y \in I$and $s, r \in R$. (110)
Using primenessof$R$, we get either $\tau([x,r]) = 0$ or $\tau(y)[g(x),\tau(x)] = 0$

Since τ is an automorphism of $R$and $\tau(I) \neq 0$, we have either $[x,r] = 0$, for all $x \in I$ and $r \in R$ or $[g(x),\tau(x)] = 0$, for all $x \in I$.
Now let $A = \{x \in I/[x,r] = 0, r \in R\}$and $B = \{x \in I/[g(x),\tau(x)] = 0\}$.
Clearly, A and B are additive proper subgroups of I whose union is I.
Since a group cannot be the set theoretic union of two proper subgroups.
Hence either $A = I$or $B = I$.
If $A = I$, then $[x,r] = 0$ implies that R is commutative
On the other hand if $B = I$, then $[g(x),\tau(x)] = 0$, for all $x \in I$. (111)
We replacing $y$by yr, $r \in R$ in equation (104), we get
$\tau(xyr)g(z) + \tau(yr)[F(x),z]_{\sigma,\tau} + \tau(yrzx) - \tau(yrx)\sigma(z) = 0$, for all $x,y,z \in I$and $r \in R$. (112)

Right multiplying equation (104) by $\tau(r)$, we get
$\tau(xy)g(z)\tau(r) + \tau(y)[F(x),z]_{\sigma,\tau}\tau(r) + \tau(yzx)\tau(r) - \tau(yx)\sigma(z)\tau(r) = 0$, for all $x,y,z \in I$and $r \in R$.(113)
We subtracting equation (112) from equation (113), we get

$$\tau(xy)[g(z),\tau(r)] + \tau(y)\big[[F(x),z]_{\sigma,\tau},\tau(r)\big] + \tau(yzx)\tau(r) - \tau(yx)\sigma(z)\tau(r) - \tau(yrzx) + \tau(yrx)\sigma(z) = 0$$

We replacing $\sigma(z)$ by $\tau(z)$ in above equation, we get

$$\tau(xy)[g(z),\tau(r)] + \tau(y)\big[[F(x),z]_{\sigma,\tau},\tau(r)\big] + \tau(yzx)\tau(r) - \tau(yx)\tau(z)\tau(r) - \tau(yrzx) + \tau(yrx)\tau(z) = 0$$

$$\tau(xy)[g(z),\tau(r)] + \tau(y)\big[[F(x),z]_{\sigma,\tau},\tau(r)\big] + \tau(y)\tau(zx)\tau(r) - \tau(y)\tau(r)\tau(zx) - \tau(y)\tau(xz)\tau(r) + \tau(y)\tau(r)\tau(xz) = 0$$

$$\tau(xy)[g(z),\tau(r)] + \tau(y)\big[[F(x),z]_{\sigma,\tau},\tau(r)\big] + \tau(y)\big(\tau(zx)\tau(r) - \tau(r)\tau(zx)\big) - \tau(y)\big(\tau(xz)\tau(r) - \tau(r)\tau(xz)\big) = 0$$

$$\tau(xy)[g(z),\tau(r)] + \tau(y)\big[[F(x),z]_{\sigma,\tau},\tau(r)\big] + \tau(y)[\tau(zx),\tau(r)] - \tau(y)[\tau(xz),\tau(r)] = 0$$

$$\tau(xy)[g(z),\tau(r)] + \tau(y)\big[[F(x),z]_{\sigma,\tau},\tau(r)\big] + \tau(y)([\tau(zx) - \tau(xz),\tau(r)]) = 0$$

$\tau(xy)[g(z),\tau(r)] + \tau(y)\big[[F(x),z]_{\sigma,\tau},\tau(r)\big] + \tau(y)\big(\big[[\tau(z),\tau(x)],\tau(r)\big]\big) = 0$, for all $x,y,z \in I$and $r \in R$.(114)
We replacing z by $x$and using equation (111) in the equation (114), we get

$$\tau(y)\big[[F(x),x]_{\sigma,\tau},\tau(r)\big] = 0$$

We replacing $y$by ys, $s \in R$ in the above equation, we get

$$\tau(ys)\big[[F(x),x]_{\sigma,\tau},\tau(r)\big] = 0$$

$\tau(y)R\big[[F(x),x]_{\sigma,\tau},\tau(r)\big] = 0$, for all $x,y \in I$and $r \in R$.
Using primenessof$R$, we get $\big[[F(x),x]_{\sigma,\tau},\tau(r)\big] = 0$ implies that $[F(x),x]_{\sigma,\tau} \in Z(R)$, for all $x \in I$. (115)
We replacing x by xu in equation (104), we get
$\tau(xuy)g(z) + \tau(y)[F(xu),z]_{\sigma,\tau} + \tau(yzxu) - \tau(yxu)\sigma(z) = 0$, for all $x,y,z,u \in I$.

$$\tau(xuy)g(z) + \tau(y)[F(x)\sigma(u) + \tau(x)d(u),z]_{\sigma,\tau} + \tau(yzxu) - \tau(yxu)\sigma(z) = 0$$

$$\tau(xuy)g(z) + \tau(y)[F(x)\sigma(u),z]_{\sigma,\tau} + \tau(y)[\tau(x)d(u),z]_{\sigma,\tau} + \tau(yzxu) - \tau(yxu)\sigma(z) = 0$$

$\tau(xuy)g(z) + \tau(y)(F(x)[\sigma(u),\sigma(z)] + [F(x),z]_{\sigma,\tau}\sigma(u)) + \tau(y)(\tau(x)[d(u),z]_{\sigma,\tau} + [\tau(x),\tau(z)]d(u)) + \tau(yzxu) - \tau(yxu)\sigma(z) = 0$, for all $x,y,z,u \in I$. (116)
Right multiplying equation (104) by $\tau(u)$, we get
$\tau(xy)g(z)\tau(u) + \tau(y)[F(x),z]_{\sigma,\tau}\tau(u) + \tau(yzx)\tau(u) - \tau(yx)\sigma(z)\tau(u) = 0$, for all $x,y,z,u \in I$. (117)
We subtracting equation (117) from equation (116), we get
$\tau(x)\tau(u)\tau(y)g(z) - \tau(x)\tau(y)g(z)\tau(u) + \tau(y)F(x)[\sigma(u),\sigma(z)] + \tau(y)[F(x),z]_{\sigma,\tau}\sigma(u) - \tau(y)[F(x),z]_{\sigma,\tau}\tau(u) + \tau(y)\tau(x)[d(u),z]_{\sigma,\tau} + \tau(y)[\tau(x),\tau(z)]d(u) - \tau(yxu)\sigma(z) + \tau(yx)\sigma(z)\tau(u) = 0$, for all $x,y,z \in I$.
We replacing $\sigma(z)$ by $\tau(z)$ and $\sigma(u)$ by $\tau(u)$in above equation, we get

$$\tau(x)\tau(u)\tau(y)g(z) - \tau(x)\tau(y)g(z)\tau(u) + \tau(y)F(x)[\tau(u),\tau(z)] + \tau(y)\tau(x)[d(u),z]_{\sigma,\tau} + \tau(y)[\tau(x),\tau(z)]d(u) - \tau(yxu)\tau(z) + \tau(yx)\tau(z)\tau(u) = 0$$

$$\tau(x)[\tau(u),\tau(y)g(z)] + \tau(y)F(x)[\tau(u),\tau(z)] + \tau(y)\tau(x)[d(u),z]_{\sigma,\tau} + \tau(y)[\tau(x),\tau(z)]d(u) + \tau(yx)[\tau(z),\tau(u)] = 0$$

$\tau(x)\tau(y)[\tau(u),g(z)] + \tau(x)[\tau(u),\tau(y)]g(z) + \tau(y)F(x)[\tau(u),\tau(z)] + \tau(y)\tau(x)[d(u),z]_{\sigma,\tau} + \tau(y)[\tau(x),\tau(z)]d(u) + \tau(yx)[\tau(z),\tau(u)] = 0$, for all $x,y,z,u \in I$.(118)
We replacing z by u and using equation (111) in the equation (118), we get
$\tau(x)[\tau(u),\tau(y)]g(u) + \tau(y)\tau(x)[d(u),u]_{\sigma,\tau} + \tau(y)[\tau(x),\tau(u)]d(u) = 0$, for all $x,y,u \in I$.
We replacing y by u in the above equation, we get
$\tau(u)\tau(x)[d(u),u]_{\sigma,\tau} + \tau(u)[\tau(x),\tau(u)]d(u) = 0$, for all $x,u \in I$.
We replacing u by x in the above equation, we get
$\tau(x)\tau(x)[d(x),x]_{\sigma,\tau} = 0$, for all $x \in I$.
Since $\tau(I) \neq 0$, then we have $[d(x),x]_{\sigma,\tau} = 0$, for all $x \in I$. (119)

Similarly, we can obtain the same conclusion when $G(xy)-[F(x),y]_{\sigma,\tau}-\tau(yx)=0$, for all $x,y\in I$. Thus theproof is completed.

Theorem 12: Let R be a prime ring and I be a non-zero idealon R. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation$g$ and$d$ on$R$ respectively and $\tau(I)g(I)\neq 0$. If $G(xy)\pm [F(x),y]_{\sigma,\tau}\pm [x,y]_{\sigma,\tau}=0$, for all $x,y\in I$, then either$R$ is commutative or $[g(x),\tau(x)]=0$, $[F(x),x]_{\sigma,\tau}\in Z(R)$ and $[d(x),x]_{\sigma,\tau}=0$, for all $x\in I$.

