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Comparing two Quantiles: the Burr Type X and Weibull Cases

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Abstract: *Interval estimation and hypothesis testing for the difference between two quantiles are investigated, using simulation, in this article. The underlying distributions that will be considered are the Weibull and Burr-Type-X. The estimation procedures will be based on the generalized confidence interval procedure, while the hypothesis testing will be based on the generalized p-values procedure. Simulation will be carried to check on the accuracy of both procedures.*

I. Introduction

A statistical comparison between two populations based on their mean, variances or proportions is a common practice in the literature. This is carried out to check on the superiority of one population over the other. In this article we will make such a comparison between two populations based on their quantiles. In probability and statistics, the quantile function of the probability distribution of a random variable specifies, for a given probability, the value which the random variable will be at, or below, with that probability. A comparison between two quantiles for the Normal and Exponential distributions had been done; see Guo and Krishnamoorthy, (2005). The quantile function is one way of prescribing a probability distribution. It is an alternative to the probability density or mass function, to the cumulative distribution function, and to the characteristic function. The quantile function of a probability distribution is the inverseF^{-1} of its cumulative distribution function (cdf) *F*. Assuming a continuous and strictly monotonic distribution function, the quartile function returns the value below, which the random variable drawn from the given distribution would fall $p\times 100$ percent of the time. That is, it returns the value of *x* such that

$$F(x) = \Pr(X \leq x) = p. \tag{1.1}$$

If the probability distribution is discrete rather than continuous then there may be gaps between values in the domain of its cdf, while if the cdf is only weakly monotonic there may be "flat spots" in its range. In either case, the quantile function is

$$Q(p) = F^{-1}(p) = \inf \{x \in R : p \leq F(x)\}, \tag{1.2}$$

for a probability $0 < p < 1$, and the quantile function returns the minimum value of *x* for which the previous probability statement holds. The Q(p), or ξ_p , (Hogg and Craig, 1995) is the quantile of order p, and thus $\xi_{0.5}$ is the median of the distribution.

Consider two independent random variables X and Y. Let x_p and y_p denote the pthquantile of X and Y respectively. That is,

$$x_p = \inf\{x: P(X \leq x) \geq p\} \quad \text{and} \quad y_p = \inf\{y: P(Y \leq y) \geq p\}. \tag{1.3}$$

The problem of interest here is to make a statistical inference about x_p - y_p based on samples of sizes m and n observations on X and Y, respectively.

In this article, we used simulation for comparing the quantiles of Burr–Type-X, and the quantiles of Weibull distributions. The generalized p-value has been introduced by Tsui and Weerahandi (1989), and the generalized confidence interval by Weerahandi (1993). Using this approach, we will give an inferential procedure for the difference between two Burr-Type-X quantiles in the following section. The performance of the procedure will be evaluated numerically through simulation. In Section 3 we present the weibulldistribution sand the derivation of its quantiles for different values of the involved parameters. In section 4 we present the simulation results on the difference for x_p - y_p , when the underlying distributions are taken to be Burr-Type-X and Weibull with different parameters. Section 5 will contain the conclusions and recommendations.

II. Burr-Type-*X* Distribution

Burr (1942) introduced 12 different forms of cumulative distribution functions for modeling lifetime data, or survival data. Out of those 12 distributions, *Burr-Type-X* and *Burr-Type-XII* have received the maximum attention. Several authors have considered different aspects of these two distributions. In this article we will present the generalized inferential procedures for the difference between the quantiles of two Burr-Type-X distributions. It is to be noted that the probability density function (pdf) of the Burr-Type-X distribution is given as follows:

$$f(x)=2\theta x e^{-x^2}(1-e^{-x^2})^{\theta-1}, \qquad x>0, \theta>0. \qquad (2.1)$$

Moreover, for the one-parameter *Burr –Type-X* distribution, the cumulative distribution function *F* is given by

$$F(x|\theta)=\left\{1-\exp(-x^2)\right\}^{\theta} \qquad x>0,\ \theta>0., \quad (2.2)$$

For any given 0 < p < 1, the pth quantile is the positive root of F(x) = p, i.e.

