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Visual Conditions of Curvature and it's Applications in Finsler Geometry with Euclidean geometry

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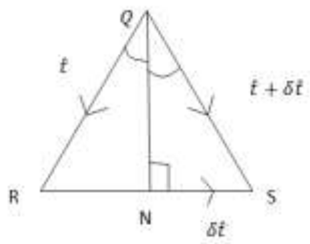
Abstract:- *In the present paper we introduce some Finsler space with Euclidean space and many types of curvature forms. Finally we give some visual conditions of curvature and give the application of Finsler space and metric with curvature in theorem (13.1), (13.2), (13.3)and (13.4).*
Keywords: -*Finsler metrics, Euclidean space, Riemannian space, Randers metric, Kropina metric, Matsumoto metric, E – Curvature, Finsler space.*

I. Introduction

The anions carton in natural elegant manner and corresponding Finsler connections. In the present age the models of Finsler geometry has much importance in applications. Therefore we give some special spaces which are based with their metrics torsion tensors and curvature Tensors.

1.1. Geometrical meaning of curvature:- [9] Let $\hat{t}$and $\hat{t}+\delta\hat{t}$be unit tangent vector at two consective point P and Q respectively and $\delta\Psi$be the angle between these vectors.



L$et\hat{t} = \overrightarrow{QR}, \hat{t}+\delta\hat{t} = \overrightarrow{QS}$ Then $\overrightarrow{RS} = \delta\hat{t}$

Now $RN + NS = QR\sin\frac{\delta\Psi}{2} + QS\sin\frac{\delta\Psi}{2}$

$= 2QR\sin\frac{\delta\Psi}{2}\,(QR = QS)$

$\Rightarrow RS = 2QR\sin\frac{\delta\Psi}{2} \rightarrow |\overrightarrow{RS}| = 2|\overrightarrow{QR}|\sin\frac{\delta\Psi}{2}$

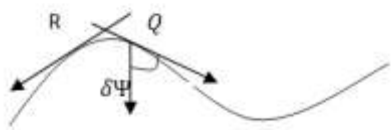
$\Rightarrow |\delta\hat{t}| = 2|\widehat{t}|\left|\sin\frac{\delta\Psi}{2}\right| \Rightarrow \frac{|\delta\hat{t}|}{\delta\Psi} = \frac{\sin\frac{\delta\Psi}{2}}{\delta\Psi}$

$\Rightarrow \left|\frac{\delta\hat{t}}{\delta\Psi}\right| = \frac{\ln\frac{\delta\Psi}{2}}{\frac{\delta\Psi}{2}}\, takingthelimitQ \rightarrow P$

$\Rightarrow \delta\Psi \rightarrow 0\ where \left|\frac{d\hat{t}}{d\Psi}\right| = 1$

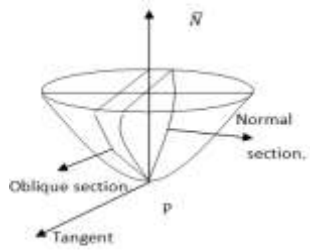
$\Rightarrow \left|\frac{d\hat{t}}{ds}.\frac{ds}{d\Psi}\right| = 1 \Rightarrow \left|\frac{d\hat{t}}{ds}\right|\left|\frac{ds}{s\Psi}\right| = 1$

$\Rightarrow K\left|\frac{ds}{d\Psi}\right| = 1 \Rightarrow K = \left|\frac{d\Psi}{ds}\right| = \frac{d\Psi}{ds}$



Some Important case: -
1.11- A given curve to be a straight lineif$K = 0$.
1.12-A curve to be helix is that if curvature and torsion are in a constant ratio.
1.13-At any point of the surface in conjugate direction the sum radii of normal curvature isconstant.

1.2. Normal Curvature: - curvature of the normal section is called normal curvature denoted by K_n .Its reciprocal is called radius of normal curvature; denoted byρ_n.



