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Analysis of Successive Occurrence of Digit 0 in Natural Numbers Less Than 10^n

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Abstract: *All successive natural numbers less than 10^n, for any positive integer n, are extensively analyzed of successive occurrence of digit 0. The formula for the number of successive occurrences of 0's is given. Formulae for the very first instance of successive 0's and their last occurrence are also provided.*
Keywords: *Digit 0, natural numbers, successive occurrences.*
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I. Introduction

Number Theory is a branch of mathematics involving study of natural numbers

$$1, 2, 3, \cdots$$

As has been aptly mentioned [2], [3], and [5], it is amongst the concepts of Mathematics most widely used anywhere.

There are many interesting properties of numbers [7]. The very first digit 0 is under consideration here. Its general occurrences in natural numbers [5] and in prime numbers [6] are already analyzed in detail.

Throughout this paper, the term number means the natural number. Like in earlier works, we have considered the ranges $1 – 10^n$, omitting the last number 10^n, for $n \in N$, i.e., the numbers under consideration are m, with $1 \leq m < 10^n$. The reason for dropping last number 10^n is that it is a number with next higher number of significant digits. The numbers in the selected range contain n or fewer significant digits.

Only significant 0's, i.e., 0's coming after some non-zero digit(s), are considered in this analysis as any number of pre-fixed 0's don't have any significance.

II. Successive Occurrence of Digit 0

0's prominence in any number system having place value character is clear [1]. In this work, the successive occurrence of digit 0 is analyzed in the range of $1 – 10^n$, except the last number 10^n, for all natural numbers n. All occurrences of 0's in similar ranges have already been formulated [5].

Theorem 1: If r and n are positive integers with $r < n$, then the number of numbers containing exactly r number of significant digit 0's in the range $1 \leq m < 10^n$ is

$$ {}_0^A O_r^n = \sum_{j=r+1}^{n} {}^{j-1}C_r 9^{j-r} ,$$

where the notation ${}_0^A O_r^n$ stands for number of numbers less than 10^n which contain r number of 0's.

For ranges as high as one quintillion, i.e., 10^{18}, we have determined the count of successive occurrence of multiple 0's by using modern computer Java program.

Table 1: Number of Numbers with One and Two Successive 0's in their Digits

Sr. No.	Numbers Range <	Number of Numbers with single (Successive) 0	Number of Numbers with 2 Successive 0's
1.	10^1	0	0
2.	10^2	9	0
3.	10^3	171	9
4.	10^4	2,358	171
5.	10^5	28,602	2,358
6.	10^6	323,847	28,602
7.	10^7	3,512,493	323,847
8.	10^8	36,993,276	3,512,493
9.	10^9	381,367,044	36,993,276
10.	10^{10}	3,868,151,445	381,367,044
11.	10^{11}	38,735,995,455	3,868,151,445
12.	10^{12}	383,927,651,154	38,735,995,455

Sr. No.	Numbers Range <	Number of Numbers with single (Successive) 0	Number of Numbers with 2 Successive 0's
13.	10^{13}	3,773,082,088,926	383,927,651,154
14.	10^{14}	36,817,337,857,203	3,773,082,088,926
15.	10^{15}	357,092,432,226,657	36,817,337,857,203
16.	10^{16}	3,445,459,413,646,392	357,092,432,226,657
17.	10^{17}	33,093,782,435,275,848	3,445,459,413,646,392
18.	10^{18}	316,605,871,329,607,538	33,093,782,435,275,848

The first range $1 \le m < 10^1 = 10$, just doesn't any 0 at all. There occur only 9 numbers which are all nine digits except 0.

In the second range $1 \le m < 10^2 = 100$, single 0 comes 9 times. Its instances, as specified in [5], are in numbers

10, 20, 30, 40, 50, 60, 70, 80, and 90,

which are all at unit's places. So, first occurrence is ${}^{2\text{-}1}C_1 9^{2\text{-}1} = 9$ times. Single occurrence is qualified to be successive by absence of non-successive character!

In the third range, $1 \le m < 10^3 = 1{,}000$, as stated in [5], single 0 occurs 171 times. It occurs in earlier 9 numbers

10, 20, 30, · · · , 90,

and then additionally in

110, 120, 130, · · · , 190, 210, 220, 230, · · · , 290, ⋯ , 910, 920, 930, · · · , 990,
101, 102, 103, · · · , 109, 201, 202, 203, · · · , 209, ⋯ , 901, 902, 903, · · · , 909.