$p=(1-\exp(-x^2))^{\theta}$, or

$$x=\sqrt{\ln[1/(1-p^{1/\theta})]} \qquad (2.3)$$

Thus if X_i ~ f(x| θ_i), i =1, 2, then the p_i th quantile of X_i can be expressed as

$$\eta_i=\sqrt{\ln[1/(1-p_i^{1/\theta_i})]}, \quad i = 1, 2. \qquad (2.4)$$

Theorem 2.1 Let X~ *Burr-Type-X* (θ), with pdf given by (2.1), then the random variable $U=-\ln\left(1-e^{-x^2}\right)$ will have the one-parameter exponential distribution with mean $\lambda=1/\theta$, i.e., U will have the following pdf

$$h(u)=\begin{cases}\dfrac{e^{-u/\lambda}}{\lambda}, & u>0,\\ 0, & otherwise.\end{cases} \qquad (2.5)$$

Proof:

By putting y = u(x) = $-\ln(1-e^{-x^2})=\ln\dfrac{1}{1-e^{-x^2}}$, we can see that

$$x = \sqrt{\ln\frac{1}{1-e^{-y}}}=w(y) \qquad (2.6)$$

On differentiating (2.6) with respect to y, we have

$$w'(y)=-\frac{e^{-y}}{2(1-e^{-y})}\{-\ln(1-e^{-y})\}^{-1/2} \qquad (2.7)$$

By using the transformation formula namely g(y) f(w(y)).|w'(y)|, we reach at the pdf of y as Given by

g(y) = $\theta e^{-\theta y}$, $0<\theta$, with $\lambda=1/\theta$.

Thus the proof of **Theorem 2.1** is complete.

Theorem 2.2 Because η_i , i = 1, 2, are positive, testing $H_0:\eta_1\le\eta_2$ *vs.* $H_1:\eta_1>\eta_2$ is equivalent to

$$H_0:\lambda_2/\lambda_1\le c \quad vs. \quad H_1:\lambda_2/\lambda_1>c, \qquad (2.8)$$

where c = $\ln p_1 / \ln p_2$, and λ_i I = 1, 2 as defined in Theorem 2.1. In the remaining of the article we will be referring to the means of the exponential distributions as cited in Theorem 2.1 above.

Proof:

From (2.4), we see that $\eta_1 \leq \eta_2$ if and only if $\eta_1^2 \leq \eta_2^2$ if and only if $-\eta_2^2 \leq -\eta_1^2$ iff $1-e^{-\eta_1^2} \leq 1-e^{-\eta_2^2}$ iff $\ln(1-e^{-\eta_1^2}) \leq \ln(1-e^{-\eta_2^2})$ iff $-(1/\theta_2)\ln p_2 \leq -(1/\theta_1)\ln p_1$ iff $-(\lambda_2)\ln p_2 \leq -(\lambda_1)\ln p_1$ iff $\lambda_2 / \lambda_1 \leq \ln p_1 / \ln p_2$ = c. In addition, we can easily see that the above test is equivalent to

$$H_0 : \theta_1 / \theta_2 \leq c \quad vs \quad H_1 : \theta_1 / \theta_2 > c .$$

Hence the proof for **Theorem 2.2** is complete.

Utilizing **Theorems 2.1 and 2.2,** we find that all we need is to generate samples from exponential distributions with parameters λ_1 and λ_2 respectively. Let $X_{i1},\ldots,X_{in_i}$ be a sample from F(u| λ_i), i = 1, 2. Define $Y_i = \sum_{j=1}^{n_i} X_{ij}, i = 1, 2$. Notice that Y_1 and Y_2 are independent with $2Y_i / \lambda_i \sim \chi^2_{2n_i}$, i = 1, 2, and hence Y_1 / Y_2 is distributed as a constant times an F random variable, (see Guo, and Krishnamoorthy (2005)). Thus it can be seen that the p-value for testing (2.8) is given by

$$P(F_{2n_1,2n_2} < cn_2 y_1 / (n_1 y_2)), \qquad (2.9)$$

where $F_{a,b}$ denotes the F distribution with a degrees of freedom for the numerator, and b degrees of freedom for the denominator. Thus the null hypothesis in (2.8) will be rejected whenever this p-value is less than α .
To find the above p-value of the test, and due to the small degrees of freedom that are tabulated for the F-Distribution, we chose to have small samples for the above cases with sizes (n_1, n_2) = (10, 15) and (15, 10) to go with the parameters values (2, 3) and (3, 5) respectively. The results of the simulation are tabulated in Table 1. From **Table 1**, we have for Formula (2.9) the following values:

$cn_2 y_1 / (n_1 y_2)$, *for different values of* c , for the first case we have: (n_1, n_2) = (10, 15), (2, 3)

C =1 $cn_2 y_1 / (n_1 y_2)$ = 0.5705, and the p-value > 0.10,

C =3 $cn_2 y_1 / (n_1 y_2)$ = 1.7115, and the p-value: .05 < P < 0.10.