Obviously $N = \hat{n} and \frac{d\hat{t}}{ds} = K\hat{n}$
Hence $r'' = K\hat{n} = K_n\overline{N}$

$\Rightarrow N.r'' = K_n\overline{N}\overline{N}$

$\Rightarrow K_n = \overline{N}r''$

In form of fundamental (metric)coefficient ,

$K_n = \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2}$

Where$\underline{r}$ =r (u, v) be the surface, H =$\sqrt{EF - G^2}$, E =$r_1\, r_1$, G =$r_2\, r_2$, F =$r_1\, r_2$

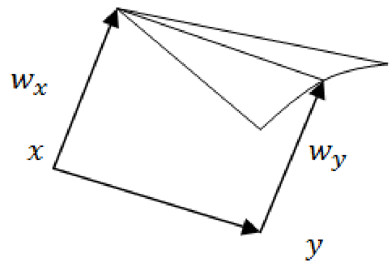
L =$r_{11}\overline{N}$, M =$r_{12}\overline{N}$, N =$r_{22}\overline{N}$ and $\overline{N} = \frac{r_1 \times r_2}{|r_1 \times r_2|} = \frac{r_1 \times r_2}{H}$
Also if K is Normal Curvature at (u, v) in direction$\langle du, dv\rangle$. Then

$$K = L\frac{du^2 + 2Mdudv + Ndv^2}{du^2 + 2Fdudv + Gdv^2}$$

1.3. Geodesic curvature: - The vector$\overline{K_g} = \lambda r_1 + \mu r_2$ is called the curvature vector and its Magnitude is GeodesicCurvature [9].

1.4.Parallel Transport:-we are given two much closed point x and y in a Riemannian manifold is there a way to complexes a tangent vector at x and a tangent vector at $y$even though they live, a priori, in different vector space .This is done by via parallel transport [10]

Visual Introductionto Curvature, Parallel Transport of w_xAlong V.



Let $w_x be$ a tangent vector at$x$, we are looking for a tangent vector w_y at$y$which would be the same as w_x since$x$and y are very closed , we may assume the$y$ is the end point of small tangent vector v at$x$. For simplicity, we will assume the w_y is orthogonal to v and that the norm of w_y is very small then there exist a particular tangent vector w_y at y .It is the one whose end point is closest to the end point of w_x , given the restrictions that w_y be orthogonal to v . The vector $w_y$is the best condition to be the same as w_x **.**
Move generally parallel transport of a vector w along any smooth curve starting at $x$can be define by decomposing the curve into small intervals and performing successive parallel transports along these subintervals.

II:-Curvature:-

The Study of curvature is based on Riemann curvature tensor
(2.1)$R^{\mu}_{\upsilon\alpha\beta} = \frac{\partial\Gamma\upsilon^{\mu}\beta}{\partial x^2} - \frac{\partial\Gamma\upsilon^{\mu}\alpha}{\partial x} + \Gamma^{\mu}_{\rho\alpha}\Gamma^{\rho}_{\upsilon\beta} - \Gamma^{\mu}_{\rho\beta}\Gamma^{\rho}_{\upsilon\alpha}$
is a co-ordinate frame. The covariant component of the Riemann tensor are connected by several symmetries (2.2)

$$\left.\begin{aligned} R_{\alpha\beta\gamma\delta} &= R_{\gamma\delta\alpha\beta} \\ R_{\alpha\beta\gamma\delta} &= -R_{\beta\gamma\delta\alpha} \\ R_{\alpha\beta\gamma\delta} &= -R_{\alpha\beta\delta\gamma} \\ R_{\alpha[\beta\gamma\delta]} &= 0 \end{aligned}\right\}$$

The (symmetric) Ricci tensor and Ricci Scalar are formed from the Riemann tensor (2.3)$R_{\alpha\beta} = R^{\mu} \propto \mu\beta$,$R = R^{\propto} \propto$
Weyl tensor
(2.4)$e_{\lambda\mu\upsilon k} = R_{\lambda\mu\upsilon k} - \frac{1}{2}\left(g_{\lambda\beta} R_{\mu k} - g_{\lambda k} R_{\mu\upsilon} - g_{\mu\upsilon} R_{\lambda k} + g_{\mu k} R_{\lambda\upsilon}\right) + \frac{1}{6}\left(g_{\lambda\upsilon} g_{\mu k} - g_{\lambda k} g_{\mu\upsilon}\right)R$
is also called theconformal tensor due to its invariance under conformal transformation. Its vanishes if and only if the metric is conform ally flat the extrinsic curvature tensor of a hyper surface which has unit normal a and which is spanned by basis vectore_j, e_j ...is denoted by k, with components
(2.5)$K_{ij} = -e_i\nabla_i n$