Proof: First we have $G(xy)+[F(x),y]_{\sigma,\tau}+[x,y]_{\sigma,\tau}=0$, for all $x,y\in I$. (120)
We replacing y by yz in equation (120), we get

$$G(xyz)+[F(x),yz]_{\sigma,\tau}+[x,yz]_{\sigma,\tau}=0$$
$$G(xy)\sigma(z)+\tau(xy)g(z)+[F(x),y]_{\sigma,\tau}\sigma(z)+\tau(y)[F(x),z]_{\sigma,\tau}+[x,y]_{\sigma,\tau}\sigma(z)+\tau(y)[x,z]_{\sigma,\tau}=0$$
$$\left(G(xy)+[F(x),y]_{\sigma,\tau}+[x,y]_{\sigma,\tau}\right)\sigma(z)+\tau(xy)g(z)+\tau(y)[F(x),z]_{\sigma,\tau}+\tau(y)[x,z]_{\sigma,\tau}=0$$

Using equation (120), it reduces to
$\tau(xy)g(z)+\tau(y)[F(x),z]_{\sigma,\tau}+\tau(y)[x,z]_{\sigma,\tau}=0$, for all $x,y,z\in I$.(121)
We replacing $y$by ry, $r\in R$ in equation (121), we get
$\tau(xry)g(z)+\tau(ry)[F(x),z]_{\sigma,\tau}+\tau(ry)[x,z]_{\sigma,\tau}=0$, for all $x,y,z\in I$and $r\in R$.(122)
Left multiplying equation (121) by $\tau(r)$, we get
$\tau(r)\tau(xy)g(z)+\tau(r)\tau(y)[F(x),z]_{\sigma,\tau}+\tau(r)\tau(y)[x,z]_{\sigma,\tau}=0$, for all $x,y,z\in I$and $r\in R$. (123)
We subtracting equation (123) from equation (122), we get
$\tau[x,r]\tau(y)g(z)=0$, for all $x,y,z\in I$and $r\in R$. (124)
The equation (124) is same as equation (107) in theorem 11. Thus, by same argument of theorem 11, we can conclude the result here.
Similarly, we can obtain the same conclusion when $G(xy)-[F(x),y]_{\sigma,\tau}-[x,y]_{\sigma,\tau}=0$, for all $x,y\in I$. Thus theproof is completed.

Theorem 13: Let R be a prime ring and I be a non-zero idealon R. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation$g$ and$d$ on$R$respectively and $\tau(I)g(I)\neq 0$. If $G(xy)\pm [F(x),y]_{\sigma,\tau}\pm (xoy)_{\sigma,\tau}=0$, for all $x,y\in I$, then either$R$ is commutative or $[g(x),\tau(x)]=0$, $[F(x),x]_{\sigma,\tau}\in Z(R)$ and $[d(x),x]_{\sigma,\tau}=0$, for all $x\in I$.
Proof: First we have $G(xy)+[F(x),y]_{\sigma,\tau}+(xoy)_{\sigma,\tau}=0$, for all $x,y\in I$. (125)
We replacing y by yz in equation (125), we get

$$G(xyz)+[F(x),yz]_{\sigma,\tau}+(xoyz)_{\sigma,\tau}=0$$
$$G(xy)\sigma(z)+\tau(xy)g(z)+[F(x),y]_{\sigma,\tau}\sigma(z)+\tau(y)[F(x),z]_{\sigma,\tau}+(xoy)_{\sigma,\tau}\sigma(z)-\tau(y)[x,z]_{\sigma,\tau}=0$$
$$\left(G(xy)+[F(x),y]_{\sigma,\tau}+(xoy)_{\sigma,\tau}\right)\sigma(z)+\tau(xy)g(z)+\tau(y)[F(x),z]_{\sigma,\tau}-\tau(y)[x,z]_{\sigma,\tau}=0$$

Using equation (125), it reduces to
$\tau(xy)g(z)+\tau(y)[F(x),z]_{\sigma,\tau}-\tau(y)[x,z]_{\sigma,\tau}=0$, for all $x,y,z\in I$. (126)
We replacing $y$by ry, $r\in R$ in equation (126), we get
$\tau(xry)g(z)+\tau(ry)[F(x),z]_{\sigma,\tau}-\tau(ry)[x,z]_{\sigma,\tau}=0$, for all $x,y,z\in I$and $r\in R$. (127)
Left multiplying equation (126) by $\tau(r)$, we get
$\tau(r)\tau(xy)g(z)+\tau(r)\tau(y)[F(x),z]_{\sigma,\tau}-\tau(r)\tau(y)[x,z]_{\sigma,\tau}=0$, for all $x,y,z\in I$and $r\in R$. (128)
We subtracting equation (128) from equation (127), we get
$\tau[x,r]\tau(y)g(z)=0$, for all $x,y,z\in I$and $r\in R$. (129)
The equation (129) is same as equation (107) in theorem 11. Thus, by same argument of theorem 11, we can conclude the result here.
Similarly, we can obtain the same conclusion when $G(xy)-[F(x),y]_{\sigma,\tau}-(xoy)_{\sigma,\tau}=0$, for all $x,y\in I$. Thus theproof is completed.

Theorem 14: Let R be a prime ring and I be a non-zero idealon R. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation g and d on R respectively and $\tau(I)\neq 0$. If $G(xy)\pm [F(x),y]_{\sigma,\tau}=0$, for all $x,y\in I$, then either R is commutative or $[g(x),\tau(x)]=0$,$[F(x),x]_{\sigma,\tau}\in Z(R)$ and $[d(x),x]_{\sigma,\tau}=0$, for all $x\in I$.
Proof: First we have $G(xy)+[F(x),y]_{\sigma,\tau}=0$, for all $x,y\in I$. (130)
We replacing y by yz in equation (130), we get

$$G(xyz)+[F(x),yz]_{\sigma,\tau}=0$$
$$G(xy)\sigma(z)+\tau(xy)g(z)+[F(x),y]_{\sigma,\tau}\sigma(z)+\tau(y)[F(x),z]_{\sigma,\tau}=0$$

$$\left(G(xy)+[F(x),y]_{\sigma,\tau}\right)\sigma(z)+\tau(xy)g(z)+\tau(y)[F(x),z]_{\sigma,\tau}=0$$

Using equation (130), it reduces to

$\tau(xy)g(z)+\tau(y)[F(x),z]_{\sigma,\tau}=0$, for all $x,y,z\in I$. (131)

We replacing $y$by ry, in equation (131), we get

$\tau(xry)g(z)+\tau(ry)[F(x),z]_{\sigma,\tau}=0$, for all $x,y,z,r\in I$. (132)

Left multiplying equation (131) by $\tau(r)$, we get

$\tau(r)\tau(xy)g(z)+\tau(r)\tau(y)[F(x),z]_{\sigma,\tau}=0$, for all $x,y,z,r\in I$. (133)

We subtracting equation (132) from equation (133), we get

$\tau[x,r]\tau(y)g(z)=0$, for all $x,y,z,r\in I$(134)

We replacing y by yx in the equation (134), we get

$\tau[x,r]\tau(yx)g(z)=0$, for all $x,y,z\in I$and $r\in R$. (135)

Right multiplying equation (134) by $\tau(x)$, we get

$\tau[x,r]\tau(y)g(z)\tau(x)=0$, for all $x,y,z\in I$and $r\in R$. (136)

We subtracting equation (135) from equation (136), we get

$\tau[x,r]\tau(y)[g(z),\tau(x)]=0$, for all $x,y,z\in I$and $r\in R$.

We replacing $y$by sy, $s\in R$ and z by x in the above equation, we get

$$\tau([x,r])\tau(s)\tau(y)[g(x),\tau(x)]=0$$

$\tau([x,r])R\tau(y)[g(x),\tau(x)]=0$, for all $x,y\in I$and $r\in R$.(137)

Using primenessof$R$, we get either $\tau([x,r])=0$ or $\tau(y)[g(x),\tau(x)]=0$

Since τ is an automorphism of $R$and $\tau(I)\neq 0$, we have either $[x,r]=0$, for all $x\in I$ and $r\in R$ or $[g(x),\tau(x)]=0$, for all $x\in I$.

Now let $A=\{x\in I/[x,r]=0, r\in R\}$and $B=\{x\in I/[g(x),\tau(x)]=0\}$.

Clearly, A and B are additive proper subgroups of I whose union is I.

Since a group cannot be the set theoretic union of two proper subgroups.

Hence either $A=I$or $B=I$.

If $A=I$, then $[x,r]=0$ implies that R is commutative

On the other hand if $B=I$, then $[g(x),\tau(x)]=0$, for all $x\in I$. (138)

We replacing y by yr in equation (131), we get

$\tau(xyr)g(z)+\tau(yr)[F(x),z]_{\sigma,\tau}=0$, for all $x,y,z,r\in I$. (139)

Right multiplying equation (131) by $\tau(r)$, we get

$\tau(xy)g(z)\tau(r)+\tau(y)[F(x),z]_{\sigma,\tau}\tau(r)=0$, for all $x,y,z,r\in I$. (140)

We subtracting equation (139) from equation (140) and using equation (138), we get

$$\tau(y)\left[[F(x),z]_{\sigma,\tau},\tau(r)\right]=0$$

Using primenessof$R$, we get

$$\left[[F(x),z]_{\sigma,\tau},\tau(r)\right]=0$$

That is $[F(x),x]_{\sigma,\tau}\in Z(R)$, for all $x\in I$. (141)

We replacing x by xu in equation (131), we get

$$\tau(xuy)g(z)+\tau(y)[F(xu),z]_{\sigma,\tau}=0$$

$$\tau(xuy)g(z)+\tau(y)[F(x)\sigma(u)+\tau(x)d(u),z]_{\sigma,\tau}=0$$

$$\tau(xuy)g(z)+\tau(y)[F(x)\sigma(z),z]_{\sigma,\tau}+\tau(y)[\tau(x)d(u),z]_{\sigma,\tau}=0$$

$\tau(xuy)g(z)+\tau(y)[F(x),z]_{\sigma,\tau}\sigma(u)+\tau(y)F(x)[\sigma(u),\sigma(z)]+\tau(y)\tau(x)[d(u),z]_{\sigma,\tau}+\tau(y)[\tau(x),\tau(z)]d(u)=0$, for all $x,y,z,u\in I$. (142)

Right multiplying equation (131) by $\sigma(u)$, we get

$\tau(xy)g(z)\sigma(u)+\tau(y)[F(x),z]_{\sigma,\tau}\sigma(u)=0$, for all $x,y,z,u\in I$. (143)

We subtracting equation (143) from equation (142), we get

$\tau(x)\tau(u)\tau(y)g(z)-\tau(x)\tau(y)g(z)\sigma(z)+\tau(y)F(x)[\sigma(u),\sigma(z)]+\tau(y)\tau(x)[d(u),z]_{\sigma,\tau}+\tau(y)[\tau(x),\tau(z)]d(u)=0$, for all $x,y,z,u\in I$. (144)

We replacing u by z and $\sigma(z)$ by $\tau(z)$ in the equation (144), we get

$$\tau(x)\tau(z)\tau(y)g(z)-\tau(x)\tau(y)g(z)\tau(z)+\tau(y)\tau(x)[d(z),z]_{\sigma,\tau}+\tau(y)[\tau(x),\tau(z)]d(z)=0$$

$$\tau(x)[\tau(z),\tau(y)g(z)]+\tau(y)\tau(x)[d(z),z]_{\sigma,\tau}+\tau(y)[\tau(x),\tau(z)]d(z)=0$$

$\tau(x)\tau(y)[\tau(z),g(z)]+\tau(x)[\tau(z),\tau(y)]g(z)+\tau(y)\tau(x)[d(z),z]_{\sigma,\tau}+\tau(y)[\tau(x),\tau(z)]d(z)=0$, for all $x,y,z\in I$.