Thus pending on the value of c, and on comparing the p-value to that of the level of significance, decision can be made on rejecting H_0 or not. Similarly,
$cn_2 y_1 / (n_1 y_2)$, *for different values of* c , for the second case we have (n_1, n_2) = (15, 10), (3, 5)

C =1 $cn_2 y_1 / (n_1 y_2)$ = 0.4167, and the p-value > 0.10,

C = 5 $cn_2 y_1 / (n_1 y_2)$ = 2.0835, and the p-value is almost 0.05.

Again, pending on the value of c, and on comparing the p-value to that of the level of significance, decision can be made on rejecting H_0 or not.

III. The Weibull Distribution

A common application of the Weibull distribution is to model the lifetimes of components such as bearings, ceramics, capacitors, and dielectrics. The Weibull distribution, in model fitting, is a strong competitor to the gamma distribution. Both the gamma and Weibull distributions are skewed, but they are valuable distributions for model fitting. Weibull distribution is commonly used as a model for life length because of the properties of its failure rate function $h(x)=(\alpha/\beta)x^{\alpha-1}$, when the pdf is given by:

$$f(x)=\begin{cases}\dfrac{\alpha}{\beta}x^{\alpha-1}e^{-\frac{x^{\alpha}}{\beta}}, & x,\alpha,\beta>0,\\ 0, & \text{otherwise.}\end{cases} \qquad (3.1)$$

This failure rate function, for $\alpha>1$, is a monotonically increasing function with no upper bound. This property gives the edge for the Weibull distribution over the gamma distribution, where the failure rate function is always bounded by $1/\beta$, when the probability density function for the gamma distribution is

$$g(y)=\frac{1}{\beta}\left(\frac{y}{\beta}\right)^{\alpha-1}\frac{e^{-y/\beta}}{\Gamma(\alpha)}, \quad y>0;\ \text{where } \Gamma(\alpha)=\int_0^{\infty}e^{-u}u^{\alpha-1}du \qquad (3.2)$$

Another property that gives the edge for the Weibull distribution over the gamma distribution is by varying the values of α and β, a wide variety of curves can be generated. Because of this, the Weibull distribution can be made to fit a wide variety of data sets.

In this paper we will consider the Weibull distribution with two parameters, namely α and β, where α is the shape parameter while $\beta^{1/\alpha}$ is the scale parameter of the distribution.

Based on the above form of the pdf for the Weibull distribution, the mean and the variance of the distribution are given as

$$E(X)=\beta^{1/\alpha}\Gamma(1+1/\alpha), \text{ and } V(x)=\beta^{1/\alpha}\left[\Gamma(1+2/\alpha)-\{\Gamma(1+1/\alpha)\}^2\right] \qquad (3.3)$$

We will use the notation X ~ Weibull (α, β) for the random variable X having a Weibull probability density function with parameters α and β, as shown above in (3.1). We will also take the shape parameter as in Hogg and Tanis (1988) to be $1<\alpha<5$. In addition, the cdf for the Weibull distribution given in (3.1) is $F(x)=1-\exp\left(-x^{\alpha}/\beta\right)$, and thus the pth quantile is the solution of the equation F(x) = p, namely,

$$\eta=[\beta\ln\{1/(1-p)\}]^{1/\alpha} \qquad (3.4)$$

Now, as it was the case in the Burr –Type-X, we have the following theorem.

Theorem 3.1 If $X\sim \text{Weibull}(\alpha,\beta_1)$ as given in (3.1), then

i) $U=X^{\alpha}$ will have an exponential distribution with parameter β_1, i.e.,

$$h(u)=\begin{cases}\dfrac{e^{-u/\beta_1}}{\beta_1}, & u>0,\\ 0, & \text{otherwise,}\end{cases}$$

And ii) $\frac{2m\bar{U}}{\beta_1} \sim \chi^2_{2m}$

iii) Similarly, if $Y \sim \text{Weibull}(\alpha, \beta_2)$, then $V = Y^{\alpha}$ will have an exponential distribution with parameter β_2,

i.e. $$g(v) = \begin{cases} \frac{1}{\beta_2} e^{-v/\beta_2}, & v > 0 \\ 0, & otherwise, \end{cases}$$

iv) $\frac{2n\bar{V}}{\beta_2} \sim \chi^2_{2n}$, and

v) $\frac{\beta_1 \bar{V}}{\beta_2 \bar{U}} \sim F_{2n,2m}$.