III. Curvatures of d-connection

The irreducible components of curvature in a space V^{n+m} provided with additional N-connection structure.For space with N-connection, we use write the corresponding formula by using "boldfaced" symbols and changethe usual differential d intro N-adapted operator (3.1)$T^{\alpha} := D\gamma^{\alpha} = \delta\upsilon^{\alpha} + \Gamma^{\gamma}_{\beta} \wedge \gamma^{\beta}$ (3.2)
$R^{\alpha}_{\beta} = D\Gamma^{\alpha}_{\beta} = \delta\Gamma^{\alpha}_{\beta} - \Gamma^{\gamma}_{\beta} \wedge \Gamma^{\alpha}_{\gamma}$

Finsler geometry and generalizations consider very general linear connection $D = D^{[h]}, D^{[v]}$ in space V^{n+m} is defined as an operator (d-tensor field) adapted to N-connection structure
(3.3)$R(x,y)z = D_x D_y z - D_y X_x z - D_{[x,y]} z$

IV. Thethreecurvatures Tensor of Carton:-

(4.1)$S^i_{jkh} = A^i_{kr} A^r_{jh} - A^i_{rh} A^r_{jk}$, is called I[st]Carton Curvature Tensor.
(4.2)$R^i_{jhk} = K^i_{jhk} + C^i_{jm} K^m_{rhk} x^m$, is called third curvature Tensor of Carton.
(4.3)$P^i_{jkh} = F \frac{\partial \Gamma^i_{jk}}{\partial x^k} + A^i_{jm} A^m_{hklr} l^r - A^i_{jhrl}$, is called second curvature Tensor of Carton.

V. Identities Satisfied by the Curvature Tensors:-

As in Riemannian geometry, the structure of the Curvature tensor is such that they satisfy a large number of identities

(5.1)$R^i_{jhk} = R^i_{jkh}$; $K^i_{jhk} = -K^i_{jhk}$

(5.2) $S^i_{jhk} = -S^i_{jkh}$

(5.3) $R^i_{jhk} = -R^i_{jkh}$

(5.4) $S_{ijkh} = g_{rj} S^r_{ikh}$

(5.5) $P_{ijkh} = g_{rj} P^r_{ikh}$

(5.6) $S_{jikh} = -S_{ijhk}$

(5.7) $P_{jihk} = -P_{ijhk}$

Note: -if curvature tensor of a Finsler space F_n (With n>2) satisfies the relation

(5.8) $K^h_{ikj} l^i = R(l_k S^h_j - l_j \delta^h_k)$

VI. The Projection Curvature Tensor

In this section we shall introduce the so-called "projective" transformation or changes which will lead us to various projective curvature tensor. Strictly speaking, this topic forms on integral part of the general geometry of paths for it is in connection with the latter that these concepts wave originally defined: nevertheless we shall see that the projective curvature tensor play an important role in the theory of special classes of Finsler spaces. Thus, while we shall derive the analysis of the first two parts of the section against the back –ground of the general geometry of paths, the reader may easily translatic, This Theory into the terminology of Finslergeometry, regarding the paths as geodesic.

VII:-Higher order curvature gravity in Finsler geometry:-

Finsler geometry is characterized in the first order of the weak approximation by an order parameter μ related to the cartontorsion tensor and desorbing the anisotropy using palatine formalizingflat Finsler space with an $f(R) = R^n$ action and where its variation is taken with respect to independent osculating metric and affine connection , the generalized conservation law and FRW field equation modulo surface terms load if $(1+\beta)\delta = \frac{3\mu}{8}(n \simeq 0.32)$ where $\beta = \frac{-3(n-1)^2}{[2n(n-2)]}$and $\delta = \frac{\mu\beta}{3}$ to the following expression of the luminosity displace as a function of the red shift z
(7.1)$D_L \simeq \frac{G^{1/\alpha}}{\alpha AC}[G_2 F_2 \left(c, 1, c+1, -\frac{B}{A}\right) - 2F_1(c, 1, c = 1 - \frac{B}{A})]$
where,$B = \frac{\mu}{3H}$, $A = 1 - B$, $C = \frac{1+\alpha}{\alpha} G = (1 + 2\mu)$,
$\alpha = 4.76$ And $22F_1(a,b,c,x)$ is the caws hyper geometric formation. [8]

