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Two Infinite Views in Mathematics

Zhangrong Lai
(College Of Chinese Language And Culture, Jinan University, China)

Abstract: *Throughout the history of mathematics, the view of actual infinity and the view of potential infinity have taken turns to dominate. In fact, these two infinite views present two aspects of the "infinity". Both of them have great significance to the development of mathematics. Therefore, we should use dialectical thought on these two views and properly use them on mathematical studies.*
Key words: *actual infinity, potential infinity, dialectical thought*

I. Introduction

Since remote times, "infinity" has always been a concept that human beings have to confront. And human's exploration on infinity never ceases, as the renowned mathematician D.Hilbert (1862—1943) has ever said: "No other question has ever moved so profoundly the spirit of man; no other idea has so fruitfully stimulated his intellect; yet no other concept stands in greater need of clarification than that of the infinity."

During people's cognition about infinity, there are gradually two views about infinity: view of actual infinity and view of potential infinity.

The basic thought in the view of actual infinity is: the infinity exists, which regards the infinity as a process or as a whole that could be completed, such as "the set of all natural numbers" really exists.

The basic thought in the view of potential infinity is: infinity is always in structuring and extending, which is a process that could never be completed. This process will never reach an end or obtain the whole set of natural numbers. However, it can accumulate more and more natural numbers, more than any specific number. Therefore, it is infinite.

The view of actual infinity deems the infinity to be static and actual; while the view of potential infinity regards infinity as dynamic and potential.

II. Historical Dispute About Two Infinite Views

The sprout of these two infinite views can be traced to ancient Greek times, when people formed two opposite views on time and space: one thought the time and space to be infinite and divisible while the other thought time and space to be consisted of indivisible segments. Ancient Greek philosopher and mathematician Zeno (B.C. 495-430) proposed four famous paradoxes to argue against the above opinion, which are the dichotomy paradox, Achilles paradox, Arrow paradox and the procession paradox. While the first two paradoxes were against the first view, the left two paradoxes were against the second view. These four paradoxes struck philosophers at that time and they could not argue back. Although Zeno expressed those paradoxes in non-mathematics language, the content involved in concepts in mathematics such as the continuity, limitation, infinite set and so on. Those problems couldn't be answered clearly at that time.

The reputed atomism theory formulator Democritus (B.C. 460-370) supported the view of actual infinity. He thought this world to be consisted of infinite simple and eternal atoms while atoms were the ultimate and indivisible material particle. Moreover, he applied this thought into mathematics, for example, he deemed the cone to be consisted of a series of indivisible thin layers and then calculated its volume.

Plato (B.C. 427-347) admitted clearly to support the view of actual infinity. He thought people could count every positive integer; therefore, all the positive integers existed, and the "the set of all positive integers" also existed. The first one in history that admitted view of potential infinity and denied view of actual infinity was student of Plato, Aristotle (B.C. 384-322). He was also the first one that clearly differentiated the two concepts. He proposed that people should differentiate the "potential" infinity with "actual" infinity. He thought that there was only the potential infinity and infinite set didn't exist. He even thought that most of the quantity could not be potential infinite large since they may exceed the scope of universe if they continued increasing. However, he thought the space to be potential infinity because space could be divisible. Differently, time was potential infinity in both directions. This view of Aristotle later on becomes the basic views of all view of potential infinity supporters.

During the next two thousand years, most of the people supported Aristotle's view of infinity. Until 16^{th} century, this situation started to change. Italian scientist Galileo (1564—1642) gave a serious contemplation on actual infinity. He supported the atomism theory proposed by Democritus and thought the area to be consisted of infinite indivisible units. He noticed that two line segments in different length could be in one-to-one correspondence; the set of all positive integers and the set of all positive even numbers could also be in one-to-one correspondence. Those thoughts imposed great impact on the set theory put forward by G.Cantor(1845－1918) in 19th century. German mathematician and astronomer J.Kepler (1571－1630) systematically calculated area and volume with the method of infinitesimal, whose essence was to determine area and volume with the sum of infinite infinitesimal elements in the same dimension. For example, he pointed out that the volume of a ball was equal to one third of radius multiplying its area. Kepler's work had significant influence, leading the method of infinitesimal to be widely applied in mathematics, and resulting in the view of actual infinity gradually to dominate.