Proof:

As it was in **Theorem 2.1** above we can write y = u(x) = X^{α}, and we can have X = $Y^{1/\alpha}$ = w(y). On differentiating with respect to y, we reach at

w'(y) = (1/α).$y^{(1-\alpha)/\alpha}$.

Henceg(y) = f(w(y)).|w'(y)| = (1/β_1).$e^{-y/\beta}{}_1$, y = u(x) > 0; and 0 otherwise.

Thus the proof of Theorem 3.1, i). For part ii), when Y = $2X^{\alpha}/\beta$, we reach at w(y) = $(\beta y/2)^{1/\alpha}$. By differentiating with respect to y, we get w'(y) = (1/α).[$(\beta y/2)^{1/\alpha-1}$].(β/2). By substitution in g(y) = f(w(y)).|w'(y)|, we have

$$g(y) = \frac{1}{2} \cdot e^{-y/2} = \frac{1}{\Gamma(r/2) \cdot 2^{r/2}} \cdot y^{r/2-1} \cdot e^{-y/2}.$$

The above function is a pdf for a Chi-square with r = 2 degrees of freedom. Thus the random variable $\frac{2m\bar{U}}{\beta_1}$ will have a Chi-Square distribution with 2m degrees of freedom, based on the sum of m independent Chi-square variables each with 2 degrees of freedom. Hence the proof of **Theorem 3.1** part ii) is complete.

Following the steps above, in the proof of **Theorem 3.1** parts i) and ii), we see that part iii) and iv) follow verbatim. Therefore, the proofs for parts iii) and iv) are done.

The proof for part v), of **Theorem 3.1**, follows by the definition of the F-Distribution. Hence the proof of **Theorem 3.1** is complete.

Theorem 3.2 From (3.4) and because η_i, i = 1, 2, are positive, testing

$$H_0 : \eta_1 \le \eta_2 \quad vs. \quad H_1 : \eta_1 > \eta_2$$

is equivalent to

$$H_0 : \beta_1 / \beta_2 \le c \quad vs \quad H_1 : \beta_1 / \beta_2 > c$$, where c = $\ln(1-p_2)/\ln(1-p_1)$.

Proof: The proof of Theorem 3.2 follows from the proof of **Theorem 2.2.**

Under H_0, $W = \frac{R_0}{1-R_0} \frac{\bar{V}}{\bar{U}} \sim F_{2n,2m}$

The *p*-value for this test is

$z = 2min\,[P_{H_0}(W>w),\ P_{H_0}(W<w)] = 2min\,[1- F(w), F(w)]$,

where *w* is the observed value of the test statistic *W* and *F* is the distribution function of *W* under H_0. The *p*-value of this test indicates how strongly H_0 is supported by the data.

IV. The Simulation Set-Up

There are two cases to consider.

I. The underlying distribution is Burr-Type-X. The following values will be used for the simulation:

A. p = 0.9 with θ = ½, 1 and 2, and on using (2.4) we find the 90th percentiles correspondingly are given by: 1.28869, 1.51743, and 1.72329.

B. p = 0.95 with θ = ½, 1 and 2, and on using (2.4) we find the 95th percentiles correspondingly are given by: 1.52575, 1.73081, and 1.91733.

II. The underlying distribution is Weibull. The following values will be used for the simulation:

A. p = 0.9, with β =1, and α = ½, 1 and 2, and on using (3.4) we find the 90th percentiles correspondingly are given by: 5.30190, 2.30259, and 1.51743.

B. p = 0.9, with β = 2, and α = ½, 1 and 2, and on using (3.4) we find the 90th percentiles correspondingly are given by: 21.20759, 4.60517, and 2.14597.

C. p = 0.95 with β = 1, and α = ½, 1 and 2, and on using (3.4) we find the 95th percentiles correspondingly are given by: 8.97441, 2.99573, and 1.73082.