This additional occurrence is ${}^{3\text{-}1}C_1 9^{3\text{-}1} = 2 \times 9^2 = 162$ times, totaling to 9 + 162 = 171 times. Since single 0 can be considered as successive, all these happen to fall in the category of successive occurrences.

In this range, double 0's occur in

100, 200, 300, · · · , 900,

which is ${}^{3\text{-}1}C_2 9^1 = 9$ times. All of them are already successive!

In the fourth range, $1 \le m < 10^4 = 10{,}000$, as determined in [5], single 0 occurs in earlier 171 numbers

10, 20, 30, · · · , 90,
110, 120, 130, · · · , 190, 210, 220, 230, · · · , 290, ⋯ , 910, 920, 930, · · · , 990,
101, 102, 103, · · · , 109, 201, 202, 203, · · · , 209, ⋯ , 901, 902, 903, · · · , 909,

and then additionally in

1110, 1120, · · · , 1190, 1210, 1220, · · · , 1290, ⋯ , 1910, 1920, · · · , 1990,
2110, 2120, · · · , 2190, 2210, 2220, · · · , 2290, ⋯ , 2910, 2920, · · · , 2990,
⋮
9110, 9120, · · · , 9190, 9210, 9220, · · · , 9290, ⋯ , 9910, 9920, · · · , 9990,
1101, 1102, · · · , 1109, 1201, 1202, · · · , 1209, ⋯ , 1901, 1902, · · · , 1909,
2101, 2102, · · · , 2109, 2201, 2202, · · · , 2209, ⋯ , 2901, 2902, · · · , 2909,
⋮
9101, 9102, · · · , 9109, 9201, 9202, · · · , 9209, ⋯ , 9901, 9902, · · · , 9909,
1011, 1012, · · · , 1019, 1021, 1022, · · · , 1029, ⋯ , 1091, 1092, · · · , 1099,
2011, 2012, · · · , 2019, 2021, 2022, · · · , 2029, ⋯ , 2091, 2092, · · · , 2099,
⋮
9011, 9012, · · · , 9019, 9021, 9022, · · · , 9029, ⋯ , 9091, 9092, · · · , 9099,

These new give count ${}^{4\text{-}1}C_1\, 9^3 = 3 \times 9^3 = 2{,}187$ times, totaling to 171 + 2,187 = 2,358 times. They too, being single, are considered successive.

Now in this range double 0's occur (successively) in earlier 9 numbers

100, 200, 300, · · · , 900,

and then additionally they occur successively in

1100, 1200, · · · , 1900, 2100, 2200, · · · , 2900, ⋯ , 9100, 9200, · · · , 9900,
1001, 1002, · · · , 1009, 2001, 2002, · · · , 2009, ⋯ , 9001, 9002, · · · , 9009.

This new count is ${}^{3\text{-}1}C_1 9^2 = 2 \times 9^2 = 162$, totaling to 9 + 162 = 171.

Above tables' expansion continues on these lines.

Higher successive 0's come in these ranges in similar systematic patterns. Taking hint from [5] and [3], we have formulated these.

Notation : The symbol ${}^{S}_{0}O^{n}_{r}$ will be used for number of numbers less than 10^n with r successive 0's.

Theorem 2 : If r and n are positive integers with $r < n$, then the number of numbers containing exactly r number of significant successive digit 0's in the range $1 \le m < 10^n$ is

$${}_0^S O_r^n = \sum_{j=2}^{n-(r-1)} {}^{j-1}C_1 9^{j-1}$$

***Proof* 1.** Let n and r be positive integers with $r < n$. We prove the result by induction by n and r. The cases of initial values of r are already discussed above, wherein the formula is seen to be valid. For any positive r, for successive instance of significant digit 0's, the minimum value of n required is $r + 1$. In this initial case, there is only one possible position for all 0's, viz., last succession, and then there are exactly 9 occurrences of r successive 0's :

$$1\underbrace{000\cdots0}_{r\text{ times}},2\underbrace{000\cdots0}_{r\text{ times}},\cdots,9\underbrace{000\cdots0}_{r\text{ times}}$$

And as per formula, this count is

$${}_0^S O_r^{n=r+1} = \sum_{j=2}^{r+1-(r-1)} {}^{j-1}C_1 9^{j-1} = {}^1C_1 9^{2-1} = 9$$

