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Goldbach's Conjecture

Ritvi Singh D/O Mr. Jasveer Singh
Student of B.Sc (Final year) ,
Govt. R.D. Girls College
Bharatpur,Rajasthan, India

Goldbach's conjecture is one of the oldest and best-known unsolved problems in number theory and all of mathematics. It states:
Every even integer greater than 2 can be expressed as sum of two primes
Proof: We can prove above conjecture with the help of mathematical induction
Let P(n) be the statement that "every even integer greater than 2 can be expressed as sum of two primes "
i.e. every number of form 2n, $n\geq 2$ can be expressed as sum of two primes
We have,
P(2) = 2(2) =4 =2+2
$\Rightarrow$4 can be expressed as sum of two primes (2 and 2)
$\Rightarrow$P(2) is true
Let P(m) be true then
$P(m) = 2m = P_1 + P_2$ ....(i)
Where P_1 and P_2 are primes
Now, we shall show that P(m+1) is true
For which we have to show that 2(m+1) can be expressed as sum of two primes
We have,
P(m+1) = 2(m+1) = 2m+2
$\Rightarrow 2(m+1) = P_1 + P_2 + 2$ ....(ii)
In order to show that 2(m+1) is sum of two primes we need to prove that either of the following possibility holds true :

- Since P_1 is prime so it is sufficient to prove that (P_2+2) is prime
- (P_1+1) and (P_2+2) both are prime
- If both the above possibilities do not holds good then we can show that if $P_3$is any prime less (or greater) than P_1 then there exist a prime P_4 such that $P_3 + P_4 = 2(m+1)$

Now using equation (ii)
$2(m+1) = P_1 + P_2 + 2$
$$\frac{2(m+1)}{a} = \frac{P_1}{a} + \frac{P_2+2}{a} \text{ .... (iii)}$$
Where a is any arbitrary number less than 2(m+1)
Since, 2(m+1)>each term of R.H.S
$\Rightarrow$factors of any term of RHS if exists will be smaller than 2(m+1)
Now there are two types of numbers smaller than 2(m+1):

- Numbers which divides 2(m+1)
- Numbers which do not divides 2(m+1)

Thus there arises following two cases:

- a is any arbitrary number which divides 2(m+1)
- a is any arbitrary number which do not divides 2(m+1)

Case – I : when a is any arbitrary number which divides 2(m+1)
Suppose a divides 2(m+1),**q**times then
$$\frac{2(m+1)}{a} = q \text{ .... (iv)}$$

Where q is any natural number
Now , P_1 is prime
⟹a do not divide P_1
Suppose a divides P_1**u** times and leaves r_1 as remainder then,
$P_1 = ua + r_1$

$$\frac{P_1}{a} = \frac{ua + r_1}{a} \quad \text{........................ (v)}$$

Substituting value of equation (iv) and (v) in (iii)

$$\frac{qa}{a} = \frac{ua + r_1}{a} + \frac{P_2 + 2}{a}$$

$qa = ua + r_1 + P_2 + 2$
$(q-u)a - r_1 = P_2 + 2$
$(q-u)a - a + a - r_1 = P_2 + 2$
$(q-u-1)a + r = P_2 + 2$
$Za + r = P_2 + 2$..................... (vi)

Where $z= (q-u-1)$
And $r=a-r_1$
Then by division algorithm we can say that $(P_2 +2)$ is not divided by a if z is whole number and $r<a$
Now, we will prove that z is whole number
Clearly,
$2(m+1)>P_1>0$

$$\Rightarrow \frac{qa}{a} > \frac{ua + r_1}{a}$$

$$\Rightarrow (q-u) > \frac{r_1}{a}$$

Also ,$r_1<a$

$$\Rightarrow \frac{r_1}{a} < 1$$

$$\Rightarrow (q-u) \geq 1$$

$$\Rightarrow (q-u-1) \geq 0 \quad \text{............................... (vii)}$$

Also, q,u,1 all are whole numbers so their difference will also be a whole number
Thus, $z=(q-u-1)$ is whole number..........................(viii)
Again, $r= (a-r_1)$
Where $0<r_1<a$
$\Rightarrow a-r_1<a$
$\Rightarrow r<a$ .... (ix)
Thus from equation (vi), (viii) and (ix) we can say that $a(P_2+2)$ is not divided by a
Note : (Here $a\neq 1$. As,
$(P_2+2)= za+r$ where $0<r<a$
Now if $a=1 \Rightarrow r=0$
$\Rightarrow P_2+2=za$
$\Rightarrow P_2 +2$ is divided by a)
Result of case I : (P_2+2) is not divided by any number (other than 1) which divides 2(m+1)