We replacing y by z and using equation (138) in the above equation, we get

$\tau(z)\tau(x)[d(z),z]_{\sigma,\tau}+\tau(z)[\tau(x),\tau(z)]d(z)=0$, for all $x,y,z\in I$.

We replacing z by x in the above equation, we get

$\tau(x)\tau(x)[d(x),x]_{\sigma,\tau}=0$, for all $x,y\in I$.

Since $\tau(I)\neq 0$, we get$[d(x),x]_{\sigma,\tau}=0$, for all $x,y\in I$. (145)

Similarly, we can obtain the same conclusion when $G(xy)-[F(x),y]_{\sigma,\tau}=0$, for all $x,y\in I$. Thus theproof is completed.

Theorem 15: Let R be a prime ring and I be a non-zero idealon R. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation$g$ and$d$ on$R$ respectively and $\tau(I)g(I)\neq 0$. If $G(xy)\pm[F(x),y]_{\sigma,\tau}\pm\sigma(xy)=0$, for all $x,y\in I$, then either$R$ is commutative or $[g(x),\tau(x)]=0$, $[F(x),x]_{\sigma,\tau}\in Z(R)$ and $[d(x),x]_{\sigma,\tau}=0$, for all $x\in I$.

Proof: First we have $G(xy)+[F(x),y]_{\sigma,\tau}+\sigma(xy)=0$, for all $x,y\in I$. (146)

We replacing y by yz in equation (146), we get

$$G(xyz)+[F(x),yz]_{\sigma,\tau}+\sigma(xyz)=0$$
$$G(xy)\sigma(z)+\tau(xy)g(z)+[F(x),y]_{\sigma,\tau}\sigma(z)+\tau(y)[F(x),z]_{\sigma,\tau}+\sigma(xy)\sigma(z)=0$$
$$\left(G(xy)+[F(x),y]_{\sigma,\tau}+\sigma(xy)\right)\sigma(z)+\tau(xy)g(z)+\tau(y)[F(x),z]_{\sigma,\tau}=0$$

Using equation (146), it reduces to

$\tau(xy)g(z)+\tau(y)[F(x),z]_{\sigma,\tau}=0$, for all $x,y,z\in I$. (147)

The equation (147) is same as equation (131) in theorem 14. Thus, by same argument of theorem 14, we can conclude the result here.

Similarly, we can obtain the same conclusion when $G(xy)-[F(x),y]_{\sigma,\tau}-\sigma(xy)=0$, for all $x,y\in I$. Thus theproof is completed.

Theorem 16: Let R be a prime ring and I be a non-zero idealon R. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation g and d on R respectively and $\tau(I)\neq 0$. If $G(xy)\pm[F(y),x]_{\sigma,\tau}\pm\sigma(yx)=0$, for all $x,y\in I$, then either R is commutative or $g(x)\sigma(x)=0$ and $d(x)\sigma(x)=0$, for all $x\in I$.

Proof: First we have $G(xy)+[F(y),x]_{\sigma,\tau}+\sigma(yx)=0$, for all $x,y\in I$. (148)

We replacing y by yx in equation (148), we get

$$G(xyx)+[F(yx),x]_{\sigma,\tau}+\sigma(yxx)=0$$
$$G(xy)\sigma(x)+\tau(xy)g(x)+[F(y)\sigma(x)+\tau(y)d(x),x]_{\sigma,\tau}+\sigma(yxx)=0$$
$$G(xy)\sigma(x)+\tau(xy)g(x)+[F(y)\sigma(x),x]_{\sigma,\tau}+[\tau(y)d(x),x]_{\sigma,\tau}+\sigma(yxx)=0$$
$$G(xy)\sigma(x)+\tau(xy)g(x)+F(y)[\sigma(x),\sigma(x)]+[F(y),x]_{\sigma,\tau}\sigma(x)+\tau(y)[d(x),x]_{\sigma,\tau}+[\tau(y),\tau(x)]d(x)+\sigma(yxx)=0$$

$(G(xy)+[F(y),x]_{\sigma,\tau}+\sigma(yx))\sigma(x)+\tau(xy)g(x)+\tau(y)[d(x),x]_{\sigma,\tau}+[\tau(y),\tau(x)]d(x)=0$, for all $x,y\in I$.
(149)

Using equation (148) in equation (149), we get

$\tau(xy)g(x)+\tau(y)[d(x),x]_{\sigma,\tau}+[\tau(y),\tau(x)]d(x)=0$, for all $x,y\in I$. (150)

We replacing y by ry in equation (150), we get

$$\tau(xry)g(x)+\tau(ry)[d(x),x]_{\sigma,\tau}+[\tau(ry),\tau(x)]d(x)=0$$

$\tau(xry)g(x)+\tau(r)\tau(y)[d(x),x]_{\sigma,\tau}+\tau(r)[\tau(y),\tau(x)]d(x)+[\tau(r),\tau(x)]\tau(y)d(x)=0$, for all $x,y,r\in I$.(151)

Left multiplying equation (150) by $\tau(r)$, we get

$\tau(r)\tau(xy)g(x)+\tau(r)\tau(y)[d(x),x]_{\sigma,\tau}+\tau(r)[\tau(y),\tau(x)]d(x)=0$, for all $x,y,r\in I$. (152)

We subtracting equation (152) from equation (151), we get

$\tau[x,r]\tau(y)g(x)+[\tau(r),\tau(x)]\tau(y)d(x)=0$, for all $x,y,r\in I$. (153)

We replacing y by xy in equation (153), we get

$\tau[x,r]\tau(xy)g(x)+[\tau(r),\tau(x)]\tau(xy)d(x)=0$, for all $x,y,r\in I$. (154)

Left multiplying equation (150) by $\tau[x,r]$, we get

$\tau[x,r]\tau(xy)g(x)+\tau[x,r]\tau(y)[d(x),x]_{\sigma,\tau}+\tau[x,r][\tau(y),\tau(x)]d(x)=0$, for all $x,y,r\in I$.(155)

We subtracting equation (154) from equation (155), we get

$$\tau[x,r]\tau(y)[d(x),x]_{\sigma,\tau}+\tau[x,r][\tau(y),\tau(x)]d(x)-[\tau(r),\tau(x)]\tau(xy)d(x)=0$$
$$\tau[x,r]\tau(y)[d(x),x]_{\sigma,\tau}+\tau[x,r][\tau(y),\tau(x)]d(x)+\tau[x,r]\tau(xy)d(x)=0$$
$$\tau[x,r](\tau(y)[d(x),x]_{\sigma,\tau}+[\tau(y),\tau(x)]d(x)+\tau(xy)d(x))=0$$
$$\tau[x,r](\tau(y)d(x)\sigma(x)-\tau(y)\tau(x)d(x)+\tau(y)\tau(x)d(x)-\tau(x)\tau(y)d(x)+\tau(xy)d(x))=0$$
$$\tau[x,r]\tau(y)d(x)\sigma(x)=0$$

We replacing $y$by sy, $s\in R$ in the above equation, we get

$\tau[x,r]R\tau(y)d(x)\sigma(x)=0$, for all $x,y,r\in I$. (156)

Using primenessof$R$, we get either $\tau([x,r])=0$ or $\tau(y)d(x)\sigma(x)=0$

Since τ is an automorphism of $R$and $\tau(I)\neq 0$, we have either $[x,r]=0$, for all $x\in I$ or $d(x)\sigma(x)=0$, for all $x\in I$.

Now let $A=\{x\in I/[x,r]=0\}$and $B=\{x\in I/d(x)\sigma(x)=0\}$.

Clearly, A and B are additive proper subgroups of I whose union is I.
Since a group cannot be the set theoretic union of two proper subgroups.
Hence either $A = I$or $B = I$.
If $A = I$, then $[x,r] = 0$ implies that R is commutative.
On the other hand if $B = I$, then $d(x)\sigma(x) = 0$, for all $x \in I$.(157)
Right multiplying equation (150) by $\sigma(x)$, we get

$$\tau(xy)g(x)\sigma(x)+\tau(y)[d(x),x]_{\sigma,\tau}\sigma(x) + [\tau(y),\tau(x)]d(x)\sigma(x) = 0$$

$\tau(xy)g(x)\sigma(x) + \tau(y)d(x)\sigma(x)\sigma(x) - \tau(y)\tau(x)d(x)\sigma(x) + [\tau(y),\tau(x)]d(x)\sigma(x) = 0$, for all $x,y \in I$.(158)
Using equation (157) in equation (158), we get

$$\tau(x)\tau(y)g(x)\sigma(x) = 0$$

Using primeness of $R$and $\tau(I) \neq 0$, we get
$g(x)\sigma(x) = 0$, for all $x \in I$. (159)
Similarly, we can obtain the same conclusion when $G(xy) - [F(y),x]_{\sigma,\tau} - \sigma(yx) = 0$, for all $x,y \in I$. Thus theproof is completed.

