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Degree Regularity on Edges of S – Valued Graph

S.Mangala Lavanya[*], S.Kiruthiga Deepa[**] And M.Chandramouleeswaran[***]

[*]*Sree Sowdambika College Of Engineering Aruppukottai-626101. Tamilnadu.India.*
[**]*Bharath Niketan College Of Engineering Aundipatti - 625536. Tamilnadu. India.*
[***]*Saiva Bhanu Kshatriya College Aruppukottai - 626101. Tamilnadu. India.*

Abstract: *Recently in [6], the authors have introduced the notion of semiring valued graphs, which is a generalization of both the crisp graph and fuzzy graph. In [3] the authors have studied the regularity conditions on S - valued graphs. In [7] the authors have studied the notion of vertex degree regularity on S - valued graphs. In this paper, we study the edge degree regularity of S - valued graphs.*
Keywords: *Semirings, Graphs, S-valued graphs, d_S - edge regular graph.*

I. Introduction

The concept of semiring was studied by several mathematicians such as Dedekind [2], Krull [5] and H.S.Vandiver [8]. Jonathan Golan [4] in his book, has introduced the notion of S - graph where S is a semiring. However, the theory was not developed further. In [6], the authors have introduced the notion of semiring valued graphs, which is a generalisation of both the crisp graph and fuzzy graph theory. In [3], the authors have studied the regularity conditions on S - valued graphs. In [7] the authors have studied the notion of vertex degree regularity on S - valued graphs. In this paper we study the edge degree regularity of S-valued graphs.

II. Preliminaries

In this section, we recall some basic definitions that are needed for our work.

Definition 2.1: [4] A semiring (S, +, .) is an algebraic system with a non-empty set S together with two binary operations + and . such that
(1) (S, +, 0) is a monoid.
(2) (S, .) is a semigroup.
(3) For all a, b, c $\in$ S , a . (b+c) = a . b + a . c and (a+b) . c = a . c + b . c
(4) 0. x = x . 0 = 0 $\forall$ x $\in$ S.

Definition 2.2: [4] Let (S, +, .) be a semiring. $\preceq$ is said to be a Canonical Pre-order if for a, b $\in$ S, a $\preceq$ b if and only if there exists an element c $\in$ S such that a + c = b.

Definition 2.3: [1] A set F of edges in a graph G = (V,E) is called an edge dominating set in G if for every edge e $\in$ E - F there exist an edge f $\in$ F such that e and f have a vertex in common.

Definition 2.4: [1] A dominating set S is a minimal edge dominating set if no proper subset of S is a edge dominating set in G.

Definition 2.5: [6] Let G = (V, E $\subset$ V $\times$V) be a given graph with V, E $\neq \phi$. For any semiring (S, +, .), a semiring-valued graph (or a S - valued graph), G^S is defined to be the graph $G^S = (V, E, \sigma, \psi)$ where $\sigma : V \to S$ and $\psi : E \to S$ is defined to be

$$\psi(x, y) = \begin{cases} \min\{\sigma(x), \sigma(y)\}, if \sigma(x) \preceq \sigma(y) or \sigma(y) \preceq \sigma(x) \\ 0, otherwise \end{cases}$$

for every unordered pair (x, y) of E $\subset$ V $\times$V. We call σ, a S - vertex set and ψ, a S - edge set of S - valued graph G^S. Henceforth, we call a S – valued graph simply as a S - graph.

Definition 2.6: [6] If $\sigma(x) = a, \forall x \in V$ and some a $\in$ S then the corresponding S - graph G^S is called a vertex regular S - graph (or simply vertex regular). An S - graph G^S is said to be an edge regular S - graph (or simply edge regular) if $\psi(x, y)$ = a for every (x, y) $\in$ E and a $\in$ S. A S - valued graph is said to be S - regular if it is both vertex and edge regular.

Definition 2.7: [3] Let G^S be a S - graph corresponding to an underlying graph G, and let a $\in$ S. G^S is said to be (a, k) - vertex regular if the following conditions are true.
(1) The crisp graph G is k - regular.
(2) $\sigma(v) = a, \forall v \in V$

Definition 2.8: [7] The Order of a S - valued graph G^S is defined as $p_S = \left(\sum_{v \in V} \sigma(v), n \right)$

where n is order of the underlying graph G.

Definition 2.9: [7] The Size of the S - valued graph G^S is defined as $q_S = \left(\sum_{(u,v) \in E} \psi(u,v), m \right)$

where m is the number of edges in the underlying graph G.