D. p = 0.95 with β = 2, and α = ½, 1 and 2, and on using (3.4) we find the 95th percentiles correspondingly are given by: 35.89765, 5.99146, 2.44775.

Table 2 below, is based on the simulation of an exponential distribution, since there is no software to generate the Burr-Type-X data, and then using the transformation

$$x = \sqrt{-\ln\left(1-e^{-u}\right)}$$

As was displayed in **Theorem 2.1**.

Table 2 has the simulation that was carried on 30 samples with sample size of 30 for each. Again the table was done on generation an exponential distribution then using the transformation to get the Burr- Type-X data. The data in each sample was ordered to show the order statistics, or the quantiles. Moreover, The Min, the Max, and the Avg. for each quantile was found. In the Burr-Type-X case the value of the parameter was taken to 1.The overall averages are displayed in the last row of the table.

Table 3 has the simulation that was carried on 30 samples with sample size of 30 for each, using MINITAB software to generate the data. As it was the case in **Table 2**, the same procedure was done in Table 3. Table 3 has the Weibull Simulated data when β = 1 and θ = ½, 1, and 2 respectively, and listing the Avg., Min, and Max of the order statistics. The overall averages are displayed in the last row.

Table 4 is a copy of **Table 3** with one difference in this case that β = 2, with the same values for θ.

The other tables are displaying the differences in the quantiles of the Burr-Type-X, as displayed in **Table 2,** and the choice of one value parameter θ = 0.5, using the Max, Min, and Avg of those quantiles.

Table 5 has theTabulated differences between the Burr-Type-X and the Weibull displayed as B – W for the Case of the Weibull Distribution when the parameters are β = 1 and θ = ½.

Table 6 has theTabulated differences between the Burr-Type-X and the Weibull displayed as B – W for the Case of the Weibull Distribution when the parameters are β = 2 and θ = ½.

Other tables can be computed on the other values of the parameters of the Weibull distribution and that of the Burr-Type-X. It is left to the interested reader to match **Table 5** and **Table 6.**

V. Conclusion

It is clearly understood that both the Burr-Type-x and the Weibull distributions are quite used in life testing and failure analysis. As it can be seen from the tabulated quantiles values, it is clearly the for the lower quartiles, i., e. those that are less than the median, in a sample of size 30, those quartiles are higher for the Burr-Type-x than the corresponding Weibull quantiles. The distribution between $X_p - Y_p$, where X_p and Y_p are the pth quintiles based on the Burr-Type-X and Weibull distributions respectively is yet to be displayed. On the other hand, we have noticed that the Upper Quantiles, i.e. those that are greater than the median, for the Weibull distribution are greater than their corresponding quantiles for the Burr-Type-X. The investigation is still ongoing.

References

[1]. Tsui, K. W. and Weerahandi, S. (1989) Generalized p-value in significance testing of hypotheses in the presence of nuisance parameters, *J. Amer. Statist. Assoc*. 84, 602-607.

[2]. Weerahandi, S. (1993) Generalized confidence interval. *J. Amer. Statist. Assoc*. 88, 899-905.

[3]. Guo, H. and Krishnamoorthy, K. (2005) Comparison between two quantiles: The Normal and Exponential cases, Communications in Statistics-Simulation and Computation, 34, 2, 243 – 252.

[4]. Hogg, R. V. and Craig, A. T. (1995) *Introduction to Mathematical Statistics*, 5th Edition, Prentice Hall.

Table 1

First Case		Second Case			
Fist Sample		Second Sample		First Sample	Second Sample
$\lambda_1 = 2$, $n_1 = 10$		$\lambda_2 = 3$, $n_2 = 15$		$\lambda_1 = 3$, $n_1 = 15$	$\lambda_2 = 5$, $n_2 = 10$
2.2732	3.3781	3.0969	1.9168		
0.2708	1.3877	0.1937	0.6444		
10.4278		6.7499	1.6318	0.3827	
1.0386	1.1269	3.6870	10.7766		
0.3408	2.2454	0.6970	0.7284		
0.8017	0.6913	2.7802	4.2920		
8.0997	0.7480	1.6519	1.3491		
0.4844	1.1117	0.9138	4.3318		
0.5843	7.0138	2.4590	13.8120		
0.1185	12.2795	1.3425	3.9954		
	6.4118	1.9035			
	2.2427	0.4225			
	4.5914	0.2389			
	8.8505	2.3178			
	5.4277	3.0566			
24.4398		64.2564	26.3929	42.2292	
0.5705	=	0.4167	=		
$n_2y_1/(n_1y_2)$		$n_2y_1/(n_1y_2)$			