So, the formula is true in this case. Each block of successive occurrences of 0's begin with count 9.
The next value of n is $r + 2$. Now except the leading position, there are $n – 1$, in this case 2, possible positions that can be occupied by r number of successive 0's, viz., last and middle.
We actually mention these in this case:

$$1\underbrace{000\cdots0}_{r\text{ times}},2\underbrace{000\cdots0}_{r\text{ times}},\cdots,9\underbrace{000\cdots0}_{r\text{ times}} \qquad (= 9)$$

$$11\underbrace{000\cdots0}_{r\text{ times}},\cdots,19\underbrace{000\cdots0}_{r\text{ times}},21\underbrace{000\cdots0}_{r\text{ times}},\cdots,29\underbrace{000\cdots0}_{r\text{ times}},\cdots,91\underbrace{000\cdots0}_{r\text{ times}},\cdots,99\underbrace{000\cdots0}_{r\text{ times}} \qquad (= 9 \times 9)$$

$$1\underbrace{000\cdots0}_{r\text{ times}}1,\cdots,1\underbrace{000\cdots0}_{r\text{ times}}9,2\underbrace{000\cdots0}_{r\text{ times}}1,\cdots,2\underbrace{000\cdots0}_{r\text{ times}}9,\cdots,9\underbrace{000\cdots0}_{r\text{ times}}1,\cdots,9\underbrace{000\cdots0}_{r\text{ times}}9 \qquad (= 9 \times 9)$$

In addition to earlier 9 occurrences of r successive 0's in the first row, now there are twice 9^2 additional occurrences, so total occurrences of r successive 0's is given by
Previous + ${}^2C_1 9^2$ and hence equals

$$9 + {}^2C_1 9^2 = {}^1C_1 9^{r+1-r} + {}^{r+2-(r-1)-1}C_1 9^{r+2-(r-1)-1} = \sum_{j=2}^{r+2-(r-1)} {}^{j-1}C_1 9^{j-1} = {}_0^S O_r^{n=r+2}$$

asserting that the formula is true in this case either.
Continuing this indefinitely, the formula is proved for all positive integers r and $n > r$.

***Proof* 2.** We will have an easy alternative proof of this Theorem as an application of Theorem in [5].
Now suppose that the range is $1 \le m < 10^n$. We want occurrences of r successive 0's in it. Because we want all r 0's to occur one after the other, we can consider that instead of r digits, they together form only one unit block '$\underbrace{000\cdots0}_{r\text{ times}}$'. Then there remain only $n – (r – 1)$ digit units. Changing n to $n – (r – 1)$ and r to 1 in right hand side of the formula in [5] and A for all in notation by S for successive, we get the formula for occurrences of r successive 0's as

$${}_0^S O_r^n = \sum_{j=1+1}^{n-(r-1)} {}^{j-1}C_1 9^{j-1} = \sum_{j=2}^{n-(r-1)} {}^{j-1}C_1 9^{j-1} .$$

The table given above is now extended to higher occurrences of successive 0's.

Table 2: Number of Numbers with Multiple Successive 0's in their Digits

Sr. No.	Number Range <	Number of Numbers with		
		3 Successive 0's	4 Successive 0's	5 Successive 0's
1.	10^4	9	0	0
2.	10^5	171	9	0
3.	10^6	2,358	171	9
4.	10^7	28,602	2,358	171
5.	10^8	323,847	28,602	2,358
6.	10^9	3,512,493	323,847	28,602
7.	10^{10}	36,993,276	3,512,493	323,847
8.	10^{11}	381,367,044	36,993,276	3,512,493
9.	10^{12}	3,868,151,445	381,367,044	36,993,276
10.	10^{13}	38,735,995,455	3,868,151,445	381,367,044

Sr. No.	Number Range <	Number of Numbers with		
		3 Successive 0's	4 Successive 0's	5 Successive 0's
11.	10^{14}	383,927,651,154	38,735,995,455	3,868,151,445
12.	10^{15}	3,773,082,088,926	383,927,651,154	38,735,995,455
13.	10^{16}	36,817,337,857,203	3,773,082,088,926	383,927,651,154
14.	10^{17}	357,092,432,226,657	36,817,337,857,203	3,773,082,088,926
15.	10^{18}	3,445,459,413,646,392	357,092,432,226,657	36,817,337,857,203