In the second half of 17th century, I.Newton (1642－1727) and G.Leibniz (1646－1716) had respectively established the Calculus, which were both exactly based on actual infinitesimal. In their theories, the infinitesimal was deemed as an entity, a fixed object, while derivative was the quotient of two infinitesimals. The integral was thought to be a sum of infinite actual infinitesimals. Therefore, the early Calculus was also named as "infinitesimal analysis". Within one century after Calculus was formulated, this view of actual infinity was widely applied by mathematicians, leading to a golden age of actual infinity. Mathematicians applied Calculus to astronomy, mechanics, optics, thermotics, etc., and received great achievements. Moreover, this also stimulated and promoted the establishment of many new mathematics branches. The creation of Calculus was appreciated by Engels as the "supreme triumph of the human spirit".

However, on the other hand, people at that time had ambiguous cognition about the infinitesimal concept. Sometimes it was 0 while sometimes not. This logical contradiction led to a series ridiculous conclusions in Calculus (such as the famous Berkeley's Paradox), resulting in the second mathematical crisis. Mathematicians therefore were forced to reconstruct the basis for Calculus, which was mainly completed by Cauchy (1789－1857) and Weierstrass (1815－1897). Especially, Weierstrass formulated strict theory of limit starting from "$\varepsilon - \delta$ definition", which abandoned the thought of regarding infinitesimal as an entity and replaced it with the view of potential infinity: "infinitesimal was a variate in infinite approximation to 0". This theory of limit based on the view of potential infinity has laid solid foundation for Calculus, and the view of potential infinity started to take the dominant place in mathematics again.

In 1870s, Cantor created the set theory, which targeted on various infinite sets. Therefore, this theory affirmed the actual infinity. The view of actual infinity got revived in mathematics. Cantor pointed out that the essential difference between infinite set and finite set was: an infinite set could be in one-to-one correspondence with its proper subset while a finite set could not. He defined the "cardinal number" concept for infinite set and proposed his transfinite number theory. Moreover, he proved the famous Cantor Cantor's theorem: for any set A, the set of all subsets of A has a strictly greater cardinality than A itself. This indicates that infinity also has infinite "levels", and there is no a largest infinity. After Cantor's theories that presented the view of actual infinity were put forward, there was a tremendous strike in mathematics. His theories were fiercely attacked by his teacher, the leader of Berlin school of thought, L.Kronecker (1823－1891). Kronecker disapproved the view of actual infinity and only admitted objects that could be determined by finite steps. He claimed that all things need construct steps or judging criteria. However, there were many mathematicians supported Cantor firmly. Hilbert actively spread the thoughts of Cantor in Germany and said: "no one can kick us out of the paradise Cantor created for us. "

Later on, paradoxes were successively discovered in set theory; especially, B.Russell (1872－1970) published his famous Russell's paradox in 1903 which challenged the basis of set theory and caused the third mathematical crisis. A great debate about the foundations of mathematics was brought out and three mutually contradictory schools were formed: logicism, intuitionism and formalism.

The main representative for logicism was B.Russell. Its main purpose was to transform mathematics into logic. They admitted the effectiveness of all mathematics together with the set theory under the view of actual infinity. Therefore, they are pros for view of actual infinity.

The main representative for intuitionism was L.E.Brouwer (1881－1966). They disapproved non-constructive methods but only admitted concepts that could be defined by finite steps and methods that could realize. Those basic thoughts determined them to be completely pros for view of potential infinity.