Table 2 Burr-Type-X

	EXP		BURR-TYPE-X		
Max	Min	Avg	Max	Min	Avg
0.08756	0.00016	0.037572	2.47889	8.74042	3.30023
0.16019	0.00424	0.066511	1.91042	5.46531	2.74345
0.21469	0.01085	0.111029	1.64399	4.52901	2.25296
0.36328	0.02868	0.155705	1.18873	3.56586	1.93663
0.38683	0.07752	0.188968	1.13696	2.59573	1.75917
0.39722	0.12166	0.22452	1.11531	2.16674	1.60395
0.40253	0.13307	0.251656	1.10451	2.08268	1.50288
0.8236	0.14322	0.307835	0.57777	2.01413	1.32816
0.94588	0.14381	0.365812	0.49157	2.01031	1.18297
1.03742	0.19145	0.410857	0.43753	1.74733	1.08792
1.11926	0.25279	0.466331	0.39530	1.49893	0.98698

1.17134	0.25326	0.509592	0.37099	1.49730	0.91814
1.42751	0.29625	0.561814	0.27431	1.36102	0.84437
1.62464	0.34162	0.629467	0.21938	1.24001	0.76116
1.63573	0.42884	0.681832	0.21668	1.05344	0.70459
1.73345	0.46142	0.741476	0.19440	0.99530	0.64705
1.74489	0.47661	0.810269	0.19196	0.96991	0.58832
1.75423	0.47818	0.878086	0.19000	0.96735	0.53713
1.90507	0.48945	0.957135	0.16112	0.94924	0.48449
1.99784	.57513	1.057164	0.14575	0.82698	0.42685
2.02596	0.62345	1.144151	0.14141	0.76807	0.38345
2.388	0.65045	1.255315	0.09631	0.73775	0.33545
2.55999	0.72667	1.38024	0.08046	0.66071	0.28971
2.81762	0.79539	1.513887	0.06161	0.60040	0.24853
2.90225	0.96563	1.710707	0.05646	0.47924	0.19935
3.00155	0.98383	1.930947	0.05099	0.46821	0.15667
3.13472	1.07278	2.216855	0.04449	0.41864	0.11536
3.57786	1.30946	2.537553	0.02833	0.31466	0.08236
4.53752	1.37642	2.997327	0.01076	0.29100	0.05121