VIII:-Riemann curvature for two dimensional space times:-

For $ds^2 = dv^2 - v^2 dv^2$, the only no vanishing Christoffel symbols are $\Gamma^v_{vu} = v and \Gamma^v_{uv} = \Gamma^u_{vu} = v^{-1}$ so that
(8.1) $R_{vuvu} = \mathrm{R}^v_{uvu} = \Gamma^v_{vu,v} + \Gamma^v_{v\alpha}\Gamma^\alpha_{uv} - \Gamma^v_{u\alpha}\Gamma^\alpha_{uv}$
$= 1 - 0 + 0 - 1 = 0$

Since there is only one independent Riemann Component in two dimensions we conclude that $R_{\alpha\beta\gamma\delta} = 0$ and the space time in flat.

IX. E-Curvature:-

Let F be a Finsler metric on manifold M. The geodesics of F are characterized locally by the equation $\frac{d^2x^i}{dt^2} = 2G^i\left(r.\frac{dx}{dt}\right) = 0$, where$G^i$ are coefficients of spray defined on M denoted by$G(x,y) = y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i}$. A Finsler metric F is called Berwald metric if $G^i = \frac{1}{2}\Gamma^i_{jk}\, y^j y^k$ are quadratic in $y \in T_x$M for any x ∈M.Taking a trace curve of berwald curvature yields mean Berwald curvature on E-curvature. The E –curvature is an important quantity defined using the spray of F it is a kind of non-Riemann quantities [1, 2] Chen and Shen obtained an equivalent condition for a Randers metric to be E-curvature and S-curvature [3]. Then they studied the relationship between isotropic E-curvature and relatively isotropic Landsberg for curvature on a Douglas manifold for [4]. Lungu got a condition for Randers spaces to be simultaneously with scalar flag curvature and with constant E-curvature [5]. For $(\alpha.\beta)$- metrics in the form $F = \alpha + \varepsilon\beta + \frac{x^2}{\beta}k$ D. Tang obtained an equivalent condition about E-curvature and S-curvature [6].Tayebi,Nankali and Peyghan proved that every m-root Carton space of E-curvature reduce to weakly Berwald spaces [7].

X. Finsler Space:-

Finsler geometry is a kind of differential geometry, which was originated by P. Finslerian 1918. It is usually considered as a generalization of Riemannian geometry. The definition of Finsler space is -Suppose that we are given a function L (x^i, y^i) of the line element (x^i, y^i) of a curve defined in R. We shall assume L as a function of class at least C^5 in all its 2n-arguments. If we define the infinitesimal distance ds between two points $P(x^i)$ and Q $(x^i + dx^i)$ of R by the relation

$$ds = L(x^i, dx^i)$$

then the manifold M^n requipped with the fundamental function L defining the Metric is called a Finsler space .if L (x^i, dx^i) satisfies the following conditions.

Condition A-The function L (x^i, y^i) is positively homogeneous of degree one in y^i i.e.
$L(x^i, ky^i) = kL(x^i, y^i)$, k > 0
Condition B-The function L (x^i, y^i) is positively if not all y^i vanish simultaneously i.e.