The founder for formalism was D.Hilbert. This school on the one hand insisted to maintain the fundamental concepts of classical mathematics and the inferential principles of classical logic (including those concepts and methods related to actual infinity) such as the concept of infinite set and application of law of excluded middle on infinite set. On the other hand, they thought credibility to only exist in finite while the concept of infinite was merely reasonable stipulation. Therefore, Hilbert divided the whole mathematics into two categories: "real mathematics" and "ideal mathematics". Mathematics that related to view of actual infinity and transfinite inference was "ideal mathematics". Besides, he also proposed the proof theory for finite standpoint, attempting to axiomatize the whole mathematics and prove the compatibility of this form system, which was the well-known "Hilbert's Programme". However, this grand plan of Hilbert was utterly denied by the "incompleteness theorem" proved by K.Godel (1906－1978) in 1931. Other assumption denied by the "incompleteness theorem" was the thought of transforming mathematics to logic holding by the logicism school. In 1960s, A.Robinson (1918－1974) extended the field of all real numbers to the nonstandard number field that included infinitely great and infinitely small via the method of model theory, on which he then conducted classical mathematical analysis and formulated "non-standard analysis". The non-standard analysis actually presents the view of actual infinity.

III. Conclusion

In mathematics, whether we should insist on the view of actual infinity or the view of potential infinity? In fact, throughout the history of mathematics, the view of actual infinity and view of potential infinity have taken turns to dominate. Through the alternation, we can see that, mathematics cannot be separated from either the view of actual infinity or the view of potential infinity. The two are compatible rather "contradictory and incompatible". For example, the theory of limit starting from "$\varepsilon - \delta$ definition" is thought to indicate the view of potential infinity; however, its foundation—the theory of real number is based on the view of actual infinity. The creation of classical mathematical analysis and nonstandard analysis also indicate that starting from different views of infinity, we can obtain the identical results through different methods. Besides, the prominent achievements of many mathematicians who object the view of actual infinity are actually rooted in the set theory under the view of actual infinity. As a consequence, modern mathematics as a whole is a theoretical system that compacts the two different views of infinity.

Potential infinity and actual infinity are actually two sides of "infinity". It is a process that extends infinitely (potential infinity) and meanwhile it is also an entity, an object that can be studied (actual infinity). While the potential infinity emphasizes "process", the actual infinity highlights "result". We should hold dialectical opinions on infinity and appropriately capitalize on the two views of infinity to conduct mathematical studies in accordance with engaged objects and thesis features.

From ancient to modern times, human have gradually enriched their cognition on infinity. While calculus opened a new era to comprehend infinity, the set theory furthermore promoted human's cognition on infinity to a new stage. We are firmly convinced that, in the course of time, human's comprehension about infinity will leap to new heights one after another!

References

[1]. M.Kline. Mathematical Thought from Ancient To modern Times [M]. Shanghai: Shanghai Science and Technology Press，2002 (in Chinese)

[2]. Zhu Wujia, etc. Research on Infinite Views (I) [J]. Journal of Nanjing University of Aeronautics and Astronautics, 2002，34（2）(in Chinese)

[3]. Li Xincan. Creators and Pioneers of Calculus [M]. Beijing: Higher Education Press, 2002(in Chinese)

[4]. Xu Lizhi. Xu Lizhi on Mathematical Methodologies [M]. Jinan: Shandong Education Press, 2001(in Chinese)

[5]. Zheng Junwen, Zhang Enhua. An Introduction to Mathematical Logic [M]. Hefei: Anhui Education Press, 1995(in Chinese)

[6]. Zhang Shunyan. Origin and Branch of Mathematics [M]. Beijing: Higher Education Press, 2000(in Chinese)

[7]. Li Wenlin. An Introduction to Mathematical History [M]. Beijing: Higher Education Press. 2002(in Chinese)

[8]. Chiefly edited by Wu Wenjun. Biography of Famous Mathematicians in the World [M]. Beijing: Science Press, 1995(in Chinese)