Theorem 17: Let R be a prime ring and I be a non-zero idealon R. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation g and d on R respectively and $\tau(I) \neq 0$. If $G(xy) \pm [F(y),x]_{\sigma,\tau} \pm [x,y]_{\sigma,\tau} = 0$, for all $x,y \in I$, then either R is commutative or $g(x)\sigma(x) = 0$ and $d(x)\sigma(x) = 0$, for all $x \in I$.
Proof: First we have $G(xy) + [F(y),x]_{\sigma,\tau} + [x,y]_{\sigma,\tau} = 0$, for all $x,y \in I$. (160)
We replacing y by yx in equation (160), we get

$$G(xyx) + [F(yx),x]_{\sigma,\tau} + [x,yx]_{\sigma,\tau} = 0$$

$$G(xy)\sigma(x) + \tau(xy)g(x) + [F(y)\sigma(x) + \tau(y)d(x),x]_{\sigma,\tau} + [x,y]_{\sigma,\tau}\sigma(x) + \tau(y)[x,x]_{\sigma,\tau} = 0$$

$$G(xy)\sigma(x) + \tau(xy)g(x) + [F(y)\sigma(x),x]_{\sigma,\tau} + [\tau(y)d(x),x]_{\sigma,\tau} + [x,y]_{\sigma,\tau}\sigma(x) + \tau(y)[x,x]_{\sigma,\tau} = 0$$

$$G(xy)\sigma(x) + \tau(xy)g(x) + F(y)[\sigma(x),\sigma(x)] + [F(y),x]_{\sigma,\tau}\sigma(x) + \tau(y)[d(x),x]_{\sigma,\tau} + [\tau(y),\tau(x)]d(x) + [x,y]_{\sigma,\tau}\sigma(x) + \tau(y)[x,x]_{\sigma,\tau} = 0$$

$\big(G(xy) + [F(y),x]_{\sigma,\tau} + [x,y]_{\sigma,\tau}\big)\sigma(x) + \tau(xy)g(x)+\tau(y)[d(x),x]_{\sigma,\tau} + [\tau(y),\tau(x)]d(x) + \tau(y)[x,x]_{\sigma,\tau} = 0$, for all $x,y \in I$. (161)
Using equation (160) in equation (161), we get
$\tau(xy)g(x)+\tau(y)[d(x),x]_{\sigma,\tau} + [\tau(y),\tau(x)]d(x) + \tau(y)[x,x]_{\sigma,\tau} = 0$, for all $x,y \in I$. (162)
We replacing y by ry in equation (162), we get

$$\tau(xry)g(x)+\tau(ry)[d(x),x]_{\sigma,\tau} + [\tau(ry),\tau(x)]d(x) + \tau(ry)[x,x]_{\sigma,\tau} = 0$$

$\tau(xry)g(x)+\tau(r)\tau(y)[d(x),x]_{\sigma,\tau} + \tau(r)[\tau(y),\tau(x)]d(x) + [\tau(r),\tau(x)]\tau(y)d(x) + \tau(ry)[x,x]_{\sigma,\tau} = 0$,
for all $x,y,r \in I$. (163)
Left multiplying equation (162) by $\tau(r)$, we get
$\tau(r)\tau(xy)g(x)+\tau(r)\tau(y)[d(x),x]_{\sigma,\tau} + \tau(r)[\tau(y),\tau(x)]d(x) + \tau(r)\tau(y)[x,x]_{\sigma,\tau} = 0$, for all $x,y,r \in I$.(164)
We subtracting equation (164) from equation (163), we get
$\tau[x,r]\tau(y)g(x) + [\tau(r),\tau(x)]\tau(y)d(x) = 0$, for all $x,y,r \in I$. (165)
The equation (165) is same as equation (153) in theorem 16. Thus, by same argument of theorem 16, we can conclude the result here.
Similarly, we can obtain the same conclusion when $G(xy) - [F(y),x]_{\sigma,\tau} - [x,y]_{\sigma,\tau} = 0$, for all $x,y \in I$. Thus theproof is completed.
Theorem 18: Let R be a prime ring and I be a non-zero idealon$R$. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation g and d on R respectively and $\tau(I) \neq 0$. If $G(xy) \pm [F(y),x]_{\sigma,\tau} \pm (xoy)_{\sigma,\tau} = 0$, for all $x,y \in I$, then either R is commutative or $g(x)\sigma(x) = 0$ and $d(x)\sigma(x) = 0$, for all $x \in I$.
Proof: First we have $G(xy) + [F(y),x]_{\sigma,\tau} + (xoy)_{\sigma,\tau} = 0$, for all $x,y \in I$. (166)
We replacing y by yx in equation (166), we get

$$G(xyx) + [F(yx),x]_{\sigma,\tau} + (xoyx)_{\sigma,\tau} = 0$$

$$G(xy)\sigma(x) + \tau(xy)g(x) + [F(y)\sigma(x) + \tau(y)d(x),x]_{\sigma,\tau} + (xoy)_{\sigma,\tau}\sigma(x) - \tau(y)[x,x]_{\sigma,\tau} = 0$$

$$G(xy)\sigma(x) + \tau(xy)g(x) + [F(y)\sigma(x),x]_{\sigma,\tau} + [\tau(y)d(x),x]_{\sigma,\tau} + (xoy)_{\sigma,\tau}\sigma(x) - \tau(y)[x,x]_{\sigma,\tau} = 0$$

$$G(xy)\sigma(x) + \tau(xy)g(x) + F(y)[\sigma(x),\sigma(x)] + [F(y),x]_{\sigma,\tau}\sigma(x) + \tau(y)[d(x),x]_{\sigma,\tau} + [\tau(y),\tau(x)]d(x) + (xoy)_{\sigma,\tau}\sigma(x) - \tau(y)[x,x]_{\sigma,\tau} = 0$$

$\big(G(xy) + [F(y),x]_{\sigma,\tau} + (xoy)_{\sigma,\tau}\big)\sigma(x) + \tau(xy)g(x)+\tau(y)[d(x),x]_{\sigma,\tau} + [\tau(y),\tau(x)]d(x) - \tau(y)[x,x]_{\sigma,\tau} = 0$, for all $x,y \in I$. (167)
Using equation (166) in equation (167), we get
$\tau(xy)g(x)+\tau(y)[d(x),x]_{\sigma,\tau} + [\tau(y),\tau(x)]d(x) - \tau(y)[x,x]_{\sigma,\tau} = 0$, for all $x,y \in I$. (168)

We replacing y by ry in equation (168), we get

$$\tau(xry)g(x)+\tau(ry)[d(x),x]_{\sigma,\tau}+[\tau(ry),\tau(x)]d(x)-\tau(ry)[x,x]_{\sigma,\tau}=0$$

$\tau(xry)g(x)+\tau(r)\tau(y)[d(x),x]_{\sigma,\tau}+\tau(r)[\tau(y),\tau(x)]d(x)+[\tau(r),\tau(x)]\tau(y)d(x)-\tau(ry)[x,x]_{\sigma,\tau}=0$, for all $x,y,r\in I$. (169)

Left multiplying equation (168) by $\tau(r)$, we get

$\tau(r)\tau(xy)g(x)+\tau(r)\tau(y)[d(x),x]_{\sigma,\tau}+\tau(r)[\tau(y),\tau(x)]d(x)-\tau(r)\tau(y)[x,x]_{\sigma,\tau}=0$, for all $x,y,r\in I$. (170)

We subtracting equation (170) from equation (169), we get

$\tau[x,r]\tau(y)g(x)+[\tau(r),\tau(x)]\tau(y)d(x)=0$, for all $x,y,r\in I$. (171)

The equation (171) is same as equation (153) in theorem 16. Thus, by using same argument of theorem 16, we can conclude the result here.

Similarly, we can obtain the same conclusion when $G(xy)-[F(y),x]_{\sigma,\tau}-(xoy)_{\sigma,\tau}=0$, for all $x,y\in I$. Thus theproof is completed.

Theorem 19: Let R be a prime ring and I be a non-zero idealon R. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation g and d on R respectively and $\tau(I)\neq 0$. If $G(xy)\pm[F(y),x]_{\sigma,\tau}\pm\sigma(xy)=0$, for all $x,y\in I$, then either R is commutative or $g(x)\sigma(x)=0$ and $d(x)\sigma(x)=0$, for all $x\in I$.

Proof: First we have $G(xy)+[F(y),x]_{\sigma,\tau}+\sigma(xy)=0$, for all $x,y\in I$. (172)

We replacing y by yx in equation (172), we get

$$G(xyx)+[F(yx),x]_{\sigma,\tau}+\sigma(xyx)=0$$

$$G(xy)\sigma(x)+\tau(xy)g(x)+[F(y)\sigma(x)+\tau(y)d(x),x]_{\sigma,\tau}+\sigma(xyx)=0$$

$$G(xy)\sigma(x)+\tau(xy)g(x)+[F(y)\sigma(x),x]_{\sigma,\tau}+[\tau(y)d(x),x]_{\sigma,\tau}+\sigma(xyx)=0$$

$$G(xy)\sigma(x)+\tau(xy)g(x)+F(y)[\sigma(x),\sigma(x)]+[F(y),x]_{\sigma,\tau}\sigma(x)+\tau(y)[d(x),x]_{\sigma,\tau}+[\tau(y),\tau(x)]d(x)+\sigma(xyx)=0$$

$\left(G(xy)+[F(y),x]_{\sigma,\tau}+\sigma(xy)\right)\sigma(x)+\tau(xy)g(x)+\tau(y)[d(x),x]_{\sigma,\tau}+[\tau(y),\tau(x)]d(x)=0$, for all $x,y\in I$. (173)

Using equation (172) in equation (173), we get

$\tau(xy)g(x)+\tau(y)[d(x),x]_{\sigma,\tau}+[\tau(y),\tau(x)]d(x)=0$, for all $x,y\in I$. (174)

The equation (174) is same as equation (150) in theorem 16. Thus, by using same argument of theorem 16, we can conclude the result here.

Similarly, we can obtain the same conclusion when $G(xy)-[F(y),x]_{\sigma,\tau}-\sigma(xy)=0$, for all $x,y\in I$. Thus theproof is completed.

Theorem 20: Let R be a prime ring and I be a non-zero idealon R. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation g and d on R respectively and $\tau(I)\neq 0$. If $G(xy)\pm[F(y),x]_{\sigma,\tau}=0$, for all $x,y\in I$, then either R is commutative or $g(x)\sigma(x)=0$ and $d(x)\sigma(x)=0$, for all $x\in I$.