8.33024 2.43475 4.137713 0.00024 0.09170 0.01609

0.16672 0.76780 0.45408

1.873629 0.623681 1.007944

TABLE 3	WEIBULL										
Case 1	Alpha = 0.5	Beta =1		Case 1	Alpha= 1	Beta =1		Case 1	Alpha= 2	Beta =1
AVG	MIN	MAX		AVG	MIN	MAX		AVG	MIN	MAX
0.00405	0.00000	0.03680		0.03245	0.00151	0.10070		0.16479	0.01655	0.32388
0.00978	0.00000	0.06700		0.07686	0.01677	0.24587		0.25426	0.04688	0.43217
0.01733	0.00030	0.07340		0.11400	0.02132	0.26470		0.32224	0.15519	0.46820
0.02834	0.00160	0.10970		0.15272	0.03502	0.40182		0.37825	0.18651	0.52110
0.03861	0.00460	0.10970		0.18861	0.04635	0.46342		0.40282	0.18882	0.55049
0.05302	0.00990	0.14504		0.22502	0.05256	0.48566		0.45008	0.19750	0.56104
0.07346	0.01450	0.17740		0.26745	0.06836	0.52613		0.48217	0.21452	0.61925
0.10485	0.03200	0.33540		0.31006	0.07329	0.61537		0.52908	0.37143	0.66581
0.15327	0.04790	0.43720		0.37927	0.12142	0.72819		0.56992	0.42110	0.75772
0.19627	0.06160	0.55990		0.41809	0.15735	0.73067		0.60769	0.44994	0.88872
0.23864	0.07750	0.61320		0.47510	0.24460	0.78485		0.64634	0.48249	0.90739
0.28365	0.07890	0.73710		0.52362	0.27113	0.84062		0.69372	0.50426	0.92301
0.35253	0.09200	1.07000		0.58566	0.28607	0.89629		0.72433	0.50934	0.93171
0.44110	0.14010	1.37860		0.64223	0.29898	0.95338		0.76448	0.51603	0.97139
0.51005	0.15230	1.55530		0.71913	0.31548	1.28910		0.79518	0.56320	1.02123
0.58255	0.15460	2.06520		0.78949	0.40421	1.31779		0.83723	0.58515	1.06023
0.69908	0.16890	2.42780		0.85182	0.43929	1.34128		0.87036	0.60426	1.17645
0.78734	0.22650	2.60350		0.91839	0.45056	1.40121		0.89919	0.65218	1.18544
0.96389	0.30220	2.66500		1.00798	0.56421	1.47137		0.94383	0.72306	1.21942
1.16112	0.31500	3.02220		1.11684	0.71307	1.58731		0.99139	0.75885	1.24151
1.34266	0.37100	3.13390		1.22509	0.77456	1.86994		1.04792	0.87662	1.24334
1.71031	0.40470	3.54490		1.33363	0.79484	2.02908		1.09725	0.92378	1.33053
2.09300	0.45160	4.69860		1.45524	0.79794	2.12657		1.15324	0.92973	1.42110
2.59592	0.55134	6.41370		1.59041	0.98928	2.19431		1.21068	0.94809	1.59557
3.11667	0.59283	6.55000		1.73997	1.10544	2.56639		1.28262	0.96983	1.69363
3.90109	0.69450	7.43040		2.01400	1.24644	3.14709		1.37485	1.10903	1.80577
5.06990	0.84350	9.66780		2.23876	1.45572	3.86585		1.47582	1.23937	1.94005
6.32271	1.04760	14.46510		2.58614	1.52300	4.33697		1.59814	1.30492	2.03744

8.98866	2.01990	19.19950		2.99914	1.68579	4.69600		1.76931	1.30737	2.31204
15.67839	2.23078	46.67890		4.14642	2.14967	8.24686		1.94996	1.36071	2.37865
1.91728	0.36961	4.73241		1.03745	0.57014	1.71749		0.87624	0.63722	1.13948

TABLE 4	WEIBULL									
Case 2	Alpha = 0.5	Beta = 2		Case 2	Alpha = 1	Beta = 2		Case 2	Alpha = 2	Beta = 2
AVG	MIN	MAX		AVG	MIN	MAX		AVG	MIN	MAX
0.00804	0.00000	0.11390		0.09209	0.00764	0.25492		0.36270	0.03930	0.70013
0.01369	0.00010	0.11850		0.15624	0.03526	0.38520		0.51646	0.14812	0.75636
0.02961	0.00050	0.12190		0.21543	0.09019	0.48850		0.68029	0.44670	0.97758
0.05891	0.00240	0.32110		0.28790	0.10534	0.53600		0.75605	0.50615	0.99368
0.08603	0.00750	0.35600		0.36951	0.13560	0.64511		0.84703	0.54072	1.06685
0.12879	0.00980	0.39750		0.43641	0.15151	0.81230		0.91269	0.57877	1.21437
0.18631	0.02540	0.42600		0.49568	0.21540	0.86230		0.99471	0.72980	1.28383
0.24405	0.03046	0.62130		0.55688	0.22036	0.90630		1.09564	0.80661	1.29691
0.33149	0.03650	0.81280		0.63830	0.27492	1.10680		1.18210	0.85878	1.45912
0.41404	0.04879	1.37650		0.70274	0.29794	1.19190		1.26168	0.94386	1.48571
0.52429	0.05670	1.41340		0.79469	0.39247	1.29120		1.34592	1.11306	1.67467
0.69607	0.08967	2.12800		0.87781	0.42888	1.40590		1.41838	1.12611	1.75776
0.84428	0.15719	2.33500		0.98376	0.55914	1.47520		1.47331	1.16386	1.80466
0.96501	0.38050	2.63400		1.10181	0.67312	1.80663		1.54504	1.20567	1.85196
1.18550	0.42770	3.12360		1.20189	0.70802	1.81109		1.61561	1.21688	2.03745
1.36342	0.54370	3.36800		1.32747	0.78259	2.07640		1.70004	1.22245	2.11232
1.63066	0.56860	4.20210		1.44358	0.89832	2.12654		1.79478	1.24868	2.27942
2.03156	0.65900	5.18830		1.56225	0.93513	2.29809		1.87614	1.35601	2.50638
2.35257	0.99778	5.34780		1.74890	1.11991	2.82507		1.96132	1.54046	2.61166
2.77950	1.02534	5.39530		1.95903	1.21357	3.34692		2.04542	1.74114	2.72600
3.25458	1.41500	5.88250		2.10801	1.25620	3.70350		2.14577	1.74452	2.75378
3.87733	1.66907	7.21070		2.28886	1.26963	4.27086		2.23606	1.77057	2.90455
4.64369	1.71460	10.12120		2.50638	1.51753	4.31245		2.35315	1.87241	3.01713
5.59030	1.76560	10.52210		2.77610	1.71522	4.60520		2.48629	1.97928	3.15240
6.81775	2.11596	11.95150		3.10106	1.76815	4.99344		2.58758	2.00934	3.22508
8.87342	3.65627	15.60220		3.52133	2.32361	5.43645		2.73429	2.05590	3.34868
10.71896	4.21840	18.80930		3.91765	2.34344	5.97370		2.90242	2.15065	3.49848
14.97397	4.25900	42.76960		4.47769	2.65755	6.79725		3.20819	2.35363	4.28409
21.29488	4.67290	45.42980		5.33055	3.06156	8.20163		3.58397	2.57407	4.71168
37.25324	8.45692	129.64000		7.19738	3.34817	14.57040		4.06476	2.83646	5.00133