Definition 2.10: [7] The Degree of the vertex v_i of the S - valued graph G^S is defined as $\deg_S(v_i) = \left(\sum_{(v_i,v_j) \in E} \psi(v_i, v_j), \ell \right)$, where ℓ is the number of edges incident with v_i.

Definition 2.11: A subset D $\subseteq$ V is said to be a weight dominating vertex set if for each $v \in$ D, $\sigma(u) \preceq \sigma(v), \forall u \in N_S[v]$.

III. Degree Regularity on Edges of S - Valued Graph

In this section, we introduce the notion of the degree of an edge in S – valued graph G^S, analogous to the notion in crisp graph theory, and discuss the regularity conditions on such edges of G^S. We start with the definition

Definition 3.1: Let G^S = (V, E, σ, ψ) be a S - valued graph. Let e $\in$ E .The open neighbourhood of e, denoted by $N_S(e)$, is defined to be the set $N_S(e) = \{(e_i, \psi(e_i)) / e \text{ and } e_i \in E \text{ are adjacent}\}$
The closed neighbourhood of e, denoted by $N_S[e]$, is defined to be the set $N_S[e] = N_S(e) \cup (e, \psi(e))$

Definition 3.2: Let G^S = (V, E, σ, ψ) be a S - valued graph. The degree of the edge e is defined as

$$\deg_S(e) = \left(\sum_{e_i \in N_S(e)} \psi(e_i), m \right)$$, where m is the number of edges adjacent to e.

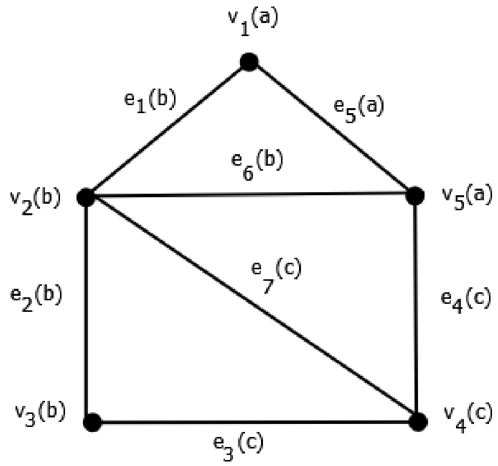
Example 3.3: Let (S = {0,a,b,c}, +, .) be a semiring with the following Cayley Tables:

+	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	a	b	b
c	c	a	b	c

.	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
c	0	b	b	b

Let $\preceq$ be a canonical pre-order in S, given by

$0 \preceq 0$, $0 \preceq a$, $0 \preceq b$, $0 \preceq c$, $a \preceq a$, $b \preceq b$, $b \preceq a$, $c \preceq c$, $c \preceq a$, $c \preceq b$
Consider the S - graph G^S,



Define $\sigma : V \to S$ by $\sigma(v_1) = \sigma(v_5) = a, \sigma(v_2) = \sigma(v_3) = b, \sigma(v_4) = c$ and
$\psi : E \to S$ by $\psi(e_1) = \psi(e_2) = \psi(e_6) = b, \psi(e_3) = \psi(e_4) = \psi(e_7) = c, \psi(e_5) = a$
Here $N_S(e_1) = \{(e_2, b), (e_5, a), (e_6, b), (e_7, c)\}, \deg_S(e_1) = (a, 4)$
$N_S(e_2) = \{(e_1, b), (e_3, c), (e_6, b), (e_7, c)\}, \deg_S(e_2) = (b, 4)$
$N_S(e_3) = \{(e_2, b), (e_4, c), (e_7, c)\}, \deg_S(e_3) = (b, 3)$
$N_S(e_4) = \{(e_3, c), (e_5, a), (e_6, b), (e_7, c)\}, \deg_S(e_4) = (a, 4)$
$N_S(e_5) = \{(e_1, b), (e_4, c), (e_6, b)\}, \deg_S(e_5) = (b, 3)$
$N_S(e_6) = \{(e_1, b), (e_2, b), (e_4, c)(e_5, a), (e_7, c)\}, \deg_S(e_6) = (a, 5)$
$N_S(e_7) = \{(e_1, b), (e_2, b), (e_3, c), (e_4, c), (e_6, b)\}, \deg_S(e_7) = (b, 5)$

Definition 3.4: If D $\subseteq$ E in G^S then the scalar cardinality of D is defined by $|D|_S = \sum_{e \in D} \psi(e)$

Definition 3.5: Let G^S = (V, E, σ, ψ) be a S - valued graph. For any e $\in$ E, the neighbourhood degree of e is defined as $N \deg_S(e) = (|N_S|_S, |N_S|)$

Remark 3.6: From definition 3.2 and 3.4 we observe that the degree of an edge is the same as its neighbourhood degree.