Case II :When a is any arbitrary number which do not divides 2(m+1)
Since a do not divides 2(m+1) then by division algorithm we can say that
$2(m+1) = sa +r_2$ ....(x)
Where s is any whole number
And $r_2 < a$
Again P_1 is prime thus P_1 is not divided by a
Thus, $P_1 = ta +r_3$.............................. (xi)

Where t is any whole number
And $r_3 < a$
Now , substituting values from equation (x) and (xi) in equation (iii) we get,

$$\frac{sa + r_2}{a} = \frac{ta + r_3}{a} + \frac{P_2 + 2}{a}$$

$$\Rightarrow \frac{(s-t)a + (r_2 - r_3)}{a} = \frac{P_2 + 2}{a}$$

$$\Rightarrow (s-t) + \frac{r_2 - r_3}{a} = \frac{P_2 + 2}{a}$$

$$\Rightarrow (s-t)a + (r_2 - r_3) = P_2 + 2$$

$$\Rightarrow wa + r_4 = P_2 + 2$$

$wa + r_4 = P_2 + 2$ ....(xii)

Since s and t are whole numbers this implies (s-t)=w is also a whole number
Now there arises following two cases

1. If w=0 then ,
$0(a) + (r_2 - r_3) = P_2 + 2$
$r_4 = P_2 + 2$ .... (xiii)

$$\frac{r_2 - r_3}{a} = \frac{P_2 + 2}{a}$$

Also, $0 < r_2, r_3 < a$
$\Rightarrow -a < r_2 - r_3 < a$
$\Rightarrow -a < r_4 < a$
Now on the basis of nature of r_4 there arises following conditions:

- **$-a < r_4 < 0$**

In this case equation (xiii) implies P_2+2 is negative integer but we know that P_2+2 is positive
Hence, if w=0 then r_4 will never be less than zero .

- **$r_4 = 0$**

This is also not possible because if $r_4=0$ then from equation (xiii) $P_2+2=0$ but P_2+2 is positive. Hence r_4 will never be zero when w=0

- **$0 < r_4 < a$**

In this case by equation (xiii) we can say that
$r_4 = P_2 + 2$

$$\frac{r_4}{a} = \frac{P_2 + 2}{a}$$

Since $r_4 < a$

$$\Rightarrow 0 < \frac{r_4}{a} < 1$$

$$\Rightarrow 0 < \frac{P_2 + 2}{a} < 1$$

$\Rightarrow$ P_2+2 is not divided by a ($\because \frac{P_2 + 2}{a}$ is a fraction smaller than 1)

Result: In this case 2(m+1) can be expressed as sum of two primes P_1 and P_2+2

***2*.If w>0**
Now there arises following conditions on the basis of nature of r_4

- **$-a < r_4 < 0$**

Now in this condition we need to check whether P_2+2 is divided by a or not
From equation (xii)
$wa + r_4 = P_2 + 2$
$wa - a + a + r_4 = P_2 + 2$
$(w-1)a + (a + r_4) = P_2 + 2$

$(w-1)a+r_5=P_2+2$
Since $w\geq 1$
$\Rightarrow (w-1)\geq 0$
Hence, (w-1)is whole number ....(xiii)
Also,
$-a<r_4<0$
$-a+a<r_4+a<0+a$
$0<r_5<a$.................. *(*xiv)
Thus , from equation (xiii) and (xiv) we can say that (P_2+2) is not divided by a

Thus in this case 2(m+1) can be expressed as sum of two primes P_1 and (P_2+2)

- **$0<r_4<a$**

Again from equation (xii)
$wa+r_4 =P_2+2$
Where w is whole number
And $r_4< a$
Then clearly from division algorithm we can say P_2+2 is not divided by a
Result: In this case 2(m+1) can be expressed as sum of two primes P_1 and (P_2+2)

- **$r_4=0$**

Since r_2 and r_3 are not zero this implies r_4 will be zero only and only if $r_2 = r_3$
Then in this case equation (xii) becomes
$(s-t)a=P_2+2$
If (s-t)=1
$\Rightarrow a=P_2+2$
$\Rightarrow P_2+2$ is divided by itself
Also (s-t) $\neq 1$ then from equation (xii) we can say that P_2+2 is not prime. But it does not mean that 2(m+1) cannot be expressed as sum of two primes as now also we have two more possibilities as told in starting of proof