Proof: First we have $G(xy)+[F(y),x]_{\sigma,\tau}=0$, for all $x,y\in I$. (175)

We replacing y by yx in equation (175), we get

$$G(xyx)+[F(yx),x]_{\sigma,\tau}=0$$

$$G(xy)\sigma(x)+\tau(xy)g(x)+[F(y)\sigma(x)+\tau(y)d(x),x]_{\sigma,\tau}=0$$

$$G(xy)\sigma(x)+\tau(xy)g(x)+[F(y)\sigma(x),x]_{\sigma,\tau}+[\tau(y)d(x),x]_{\sigma,\tau}=0$$

$$G(xy)\sigma(x)+\tau(xy)g(x)+F(y)[\sigma(x),\sigma(x)]+[F(y),x]_{\sigma,\tau}\sigma(x)+\tau(y)[d(x),x]_{\sigma,\tau}+[\tau(y),\tau(x)]d(x)=0$$

$\left(G(xy)+[F(y),x]_{\sigma,\tau}\right)\sigma(x)+\tau(xy)g(x)+\tau(y)[d(x),x]_{\sigma,\tau}+[\tau(y),\tau(x)]d(x)=0$, for all $x,y\in I$. (176)

Using equation (175) in equation (176), we get

$\tau(xy)g(x)+\tau(y)[d(x),x]_{\sigma,\tau}+[\tau(y),\tau(x)]d(x)=0$, for all $x,y\in I$. (177)

The equation (177) is same as equation (150) in theorem 16. Thus, by using same argument of theorem 16, we can conclude the result here.

Similarly, we can obtain the same conclusion when $G(xy)-[F(y),x]_{\sigma,\tau}=0$, for all $x,y\in I$. Thus theproof is completed.

Theorem 21: Let R be a prime ring and I be a non-zero idealon R. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation g and d on R respectively and $\tau(I)g(I)\neq 0$. If $G(xy)\pm(F(x)oy)_{\sigma,\tau}\pm\tau(yx)=0$, for all $x,y\in I$, theneither R is a commutative or $[g(x),\tau(x)]=0$, $[F(x),x]_{\sigma,\tau}\in Z(R)$ and $[d(x),x]_{\sigma,\tau}=0$, for all $x\in I$.

Proof: First we have $G(xy)+(F(x)oy)_{\sigma,\tau}+\tau(yx)=0$, for all $x,y\in I$. (178)

We replacing y by yz in equation (178), we get

$$G(xyz) + (F(x)oyz)_{\sigma,\tau} + \tau(yzx) = 0$$
$$G(xy)\sigma(z) + \tau(xy)g(z) + (F(x)oy)_{\sigma,\tau}\sigma(z) - \tau(y)[F(x),z]_{\sigma,\tau} + \tau(yzx) = 0$$
$$\tau(xy)g(z) - \tau(y)[F(x),z]_{\sigma,\tau} + \tau(yzx) + \big(G(xy) + (F(x)oy)_{\sigma,\tau}\big)\sigma(z) = 0$$

Using equation (178), it reduces to
$\tau(xy)g(z) - \tau(y)[F(x),z]_{\sigma,\tau} + \tau(yzx) - \tau(yx)\sigma(z) = 0$, for all $x,y,z \in I$. (179)
We replacing $y$by ry, $r \in R$ in equation (179), we get
$\tau(xry)g(z) - \tau(ry)[F(x),z]_{\sigma,\tau} + \tau(ryzx) - \tau(ryx)\sigma(z) = 0$, for all $x,y,z \in I$and $r \in R$.(180)
Left multiplying equation (179) by $\tau(r)$, we get
$\tau(r)\tau(xy)g(z) - \tau(r)\tau(y)[F(x),z]_{\sigma,\tau} + \tau(r)\tau(yzx) - \tau(r)\tau(yx)\sigma(z) = 0$,
for all $x,y,z \in I$and $r \in R$. (181)
We subtracting equation (181) from equation (180), we get
$\tau[x,r]\tau(y)g(z) = 0$, for all $x,y,z \in I$and $r \in R$. (182)
We replacing y by yx in the equation (182), we get
$\tau[x,r]\tau(yx)g(z) = 0$, for all $x,y,z \in I$and $r \in R$. (183)
Right multiplying equation (182) by $\tau(x)$, we get
$\tau[x,r]\tau(y)g(z)\tau(x) = 0$, for all $x,y,z \in I$and $r \in R$. (184)
We subtracting equation (183) from equation (184), we get
$\tau[x,r]\tau(y)[g(z),\tau(x)] = 0$, for all $x,y,z \in I$and $r \in R$.
We replacing $y$by sy, $s \in R$ and z by x in the above equation, we get

$$\tau([x,r])\tau(s)\tau(y)[g(x),\tau(x)] = 0$$

$\tau([x,r])R\tau(y)[g(x),\tau(x)] = 0$, for all $x,y \in I$and $s,r \in R$.(185)
Using primenessof$R$, we get either $\tau([x,r]) = 0$ or $\tau(y)[g(x),\tau(x)] = 0$
Since τ is an automorphism of $R$and $\tau(I) \neq 0$, we have either $[x,r] = 0$, for all $x \in I$ and $r \in R$ or $[g(x),\tau(x)] = 0$, for all $x \in I$.
Now let $A = \{x \in I/[x,r] = 0, r \in R\}$and $B = \{x \in I/[g(x),\tau(x)] = 0\}$.
Clearly, A and B are additive proper subgroups of I whose union is I.
Since a group cannot be the set theoretic union of two proper subgroups.
Hence either $A = I$or $B = I$.
If $A = I$, then $[x,r] = 0$ implies that R is commutative
On the other hand if $B = I$, then $[g(x),\tau(x)] = 0$, for all $x \in I$. (186)
We replacing $y$by yr, $r \in R$ in equation (179), we get
$\tau(xyr)g(z) - \tau(yr)[F(x),z]_{\sigma,\tau} + \tau(yrzx) - \tau(yrx)\sigma(z) = 0$, for all $x,y,z \in I$and $r \in R$.(187)
Right multiplying equation (179) by $\tau(r)$, we get
$\tau(xy)g(z)\tau(r) - \tau(y)[F(x),z]_{\sigma,\tau}\tau(r) + \tau(yzx)\tau(r) - \tau(yx)\sigma(z)\tau(r) = 0$, for all $x,y,z \in I$and $r \in R$.(188)
We subtracting equation (187) from equation (188), we get

$$\tau(xy)[g(z),\tau(r)] - \tau(y)\big[[F(x),z]_{\sigma,\tau},\tau(r)\big] + \tau(yzx)\tau(r) - \tau(yx)\sigma(z)\tau(r) - \tau(yrzx) + \tau(yrx)\sigma(z) = 0$$

We replacing $\sigma(z)$ by $\tau(z)$ in above equation, we get

$$\tau(xy)[g(z),\tau(r)] - \tau(y)\big[[F(x),z]_{\sigma,\tau},\tau(r)\big] + \tau(yzx)\tau(r) - \tau(yx)\tau(z)\tau(r) - \tau(yrzx) + \tau(yrx)\tau(z) = 0$$
$$\tau(xy)[g(z),\tau(r)] - \tau(y)\big[[F(x),z]_{\sigma,\tau},\tau(r)\big] + \tau(y)\tau(zx)\tau(r) - \tau(y)\tau(r)\tau(zx) - \tau(y)\tau(xz)\tau(r) + \tau(y)\tau(r)\tau(xz) = 0$$
$$\tau(xy)[g(z),\tau(r)] - \tau(y)\big[[F(x),z]_{\sigma,\tau},\tau(r)\big] + \tau(y)\big(\tau(zx)\tau(r) - \tau(r)\tau(zx)\big) - \tau(y)\big(\tau(xz)\tau(r) - \tau(r)\tau(xz)\big) = 0$$
$$\tau(xy)[g(z),\tau(r)] - \tau(y)\big[[F(x),z]_{\sigma,\tau},\tau(r)\big] + \tau(y)[\tau(zx),\tau(r)] - \tau(y)[\tau(xz),\tau(r)] = 0$$
$$\tau(xy)[g(z),\tau(r)] - \tau(y)\big[[F(x),z]_{\sigma,\tau},\tau(r)\big] + \tau(y)([\tau(zx) - \tau(xz),\tau(r)]) = 0$$

$\tau(xy)[g(z),\tau(r)] - \tau(y)\big[[F(x),z]_{\sigma,\tau},\tau(r)\big] + \tau(y)\big(\big[[\tau(z),\tau(x)],\tau(r)\big]\big) = 0$, for all $x,y,z \in I$and $r \in R$.(189)
We replacing z by x and using equation (186) in the equation (189), we get

$$\tau(y)\big[[F(x),x]_{\sigma,\tau},\tau(r)\big] = 0$$

We replacing $y$by ys, $s \in R$ in the above equation, we get

$$\tau(ys)\big[[F(x),x]_{\sigma,\tau},\tau(r)\big] = 0$$

$\tau(y)R\big[[F(x),x]_{\sigma,\tau},\tau(r)\big] = 0$, for all $x,y \in I$and $r \in R$.
Using primenessof$R$, we get $\big[[F(x),x]_{\sigma,\tau},\tau(r)\big] = 0$ implies that $[F(x),x]_{\sigma,\tau} \in Z(R)$, for all $x \in I$. (190)
We replacing x by xu in equation (179), we get
$\tau(xuy)g(z) - \tau(y)[F(xu),z]_{\sigma,\tau} + \tau(yzxu) - \tau(yxu)\sigma(z) = 0$, for all $x,y,z,u \in I$.

$$\tau(xuy)g(z) - \tau(y)[F(x)\sigma(u) + \tau(x)d(u),z]_{\sigma,\tau} + \tau(yzxu) - \tau(yxu)\sigma(z) = 0$$
$$\tau(xuy)g(z) - \tau(y)[F(x)\sigma(u),z]_{\sigma,\tau} - \tau(y)[\tau(x)d(u),z]_{\sigma,\tau} + \tau(yzxu) - \tau(yxu)\sigma(z) = 0$$