4.43906	1.30038	11.25800		1.80591	1.01688	3.01724		1.78959	1.32933	2.28313

Table 5		
Alpha = 0.5	Beta = 1	B - W
Avg	Min	Max
Diff	Diff	Diff
3.29618	8.74042	3.26343
2.73368	5.46531	2.67645
2.23563	4.52871	2.17956
1.90829	3.56426	1.82693
1.72056	2.59113	1.64947
1.55093	2.15684	1.45891
1.42942	2.06818	1.32548
1.22332	1.98213	0.99276
1.02970	1.96241	0.74577
0.89165	1.68573	0.52802
0.74834	1.42143	0.37378
0.63449	1.41840	0.18104
0.49184	1.26902	-0.22563
0.32006	1.09991	-0.61744
0.19454	0.90114	-0.85071
0.06450	0.84070	-1.41815
-0.11077	0.80101	-1.83948
-0.25021	0.74085	-2.06637
-0.47939	0.64704	-2.18051
-0.73427	0.51198	-2.59535
-0.95921	0.39707	-2.75045
-1.37486	0.33305	-3.20945
-1.80329	0.20911	-4.40889
-2.34739	0.04906	-6.16517
-2.91732	-0.11359	-6.35065
-3.74442	-0.22629	-7.27373
-4.95454	-0.42486	-9.55244
-6.24035	-0.73294	-14.38274
-8.93745	-1.72890	-19.14829
-15.66231	-2.13908	-46.66281

-1.46319	0.39820	-4.27833

Table 6		
Alpha = 0.5	Beta = 2	B - W
Avg	Min	Max
Diff	Diff	Diff
3.29219	8.74042	2.36499
2.72977	5.46521	1.79192
2.22336	4.52851	1.52209
1.87772	3.56346	0.86763
1.67315	2.58823	0.78096
1.47516	2.15694	0.71781
1.31657	2.05728	0.67851
1.08412	1.98367	-0.04353
0.85148	1.97381	-0.32123
0.67387	1.69854	-0.93897
0.46269	1.44223	-1.01810
0.22208	1.40763	-1.75701
0.00009	1.20383	-2.06069
-0.20385	0.85951	-2.41462
-0.48091	0.62574	-2.90692
-0.71638	0.45160	-3.17360
-1.04234	0.40131	-4.01014
-1.49442	0.30835	-4.99830
-1.86808	-0.04854	-5.18668
-2.35265	-0.19836	-5.24955
-2.87113	-0.64693	-5.74109
-3.54187	-0.93132	-7.11439
-4.35398	-1.05389	-10.04074
-5.34177	-1.16520	-10.46049
-6.61840	-1.63672	-11.89504
-8.71675	-3.18806	-15.55121
-10.60361	-3.79976	-18.76481
-14.89161	-3.94434	-42.74127
-21.24367	-4.38190	-45.41904
-37.23715	-8.36522	-129.63976
-3.98498	-0.53258	-11.09127