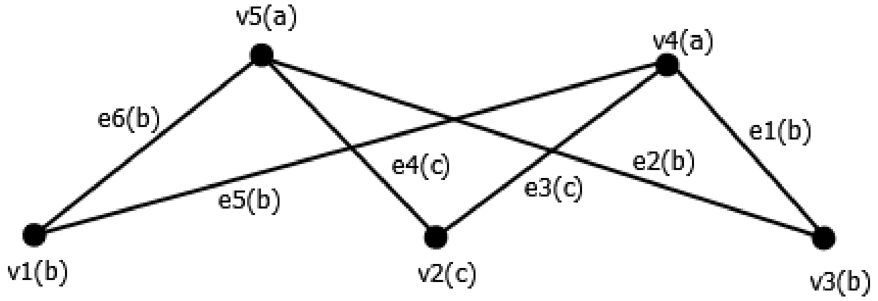
Remark 3.7: The scalar cardinality of $N_S[e]$ will be given by

(1) $|N_S[e]| = |N_S(e)| + 1$

(2) $|N_S[e]|_S = |N_S(e)|_S + \psi(e)$

Definition 3.8: An S - valued graph G^S is said to be d_S - edge regular if for any e $\in$ E,
$\deg_S(e) = (|N_S(e)|_S, |N_S(e)|)$

Example 3.9: Consider the semiring (S = {0,a,b,c}, +, .) with canonical pre-order given in example 3.3
Consider the S - graph G^S,



Define $\sigma : V \rightarrow S$ by $\sigma(v_1) = \sigma(v_3) = b, \sigma(v_2) = c, \sigma(v_4) = \sigma(v_5) = a$ and $\psi : E \rightarrow S$ by $\psi(e_1) = \psi(e_2) = \psi(e_5) = \psi(e_6) = b, \psi(e_3) = \psi(e_4) = c$

Here degree of every edge $e_i \in$ E is (b, 3).

$\therefore G^S$ is an d_S - edge regular graph, $d_S(e)$ = (b, 3).

Remark 3.10: In terms of neighbourhood of an edge, definition 2.9 can be modified as $q_S = \left(\sum_{e \in E} \psi(e), q \right)$, where q is the number of edges in the underlying graph G.

Theorem 3.11: If S is an additively idempotent semiring and G^S is S-regular then $\deg_S(e) \preceq q_S$, $\forall$ e$\in$ E.

Proof :

Let $G^S = (V,E,\sigma,\psi)$ be S-regular.

$\therefore \sigma(v_i) = a$, $\forall$i and for some a$\in$S

$\Rightarrow \psi(e_i) = a$, $\forall$i and for some a$\in$S

Since S is additively idempotent , a + a = a, $\forall$a$\in$S

Let e $\in$ E

Now, $q_S = \left(\sum_{e \in E} \psi(e), q \right) = (a, q)$, where q is the number of edges in G.

= (a; q) where q is the number of edges in G .

and $\deg_S(e) = \left(\sum_{e_i \in N_S(e)} \psi(e_i), m \right) = (a, m)$, where m is number of edges adjacent with e .

Since S is a semiring, it possess a canonical pre-order.

$\therefore a \preceq a$, $\forall$a $\in$ S

Clearly q $\geq$ m,$\therefore$ $\deg_S(e) \preceq q_S$ for all e$\in$E.

Since every (a ,k)- regular S- valued graph is S-regular, the above theorem holds good for (a, k)- regular S-valued graphs on an additively idempotent semiring.Thus we have the following

Corollary 3.12: An (a,k)- regular S- valued graph G^S on an additively idempotent semiring S satisfies $\deg_S(e_i) \preceq q_S$, $\forall$i

Theorem 3.13: Let a $\in$ S be an additively idempotent element in S.Then every (a, k)- regular S- valued graph G^S is ds- edge regular iff $\deg_S(e) \preceq$ (a,k) for some a and $\forall$e $\in$ E.

Proof :

Let $G^S = (V,E,\sigma,\psi)$ be a (a, k)-regular S- valued graph.

Assume that G^S is d_S- edge regular.