$\tau(xuy)g(z)-\tau(y)\left(F(x)[\sigma(u),\sigma(z)]+[F(x),z]_{\sigma,\tau}\sigma(u)\right)-\tau(y)(\tau(x)[d(u),z]_{\sigma,\tau}+[\tau(x),\tau(z)]d(u))+\tau(yzxu)-\tau(yxu)\sigma(z)=0$, for all $x,y,z,u\in I$. (191)
Right multiplying equation (179) by $\tau(u)$, we get
$\tau(xy)g(z)\tau(u)-\tau(y)[F(x),z]_{\sigma,\tau}\tau(u)+\tau(yzx)\tau(u)-\tau(yx)\sigma(z)\tau(u)=0$, for all $x,y,z,u\in I$. (192)
We subtracting equation (192) from equation (191), we get
$\tau(x)\tau(u)\tau(y)g(z)-\tau(x)\tau(y)g(z)\tau(u)-\tau(y)F(x)[\sigma(u),\sigma(z)]-\tau(y)[F(x),z]_{\sigma,\tau}\sigma(u)+\tau(y)[F(x),z]_{\sigma,\tau}\tau(u)-\tau(y)\tau(x)[d(u),z]_{\sigma,\tau}-\tau(y)[\tau(x),\tau(z)]d(u)-\tau(yxu)\sigma(z)+\tau(yx)\sigma(z)\tau(u)=0$, for all $x,y,z\in I$.
We replacing $\sigma(z)$ by $\tau(z)$ and $\sigma(u)$ by $\tau(u)$ in above equation, we get

$$\tau(x)\tau(u)\tau(y)g(z)-\tau(x)\tau(y)g(z)\tau(u)-\tau(y)F(x)[\tau(u),\tau(z)]-\tau(y)\tau(x)[d(u),z]_{\sigma,\tau}-\tau(y)[\tau(x),\tau(z)]d(u)-\tau(yxu)\tau(z)+\tau(yx)\tau(z)\tau(u)=0$$

$$\tau(x)[\tau(u),\tau(y)g(z)]-\tau(y)F(x)[\tau(u),\tau(z)]-\tau(y)\tau(x)[d(u),z]_{\sigma,\tau}-\tau(y)[\tau(x),\tau(z)]d(u)+\tau(yx)[\tau(z),\tau(u)]=0$$

$\tau(x)\tau(y)[\tau(u),g(z)]+\tau(x)[\tau(u),\tau(y)]g(z)-\tau(y)F(x)[\tau(u),\tau(z)]-\tau(y)\tau(x)[d(u),z]_{\sigma,\tau}-\tau(y)[\tau(x),\tau(z)]d(u)+\tau(yx)[\tau(z),\tau(u)]=0$, for all $x,y,z,u\in I$. (193)
We replacing z by u and using equation (186) in the equation (193), we get
$\tau(x)[\tau(u),\tau(y)]g(u)-\tau(y)\tau(x)[d(u),u]_{\sigma,\tau}-\tau(y)[\tau(x),\tau(u)]d(u)=0$, for all $x,y,u\in I$.
We replacing y by u in the above equation, we get
$\tau(u)\tau(x)[d(u),u]_{\sigma,\tau}+\tau(u)[\tau(x),\tau(u)]d(u)=0$, for all $x,u\in I$.
We replacing u by x in the above equation, we get
$\tau(x)\tau(x)[d(x),x]_{\sigma,\tau}=0$, for all $x\in I$.
Since $\tau(I)\neq 0$, then we have $[d(x),x]_{\sigma,\tau}=0$, for all $x\in I$. (194)
Similarly, we can obtain the same conclusion when $G(xy)-(F(x)oy)_{\sigma,\tau}-\tau(yx)=0$, for all $x,y\in I$. Thus theproof is completed.

Theorem 22: Let R be a prime ring and I be a non-zero idealon R. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation$g$ and d on$R$ respectively and $\tau(I)g(I)\neq 0$. If $G(xy)\pm(F(x)oy)_{\sigma,\tau}\pm[x,y]_{\sigma,\tau}=0$, for all $x,y\in I$, then either R is a commutative or $[g(x),\tau(x)]=0$, $[F(x),x]_{\sigma,\tau}\in Z(R)$ and $[d(x),x]_{\sigma,\tau}=0$, for all $x\in I$.

Proof: First we have $G(xy)+(F(x)oy)_{\sigma,\tau}+[x,y]_{\sigma,\tau}=0$, for all $x,y\in I$. (195)
We replacing y by yz in equation (195), we get

$$G(xyz)+(F(x)oyz)_{\sigma,\tau}+[x,yz]_{\sigma,\tau}=0$$

$$G(xy)\sigma(z)+\tau(xy)g(z)+(F(x)oy)_{\sigma,\tau}\sigma(z)-\tau(y)[F(x),z]_{\sigma,\tau}+[x,y]_{\sigma,\tau}\sigma(z)+\tau(y)[x,z]_{\sigma,\tau}=0$$

$\left(G(xy)+(F(x)oy)_{\sigma,\tau}+[x,y]_{\sigma,\tau}\right)\sigma(z)+\tau(xy)g(z)-\tau(y)[F(x),z]_{\sigma,\tau}+\tau(y)[x,z]_{\sigma,\tau}=0$, for all $x,y,z\in I$. (196)
Using equation (195) in equation (196), we get
$\tau(xy)g(z)-\tau(y)[F(x),z]_{\sigma,\tau}+\tau(y)[x,z]_{\sigma,\tau}=0$, for all $x,y,z\in I$. (197)
We replacing $y$by ry, $r\in R$ in equation (197), we get
$\tau(xry)g(z)-\tau(ry)[F(x),z]_{\sigma,\tau}+\tau(ry)[x,z]_{\sigma,\tau}=0$, for all $x,y,z,r\in I$. (198)
Left multiplying equation (197) by $\tau(r)$, we get
$\tau(r)\tau(xy)g(z)-\tau(r)\tau(y)[F(x),z]_{\sigma,\tau}+\tau(r)\tau(y)[x,z]_{\sigma,\tau}=0$, for all $x,y,z\in I$and $r\in R$. (199)
We subtracting equation (199) from equation (198), we get
$\tau[x,r]\tau(y)g(z)=0$, for all $x,y,z\in I$and $r\in R$. (200)
The equation (200) is same as equation (182) in theorem 21. Thus, by using same argument of theorem 21, we can conclude the result here.
Similarly, we can obtain the same conclusion when $G(xy)-(F(x)oy)_{\sigma,\tau}-[x,y]_{\sigma,\tau}=0$, for all $x,y\in I$. Thus theproof is completed.

Theorem 23: Let R be a prime ring and I be a non-zero idealon R. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation$g$ and$d$ on$R$ respectively and $\tau(I)g(I)\neq 0$. If $G(xy)\pm(F(x)oy)_{\sigma,\tau}\pm(xoy)_{\sigma,\tau}=0$, for all $x,y\in I$, then either R is a commutative or $[g(x),\tau(x)]=0$, $[F(x),x]_{\sigma,\tau}\in Z(R)$ and $[d(x),x]_{\sigma,\tau}=0$, for all $x\in I$.
Proof: First we have $G(xy)+(F(x)oy)_{\sigma,\tau}+(xoy)_{\sigma,\tau}=0$, for all $x,y\in I$. (201)
We replacing y by yz in equation (201), we get

$$G(xyz)+(F(x)oyz)_{\sigma,\tau}+(xoyz)_{\sigma,\tau}=0$$

$$G(xy)\sigma(z)+\tau(xy)g(z)+(F(x)oy)_{\sigma,\tau}\sigma(z)-\tau(y)[F(x),z]_{\sigma,\tau}+(xoy)_{\sigma,\tau}\sigma(z)-\tau(y)[x,z]_{\sigma,\tau}=0$$

$\big(G(xy)+(F(x)oy)_{\sigma,\tau}+(xoy)_{\sigma,\tau}\big)\sigma(z)+\tau(xy)g(z)-\tau(y)[F(x),z]_{\sigma,\tau}-\tau(y)[x,z]_{\sigma,\tau}=0$, for all $x,y,z\in I$. (202)

Using equation (201) in equation (202), we get

$\tau(xy)g(z)-\tau(y)[F(x),z]_{\sigma,\tau}-\tau(y)[x,z]_{\sigma,\tau}=0$, for all $x,y,z\in I$. (203)

We replacing $y$by ry, $r\in R$ in equation (203), we get

$\tau(xry)g(z)-\tau(ry)[F(x),z]_{\sigma,\tau}-\tau(ry)[x,z]_{\sigma,\tau}=0$, for all $x,y,z,r\in I$. (204)

Left multiplying equation (203) by $\tau(r)$, we get

$\tau(r)\tau(xy)g(z)-\tau(r)\tau(y)[F(x),z]_{\sigma,\tau}-\tau(r)\tau(y)[x,z]_{\sigma,\tau}=0$, for all $x,y,z\in I$and $r\in R$. (205)

We subtracting equation (205) from equation (204), we get

$\tau[x,r]\tau(y)g(z)=0$, for all $x,y,z\in I$and $r\in R$. (206)

The equation (206) is same as equation (182) in theorem 21. Thus, by using same argument of theorem 21, we can conclude the result here.

Similarly, we can obtain the same conclusion when $G(xy)-(F(x)oy)_{\sigma,\tau}-(xoy)_{\sigma,\tau}=0$, for all $x,y\in I$. Thus theproof is completed.

Theorem 24: Let R be a prime ring and I be a non-zero idealon R. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation$g$ and$d$ on$R$ respectively and $\tau(I)g(I)\neq 0$. If $G(xy)\pm(F(x)oy)_{\sigma,\tau}\pm\sigma(xy)=0$, for all $x,y\in I$, then either R is a commutative or $[g(x),\tau(x)]=0$, $[F(x),x]_{\sigma,\tau}\in Z(R)$ and $[d(x),x]_{\sigma,\tau}=0$, for all $x\in I$.