[Which were 2.*possibility:* 2(m+1) is *sum (*P_1+1) and (P_2+1) then we will show that P_1+1 and (P_2+1) are primes
3.*possibility:*2(m+1) is sum of P_3 and P_4 where P_3 is an prime less(or greater) than P_1 and P_4 is any natural number and then we will prove that P_4 is also prime]
Now we will check whether the second possibility holds true
$2(m+1)=(P_1+1)+(P_2+1)$
Now we will prove that (P_1+1) and (P_2+1) both are prime
But, we know that all prime numbers except 2 are odd
Also P_1 and P_2 both are prime
$\Rightarrow P_1$ andP_2 both are odd
$\Rightarrow$ $(P_1 +1)$and$(P_2 +1)$ are even
$\Rightarrow (P_1 +1)$and$(P_2 +1)$ can be prime only and only if $(P_1 +1)$and$(P_2 +1)$ both are separately equal to 2
Thus this possibility holds true only and only if $2(m+1)= (P_1 +1) + (P_2 +1)$

$$2(m+1)=2+2$$
$$2(m+1)=4$$

Now we will check whether our last possibility holds true or not. For which let us consider a prime P_3 *and* a natural number P_4 such that $P_3 + P_4 = 2(m+1)$................. *(*xv)
Since P_3 is prime , hence it will not be divided by any a(other than 1 and itself)
Thus we can write $P_3 = va +r_6$ (using division algorithm)........... *(*xvi)
Also $2(m+1) = sa +r_2$................ *(*xvii)
(as we have taken the case that 2(m+1) is not divided by a)
Now substituting value of P_3 and 2(m+1) in equation (xv) we get
$sa+ r_2 =va+r_6 +P_4$
$(s-v)a +(r_2-r_6)=P_4$
$w_1a + r_7 = P_4$
Again there arises following three conditions on the basis of nature of r_7

- **$-a<r_7<0$**
- **$0<r_7<a$**
- **$r_7=0$**

Again for first two conditions we can prove P_4 is prime in similar manner as we have proved for P_2+2
But , if $r_7 = 0$ then,
$(r_2-r_6)=0$
Since r_2 and r_6 are not zero
$\Rightarrow r_2=r_6$

Where r_6 is remainder when P_3 is divided by a
But, $r_2 = r_3$
Where r_3 is remainder when P_1 is divided by a
$\Rightarrow r_3 = r_6$

Now we will repeat above process finite number of times then we will definitely get a prime P_n which when divided by **a** leaves a remainder r_n such that $r_n \neq r_2$
Because if $r_n=r_2$ then it means that on dividing each prime number by arbitrary a we get same remainder but it is not possible.
If it would have been possible then difference between any two prime numbers will be multiple of a but we know that prime numbers do not follow any such law . For example 5,13 and 23 are primes but difference between any two is not divided by arbitrary **a** as
13-5=8
23-13=10
But there exist no arbitrary a which divides both 8 and 10
***Note:* Here a ≠2 as we have taken the case that a do not divides 2(m+1)**

Hence we can say that we can obtain a prime P_n such that $2(m+1) = P_n + P_{n+1}$
And 2(m+1) and P_n when divided by **a** do not leaves same remainder
i.e. $r_2-r_7 \neq 0$ where r_2 is remainder when 2(m+1) is divided by a
And r_n is remainder when P_n is divided by a

$$\frac{2(m+1)}{a} = \frac{P_n}{a} + \frac{P_{n+1}}{a}$$

$$\Rightarrow \frac{sa+r_2}{a} = \frac{xa+r_n}{a} + \frac{P_{n+1}}{a}$$

$$(s-x)a + \frac{r_{2} - r_n}{a} = \frac{P_n+1}{a}$$

Now, there arise only two cases which are as under :

- **(s-x)=0** In this case P_{n+1} will be prime we can prove it in similar manner as we have proved for P_2+2
- **(s-x)≠0** then again there arises two conditions

1. **$0<r_2-r_n<a$**
2. **$-a<r_2-r_n<0$**

In both cases we can prove that P_{n+1} is prime in similar manner as we have proved for P_2+2
[**here (iii) condition $r_2-r_n=0$ do not appears as $r_2 \neq r_n$ also $r_2, r_n \neq 0$)**
Hence we can say that in this condition also 2(m+1) can be expressed as sum of two primes

Combining results of case I and results of all the conditions of case II we can say that in each and every condition we can express 2(m+1) as a sum of two primes
Hence *P (*m+1) is also true
Thus by principle of mathematical induction we can prove that every even integer greater than two can be expressed as sum of two primes