Then $\deg_S(e)$ = (b,k), $\forall$ i and for some b $\in$ S

That is

$$\left(\sum_{e_i \in N_S(e)} \psi(e_i), k\right) = (b, k)$$
$$(a + a + a + ... + a, k) = (b, k)$$
$$(a, k) = (b, k) \Rightarrow a = b$$
$$\therefore \deg_S(e) = (a, k),\ for\ some\ a.$$

Conversely, Let $G^S = (V, E, \sigma, \psi)$ be a (a, k)- regular and a be an additively idempotent element in S , and $\deg_S(e) = (a, k)$ for some a and for each $e \in E$.
Let $v_1, v_2 \in V$ be such that $e = v_1v_2 \in E$.
Since G^S is (a,k) regular, $\sigma(v_1) = \sigma(v_2) = a$.
Then $\psi(e) = \min \{ \sigma(v_1), \sigma(v_2)\} = a$
This true for every edge $e_i = v_iv_j \in E, \psi(e_i) = a, \forall i$

Now $\deg_S(e) = \left(\sum_{e_i \in N_S(e)} \psi(e_i), k\right) = (a, k)$
Since $e \in E$ is arbitrary, G^S is d_S- edge regular.

Theorem 3.14: Let G^S be a complete bipartite graph with $V = (V_1, V_2)$. If $|V_1| < |V_2|$ and V_1 is a weight dominating vertex set then G^S is a d_S- edge regular graph.
Proof:
Let G^S be a complete bipartite graph with $V = (V_1, V_2)$.
Let V_1 be a weight dominating vertex set and $|V_1| < |V_2|$.
Then $\sigma(v_j) \preceq \sigma(v_i), \forall v_i \in V_1, v_j \in V_2$.

Consider $\deg_S(e_{ij}) = \left(\sum_{e_{is} \in N_S[e_{ij}]} \psi(e_{is}), |V_2|\right) = \left(\sum_{v_s \in N_S[v_i]} \sigma(v_s), |V_2|\right)$

And $\deg_S(e_{ik}) = \left(\sum_{e_{ir} \in N_S[e_{ik}]} \psi(e_{ir}), |V_2|\right) = \left(\sum_{v_r \in N_S[v_i]} \sigma(v_r), |V_2|\right)$

Here $\deg_S(e_{ij}) = \deg_S(e_{ik})$ iff $\sum \sigma(v_s) = \sum \sigma(v_r) = a$,for some $a \in S$
Therefore for any edge $e_{ij}, e_{ik} \in N_S[v_i]$, $\deg_S(e_{ij}) = \deg_S(e_{ik}) = (a, |V_2|)$
Therefore G^S is a d_S- edge regular graph.

Corollary 3.15: Let G^S be a complete bipartite graph with $V = (V_1, V_2)$.If $|V_1| = |V_2|$ and either V_1 or V_2 is a weight dominating vertex set then G^S is a d_S - edge regular graph.

Theorem 3.16: Let G^S be a star with n vertices. If its pole has the maximum weight then G^S is a d_S- edge regular graph.
Proof:
Let G^S be a star with n vertices.
Let the pole v_1 have the maximum weight .
Then $\sigma(v_j) \preceq \sigma(v_1), \forall v_j \in V - \{v_1\}$

Consider $\deg_S(e_{1j}) = \left(\sum_{e_{1s} \in N_S[e_{1j}]} \psi(e_{1s}), n-2 \right) = \left(\sum_{v_s \in N_S[v_1]} \sigma(v_s), n-2 \right)$

And $\deg_S(e_{1k}) = \left(\sum_{e_{1r} \in N_S[e_{1k}]} \psi(e_{1r}), n-2 \right) = \left(\sum_{v_r \in N_S[v_1]} \sigma(v_r), n-2 \right)$

Here $\deg_S(e_{1j}) = \deg_S(e_{1k})$ iff $\sum \sigma(v_s) = \sum \sigma(v_r) = a$,for some $a \in S$

Therefore for any edge $e_{1j}, e_{1k} \in N_S[v_1]$, $\deg_S(e_{1j}) = \deg_S(e_{1k}) = (a, n-2)$

Therefore G^S is a d_S- edge regular graph.

IV. Conclusion

Unlike the crisp graph theory, in S− valued graphs, in this paper we have introduced the notion of degree of an edge in S− valued graph. Also we have studied the degree regularity conditions on the edges of S− valued graph. In our future work, we would like to extend the study of S− valued graphs with irregularity conditions.

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