Proof: First we have $G(xy)+(F(x)oy)_{\sigma,\tau}+\sigma(xy)=0$, for all $x,y\in I$. (207)

We replacing y by yz in equation (207), we get

$$G(xyz)+(F(x)oyz)_{\sigma,\tau}+\sigma(xyz)=0$$

$$G(xy)\sigma(z)+\tau(xy)g(z)+(F(x)oy)_{\sigma,\tau}\sigma(z)-\tau(y)[F(x),z]_{\sigma,\tau}+\sigma(xy)\sigma(z)=0$$

$\big(G(xy)+(F(x)oy)_{\sigma,\tau}+\sigma(xy)\big)\sigma(z)+\tau(xy)g(z)-\tau(y)[F(x),z]_{\sigma,\tau}=0$,for all $x,y,z\in I$. (208)

Using equation (207) in equation (208), we get

$\tau(xy)g(z)-\tau(y)[F(x),z]_{\sigma,\tau}=0$, for all $x,y,z\in I$. (209)

We replacing $y$by ry, $r\in R$ in equation (209), we get

$\tau(xry)g(z)-\tau(ry)[F(x),z]_{\sigma,\tau}=0$, for all $x,y,z,r\in I$. (210)

Left multiplying equation (209) by $\tau(r)$, we get

$\tau(r)\tau(xy)g(z)-\tau(r)\tau(y)[F(x),z]_{\sigma,\tau}=0$, for all $x,y,z\in I$and $r\in R$.(211)

We subtracting equation (211) from equation (210), we get

$\tau[x,r]\tau(y)g(z)=0$, for all $x,y,z\in I$and $r\in R$. (212)

The equation (212) is same as equation (182) in theorem 21. Thus, by using same argument of theorem 21, we can conclude the result here.

Similarly, we can obtain the same conclusion when $G(xy)-(F(x)oy)_{\sigma,\tau}-\sigma(xy)=0$, for all $x,y\in I$. Thus theproof is completed.

Theorem 25: Let R be a prime ring and I be a non-zero idealon R. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation g and d on R respectively and $\tau(I)\neq 0$. If $G(xy)\pm(F(x)oy)_{\sigma,\tau}=0$, for all $x,y\in I$, then either R is commutative or $[g(x),\tau(x)]=0$,$[F(x),x]_{\sigma,\tau}\in Z(R)$ and $[d(x),x]_{\sigma,\tau}=0$, for all $x\in I$.

Proof: First we have $G(xy)+(F(x)oy)_{\sigma,\tau}=0$, for all $x,y\in I$. (213)

We replacing y by yz in equation (213), we get

$$G(xyz)+(F(x)oyz)_{\sigma,\tau}=0$$

$$G(xy)\sigma(z)+\tau(xy)g(z)+(F(x)oy)_{\sigma,\tau}\sigma(z)-\tau(y)[F(x),z]_{\sigma,\tau}=0$$

$$\big(G(xy)+(F(x)oy)_{\sigma,\tau}\big)\sigma(z)+\tau(xy)g(z)-\tau(y)[F(x),z]_{\sigma,\tau}=0$$

Using equation (213), it reduces to

$\tau(xy)g(z)-\tau(y)[F(x),z]_{\sigma,\tau}=0$, for all $x,y,z\in I$. (214)

We replacing $y$by ry, in equation (214), we get

$\tau(xry)g(z)-\tau(ry)[F(x),z]_{\sigma,\tau}=0$, for all $x,y,z,r\in I$. (215)

Left multiplying equation (214) by $\tau(r)$, we get

$\tau(r)\tau(xy)g(z)-\tau(r)\tau(y)[F(x),z]_{\sigma,\tau}=0$, for all $x,y,z,r\in I$. (216)

We subtracting equation (215) from equation (216), we get

$\tau(xry)g(z)-\tau(r)\tau(xy)g(z)=0$, for all $x,y,z,r\in I$. (217)

$\tau[x,r]\tau(y)g(z)=0$, for all $x,y,z,r\in I$. (218)

We replacing y by yx in the equation (218), we get

$\tau[x,r]\tau(yx)g(z)=0$, for all $x,y,z\in I$and $r\in R$. (219)
Right multiplying equation (218) by $\tau(x)$, we get
$\tau[x,r]\tau(y)g(z)\tau(x)=0$, for all $x,y,z\in I$and $r\in R$. (220)
We subtracting equation (219) from equation (220), we get
$\tau[x,r]\tau(y)[g(z),\tau(x)]=0$, for all $x,y,z\in I$and $r\in R$.
We replacing $y$by sy, $s\in R$ and z by x in the above equation, we get

$$\tau([x,r])\tau(s)\tau(y)[g(x),\tau(x)]=0$$

$\tau([x,r])R\tau(y)[g(x),\tau(x)]=0$, for all $x,y\in I$and $s,r\in R$. (221)
Using primenessof$R$, we get either $\tau([x,r])=0$ or $\tau(y)[g(x),\tau(x)]=0$
Since τ is an automorphism of $R$and $\tau(I)\neq 0$, we have either $[x,r]=0$, for all $x\in I$ and $r\in R$ or $[g(x),\tau(x)]=0$, for all $x\in I$.
Now let $A=\{x\in I/[x,r]=0, r\in R\}$and $B=\{x\in I/[g(x),\tau(x)]=0\}$.
Clearly, A and B are additive proper subgroups of I whose union is I.
Since a group cannot be the set theoretic union of two proper subgroups.
Hence either $A=I$or $B=I$.
If $A=I$, then $[x,r]=0$ implies that R is commutative
On the other hand if $B=I$, then $[g(x),\tau(x)]=0$, for all $x\in I$. (224)
We replacing y by yr in equation (214), we get
$\tau(xyr)g(z)-\tau(yr)[F(x),z]_{\sigma,\tau}=0$, for all $x,y,z,r\in I$. (225)
Right multiplying equation (214) by $\tau(r)$, we get
$\tau(xy)g(z)\tau(r)-\tau(y)[F(x),z]_{\sigma,\tau}\tau(r)=0$, for all $x,y,z,r\in I$. (226)
We subtracting equation (225) from equation (226) and using equation (224), we get

$$\tau(y)\big[[F(x),z]_{\sigma,\tau},\tau(r)\big]=0$$

Using primenessof$R$, we get

$$\big[[F(x),z]_{\sigma,\tau},\tau(r)\big]=0$$

That is $[F(x),x]_{\sigma,\tau}\in Z(R)$, for all $x\in I$. (227)
We replacing x by xu in equation (214), we get

$$\tau(xuy)g(z)-\tau(y)[F(xu),z]_{\sigma,\tau}=0$$
$$\tau(xuy)g(z)-\tau(y)[F(x)\sigma(u)+\tau(x)d(u),z]_{\sigma,\tau}=0$$
$$\tau(xuy)g(z)-\tau(y)[F(x)\sigma(z),z]_{\sigma,\tau}-\tau(y)[\tau(x)d(u),z]_{\sigma,\tau}=0$$

$\tau(xuy)g(z)-\tau(y)[F(x),z]_{\sigma,\tau}\sigma(u)-\tau(y)F(x)[\sigma(u),\sigma(z)]-\tau(y)\tau(x)[d(u),z]_{\sigma,\tau}-$
$\tau(y)[\tau(x),\tau(z)]d(u)=0$, for all $x,y,z,u\in I$. (228)
Right multiplying equation (214) by $\sigma(u)$, we get
$\tau(xy)g(z)\sigma(u)-\tau(y)[F(x),z]_{\sigma,\tau}\sigma(u)=0$, for all $x,y,z,u\in I$. (229)
We subtracting equation (229) from equation (228), we get
$\tau(x)\tau(u)\tau(y)g(z)-\tau(x)\tau(y)g(z)\sigma(u)-\tau(y)F(x)[\sigma(u),\sigma(z)]-\tau(y)\tau(x)[d(u),z]_{\sigma,\tau}-$
$\tau(y)[\tau(x),\tau(z)]d(u)=0$, for all $x,y,z,u\in I$. (230)
We replacing u by z and $\sigma(z)$ by $\tau(z)$ in the equation (230), we get

$$\tau(x)\tau(z)\tau(y)g(z)-\tau(x)\tau(y)g(z)\tau(z)-\tau(y)\tau(x)[d(z),z]_{\sigma,\tau}-\tau(y)[\tau(x),\tau(z)]d(z)=0$$
$$\tau(x)[\tau(z),\tau(y)g(z)]-\tau(y)\tau(x)[d(z),z]_{\sigma,\tau}-\tau(y)[\tau(x),\tau(z)]d(z)=0$$

$\tau(x)\tau(y)[\tau(z),g(z)]+\tau(x)[\tau(z),\tau(y)]g(z)-\tau(y)\tau(x)[d(z),z]_{\sigma,\tau}-\tau(y)[\tau(x),\tau(z)]d(z)=0$, for all $x,y,z\in I$.
We replacing y by z and using equation (224) in the above equation, we get
$\tau(z)\tau(x)[d(z),z]_{\sigma,\tau}+\tau(z)[\tau(x),\tau(z)]d(z)=0$, for all $x,y,z\in I$.
We replacing z by x in the above equation, we get
$\tau(x)\tau(x)[d(x),x]_{\sigma,\tau}=0$, for all $x,y\in I$.
Since $\tau(I)\neq 0$, we get
$[d(x),x]_{\sigma,\tau}=0$, for all $x,y\in I$. (231)
Similarly, we can obtain the same conclusion when $G(xy)-(F(x)oy)_{\sigma,\tau}=0$, for all $x,y\in I$. Thus theproof is completed.