Table 2: Continued …

Sr. No.	Number Range <	Number of Numbers with			
		6 Successive 0's	7 Successive 0's	8 Successive 0's	9 Successive 0's
1.	10^{7}	9	0	0	0
2.	10^{8}	171	9	0	0
3.	10^{9}	2,358	171	9	0
4.	10^{10}	28,602	2,358	171	9
5.	10^{11}	323,847	28,602	2,358	171
6.	10^{12}	3,512,493	323,847	28,602	2,358
7.	10^{13}	36,993,276	3,512,493	323,847	28,602
8.	10^{14}	381,367,044	36,993,276	3,512,493	323,847
9.	10^{15}	3,868,151,445	381,367,044	36,993,276	3,512,493
10.	10^{16}	38,735,995,455	3,868,151,445	381,367,044	36,993,276
11.	10^{17}	383,927,651,154	38,735,995,455	3,868,151,445	381,367,044
12.	10^{18}	3,773,082,088,926	383,927,651,154	38,735,995,455	3,868,151,445

Table 2: Continued …

Sr. No.	Number Range <	Number of Numbers with			
		10 Successive 0's	11 Successive 0's	12 Successive 0's	13 Successive 0's
1.	10^{11}	9	0	0	0
2.	10^{12}	171	9	0	0
3.	10^{13}	2,358	171	9	0
4.	10^{14}	28,602	2,358	171	9
5.	10^{15}	323,847	28,602	2,358	171
6.	10^{16}	3,512,493	323,847	28,602	2,358
7.	10^{17}	36,993,276	3,512,493	323,847	28,602
8.	10^{18}	381,367,044	36,993,276	3,512,493	323,847

Table 2: Continued …

Sr. No.	Number Range <	Number of Numbers with Successive				
		14 Successive 0's	15 Successive 0's	16 Successive 0's	17 Successive 0's	18 Successive 0's
1.	10^{14}	0	0	0	0	0
2.	10^{15}	9	0	0	0	0
3.	10^{16}	171	9	0	0	0
4.	10^{17}	2,358	171	9	0	0
5.	10^{18}	28,602	2,358	171	9	0

III. First Occurrence of Successive Digit 0's

The first number containing 0 is 10. In fact, 0 becomes significant from 10 onwards only. It being single occurrence is successive. For 2 successive 0's, the first instance is 100, for 3 successive 0's, it is 1000 and so on. It's simple formulation follows.

Formula 1 : If n and r are natural numbers, then the first occurrence of successive r zeros in numbers in range $1 \leq m < 10^n$ is

$$f = \begin{cases} - & , \text{if } r \geq n \\ 10^r & , \text{if } r < n \end{cases}.$$

IV. Last Occurrence of Successive Digit 0's

The last numbers in our ranges containing multiple successive 0's are as follows.

Table 3: Last Numbers in Range with Multiple Successive 0's in their Digits

Sr. No.	Last number with Successive ↓	Number Range < ↓								
		10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
1.	1 0	-	90	990	9,990	99,990	999,990	9,999,990	99,999,990	999,999,990
2.	2 0's	-	-	900	9,900	99,900	999,900	9,999,900	99,999,900	999,999,900

Sr. No.	Last number with Successive ↓	Number Range < ↓								
		10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
3.	3 0's	-	-	-	9,000	99,000	999,000	9,999,000	99,999,000	999,999,000
4.	4 0's	-	-	-	-	90,000	990,000	9,990,000	99,990,000	999,990,000
5.	5 0's	-	-	-	-	-	900,000	9,900,000	99,900,000	999,900,000
6.	6 0's	-	-	-	-	-	-	9,000,000	99,000,000	999,000,000
7.	7 0's	-	-	-	-	-	-	-	90,000,000	990,000,000
8.	8 0's	-	-	-	-	-	-	-	-	900,000,000

They fit in a formula.

Formula 2 : If n and r are natural numbers, then the last occurrence of r successive 0's in numbers in range $1 \le m < 10^n$ is

$$l = \begin{cases} - & \text{, if } r \ge n \\ 10^n - 10^r & \text{, if } r < n \end{cases}.$$

It is no surprise that formulae 1 and 2 given here are just the same as the formulae in [5]!

The gradually progressing integer sequence in the tables for count of occurrences of higher number of successive 0's is peculiar.

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