Theorem 26: Let R be a prime ring and I be a non-zero idealon R. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation g and d on R respectively and $\tau(I)\neq 0$. If $G(xy)\pm(F(y)ox)_{\sigma,\tau}\pm\sigma(yx)=0$, for all $x,y\in I$, then either R is commutative or $g(x)\sigma(x)=0$ and $d(x)\sigma(x)=0$, for all $x\in I$.
Proof: First we have $G(xy)+(F(y)ox)_{\sigma,\tau}+\sigma(yx)=0$, for all $x,y\in I$. (232)
We replacing y by yx in equation (232), we get

$$G(xyx) + (F(yx)ox)_{\sigma,\tau} + \sigma(yxx) = 0$$
$$G(xy)\sigma(x) + \tau(xy)g(x) + ((F(y)\sigma(x) + \tau(y)d(x))ox)_{\sigma,\tau} + \sigma(yxx) = 0$$
$$G(xy)\sigma(x) + \tau(xy)g(x) + ((F(y)\sigma(x))ox)_{\sigma,\tau} + ((\tau(y)d(x))ox)_{\sigma,\tau} + \sigma(yxx) = 0$$
$$G(xy)\sigma(x) + \tau(xy)g(x) + F(y)[\sigma(x),\sigma(x)] + (F(y)ox)_{\sigma,\tau}\sigma(x) + \tau(y)(d(x)ox)_{\sigma,\tau} - [\tau(y),\tau(x)]d(x) + \sigma(yxx) = 0$$
$\left(G(xy) + (F(y)ox)_{\sigma,\tau} + \sigma(yx)\right)\sigma(x) + \tau(xy)g(x)+\tau(y)(d(x)ox)_{\sigma,\tau} - [\tau(y),\tau(x)]d(x) = 0$, for all $x,y \in I$. (233)

Using equation (232) in equation (233), we get
$\tau(xy)g(x)+\tau(y)(d(x)ox)_{\sigma,\tau} - [\tau(y),\tau(x)]d(x) = 0$, for all $x,y \in I$. (234)
We replacing y by ry in equation (234), we get
$$\tau(xry)g(x)+\tau(ry)(d(x)ox)_{\sigma,\tau} - [\tau(ry),\tau(x)]d(x) = 0$$
$\tau(xry)g(x)+\tau(r)\tau(y)(d(x)ox)_{\sigma,\tau} - \tau(r)[\tau(y),\tau(x)]d(x) - [\tau(r),\tau(x)]\tau(y)d(x) = 0$, for all $x,y,r \in I$. (235)
Left multiplying equation (234) by $\tau(r)$, we get
$\tau(r)\tau(xy)g(x)+\tau(r)\tau(y)(d(x)ox)_{\sigma,\tau} - \tau(r)[\tau(y),\tau(x)]d(x) = 0$, for all $x,y,r \in I$. (236)
We subtracting equation (236) from equation (235), we get
$\tau[x,r]\tau(y)g(x) - [\tau(r),\tau(x)]\tau(y)d(x) = 0$, for all $x,y,r \in I$. (237)
We replacing y by xy in equation (237), we get
$\tau[x,r]\tau(xy)g(x) - [\tau(r),\tau(x)]\tau(xy)d(x) = 0$, for all $x,y,r \in I$. (238)
Left multiplying equation (234) by $\tau[x,r]$, we get
$\tau[x,r]\tau(xy)g(x)+\tau[x,r]\tau(y)[d(x)ox]_{\sigma,\tau} - \tau[x,r][\tau(y),\tau(x)]d(x) = 0$, for all $x,y,r \in I$.(239)
We subtracting equation (238) from equation (239), we get
$$\tau[x,r]\tau(y)[d(x)ox]_{\sigma,\tau} - \tau[x,r][\tau(y),\tau(x)]d(x) + [\tau(r),\tau(x)]\tau(xy)d(x) = 0$$
$$\tau[x,r]\tau(y)[d(x)ox]_{\sigma,\tau} - \tau[x,r][\tau(y),\tau(x)]d(x) - \tau[x,r]\tau(xy)d(x) = 0$$
$$\tau[x,r](\tau(y)[d(x)ox]_{\sigma,\tau} - [\tau(y),\tau(x)]d(x) - \tau(xy)d(x)) = 0$$
$$\tau[x,r](\tau(y)d(x)\sigma(x) + \tau(y)\tau(x)d(x) - \tau(y)\tau(x)d(x) + \tau(x)\tau(y)d(x) - \tau(xy)d(x)) = 0$$
$\tau[x,r]\tau(y)d(x)\sigma(x) = 0$, for all $x,y,r \in I$. (240)
We replacing $y$by sy, $s \in R$ in the above equation, we get
$\tau[x,r]R\tau(y)d(x)\sigma(x) = 0$, for all $x,y,r \in I$. (241)
Using primenessof$R$, we get either $\tau([x,r]) = 0$ or $\tau(y)d(x)\sigma(x) = 0$
Since τ is an automorphism of $R$and $\tau(I) \neq 0$, we have either $[x,r] = 0$, for all $x \in I$ or $d(x)\sigma(x) = 0$, for all $x \in I$.
Now let $A = \{x \in I/[x,r] = 0\}$and $B = \{x \in I/d(x)\sigma(x) = 0\}$.
Clearly, A and B are additive proper subgroups of I whose union is I.
Since a group cannot be the set theoretic union of two proper subgroups.
Hence either $A = I$or $B = I$.
If $A = I$, then $[x,r] = 0$ implies that R is commutative.
On the other hand if $B = I$, then $d(x)\sigma(x) = 0$, for all $x \in I$. (242)
Right multiplying equation (234) by $\sigma(x)$, we get
$$\tau(xy)g(x)\sigma(x) + \tau(y)(d(x)ox)_{\sigma,\tau}\sigma(x) - [\tau(y),\tau(x)]d(x)\sigma(x) = 0$$
$\tau(xy)g(x)\sigma(x) + \tau(y)d(x)\sigma(x)\sigma(x) + \tau(y)\tau(x)d(x)\sigma(x) - [\tau(y),\tau(x)]d(x)\sigma(x) = 0$, for all $x,y \in I$. (243)
Using equation (242) in equation (243), we get
$$\tau(x)\tau(y)g(x)\sigma(x) = 0$$
Using primeness of $R$and $\tau(I) \neq 0$, we get
$g(x)\sigma(x) = 0$, for all $x \in I$. (244)
Similarly, we can obtain the same conclusion when $G(xy) - (F(y)ox)_{\sigma,\tau} - \sigma(yx) = 0$, for all $x,y \in I$. Thus theproof is completed.

Theorem 27: Let R be a prime ring and I be a non-zero idealon R. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation g and d on R respectively and $\tau(I) \neq 0$. If $G(xy) \pm (F(y)ox)_{\sigma,\tau} \pm \sigma(xy) = 0$, for all $x,y \in I$, then either R is commutative or $g(x)\sigma(x) = 0$ and $d(x)\sigma(x) = 0$, for all $x \in I$.
Proof: First we have $G(xy) + (F(y)ox)_{\sigma,\tau} + \sigma(xy) = 0$, for all $x,y \in I$. (245)
We replacing y by yx in equation (245), we get
$$G(xyx) + (F(yx)ox)_{\sigma,\tau} + \sigma(xyx) = 0$$
$$G(xy)\sigma(x) + \tau(xy)g(x) + ((F(y)\sigma(x) + \tau(y)d(x))ox)_{\sigma,\tau} + \sigma(xyx) = 0$$

$$G(xy)\sigma(x)+\tau(xy)g(x)+((F(y)\sigma(x))ox)_{\sigma,\tau}+((\tau(y)d(x))ox)_{\sigma,\tau}+\sigma(xyx)=0$$
$$G(xy)\sigma(x)+\tau(xy)g(x)+F(y)[\sigma(x),\sigma(x)]+(F(y)ox)_{\sigma,\tau}\sigma(x)+\tau(y)(d(x)ox)_{\sigma,\tau}-[\tau(y),\tau(x)]d(x)+\sigma(xyx)=0$$
$(G(xy)+(F(y)ox)_{\sigma,\tau}+\sigma(xy))\sigma(x)+\tau(xy)g(x)+\tau(y)(d(x)ox)_{\sigma,\tau}-[\tau(y),\tau(x)]d(x)=0$, for all $x,y\in I$. (246)

Using equation (245) in equation (246), we get
$\tau(xy)g(x)+\tau(y)(d(x)ox)_{\sigma,\tau}-[\tau(y),\tau(x)]d(x)=0$, for all $x,y\in I$. (247)
The equation (247) is same as equation (234) in theorem 26. Thus, by using same argument of theorem 26, we can conclude the result here.
Similarly, we can obtain the same conclusion when $G(xy)-(F(y)ox)_{\sigma,\tau}-\sigma(xy)=0$, for all $x,y\in I$. Thus theproof is completed.

Theorem 28: Let R be a prime ring and I be a non-zero idealon R. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation g and d on R respectively and $\tau(I)\neq 0$. If $G(xy)\pm(F(y)ox)_{\sigma,\tau}=0$, for all $x,y\in I$, then either R is commutative or $g(x)\sigma(x)=0$ and $d(x)\sigma(x)=0$, for all $x\in I$.

Proof: First we have $G(xy)+(F(y)ox)_{\sigma,\tau}=0$, for all $x,y\in I$. (248)
We replacing y by yx in equation (248), we get
$$G(xyx)+(F(yx)ox)_{\sigma,\tau}=0$$
$$G(xy)\sigma(x)+\tau(xy)g(x)+((F(y)\sigma(x)+\tau(y)d(x))ox)_{\sigma,\tau}=0$$
$$G(xy)\sigma(x)+\tau(xy)g(x)+((F(y)\sigma(x))ox)_{\sigma,\tau}+((\tau(y)d(x))ox)_{\sigma,\tau}=0$$
$$G(xy)\sigma(x)+\tau(xy)g(x)+F(y)[\sigma(x),\sigma(x)]+(F(y)ox)_{\sigma,\tau}\sigma(x)+\tau(y)(d(x)ox)_{\sigma,\tau}-[\tau(y),\tau(x)]d(x)=0$$
$\left(G(xy)+(F(y)ox)_{\sigma,\tau}\right)\sigma(x)+\tau(xy)g(x)+\tau(y)(d(x)ox)_{\sigma,\tau}-[\tau(y),\tau(x)]d(x)=0$, for all $x,y\in I$. (249)
Using equation (248) in equation (249), we get
$\tau(xy)g(x)+\tau(y)(d(x)ox)_{\sigma,\tau}-[\tau(y),\tau(x)]d(x)=0$, for all $x,y\in I$. (250)
The equation (250) is same as equation (234) in theorem 26. Thus, by using same argument of theorem 26, we can conclude the result here.
Similarly, we can obtain the same conclusion when $G(xy)-(F(y)ox)_{\sigma,\tau}-\sigma(xy)=0$, for all $x,y\in I$. Thus theproof is completed.

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