$$L(x^i, y^i) > 0 \; with \sum_i (y^i)^2 \neq 0$$

Condition C-The quadratic form

$$\dot{\partial}_j L^2(x,y)\varepsilon^i\varepsilon^j = \frac{\partial^2 L^2(x,y)}{\partial y^i \partial y^j}\varepsilon^i\varepsilon^j$$

is assumed to be positive definite for any variable ε^i.
Form Euler's theorem on homogenous functions, we have$\dot{\partial}_i L(x,y)\dot{y}^i = L\,(x,y)$

$$\dot{\partial}_i\dot{\partial}_j L^2(x,y)y^i = 0$$

We put $g_{ij}(x,y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2(x,y)$
Using the theory of quadratic form and the condition C, we deduce that-

$$g(x,y) = g_{ij}(x,y) > 0$$

For all line elements , if the function L is of particular form

$$L(x^i, dx^i) = [g_{ij}(x^k)dx^i dx^j]^{1/2}$$

Where the coefficients $g_{ij}(x^k)$ are independent ofdx^i, the metric defined by this function is called Riemannian metric and manifold M^n is called a Riemannian space. Throughout the paper, $F^n\, or\, (M^n, L)$ will denote the n-dimensional finsler space, where as n-dimensional Riemannian space will be denoted byR^n.

XII:-Some Special Form of Finsler Space:-

(11.1):- Berwald space. If the connection coefficientG^i_{jk} .of the Berwald's connection $B\Gamma$given by

$$G^i_{jk} = \partial_j G^i_k$$

are functions of position alone, the space is called a Berwald space.
(11.2):-Landsberg space. A Finsler space is called a Landsberg space if the Berwaldconnection BΓ is h-metrical i.e. $gij(k) = 0.$

(11.3):-C-reducible Finsler space.A Finsler space of dimension n, more than two, is called C-reducible if C_{ijk} is written in the form

$$C_{ijk} = \frac{1}{n+1}\left(h_{ij} C_k + h_{jk} C_i + h_{ki} C_j\right)$$

where$C_{ijk} = C_{ijk}\, g^{jk}$ is the torsion vector and h_{ij} is the angular metric tensor given by$h_{ij} = g_{ij} - l_i l_j$

(11.4) - Semi C-reducible Finsler space. A Finsler space of dimension n, more than two, is called semi C-reducible if C_{ijk} is written in the form

$$C_{ijk} = \frac{p}{n+1}\left(h_{ij} C_k + h_{jk} C_i + h_{ki} C_j\right) + \frac{q}{C^2} C_i C_j C_k$$

where$C^2 = g^{ij} C_i C_j$ and$p + q = 1$.

(11.5):-P-reducible Finsler space. A Finsler space of dimension n, more than two, is called P-reducible if (v) hv torsion tensor P_{ijk} of $C\Gamma$is written in the form

$$P_{ijk} = \frac{1}{n+1}\left(h_{ij} C_{k0} + h_{jk} C_{i0} + h_{ki} C_{j0}\right)$$

(11.6):-C2-like Finsler space. A Finsler space is called C2-like Finsler space if

$$C_{ijk} = \frac{1}{C^2} C_i C_j C_k$$

(11.7):-C3-like Finsler space. A Finsler space is called C3-like Finsler space if

$$C_{ijk} = S_{(ijk)}\left\{h_{ij}\, a_k + C_i C_j\, b_k\right\},$$

wherea_k and $b_k$are components of arbitrary indicatory tensors.

(11.8):-S3-like Finsler space. A Finsler space F^n with fundamental function $L(x, y)$ is calledS_3-like Finsler space if v-curvature tensor$S_{\hbar ijk}$ of CΓ is written in the form $L^2 S_{hijk} = S\left(h_{hj} h_{ik} - h_{hk} h_{ij}\right)$.

(11.9):-S4-like Finsler space. A Finsler space F^n is called S4-like Finsler space if v-curvature tensor $S_{\hbar ijk}$ of CΓis written in the form

$$S_{hijk} = h_{hj} M_{ik} + M_{hj} h_{ik} - h_{hk} M_{ij} - M_{ij} - h_{ij}$$

whereM_{ij} are components of a symmetric covariant tensor of second order and are(-2) p-homogeneous in y^i satisfying $M_{0j} = 0$.

(11.10):-R3-like Finsler space. A Finsler space of dimension more than three, is called R3-like Finslerspace if h-curvature tensor R_{hijk} ofCΓ written in the form

$R_{hijk} = g_{hj} L_{ik} + L_{hj}\, g_{ik} - g_{hk} L_{ij} - L_{hk}\, g_{ij}$.

whereL_{ij} are components of a covariant tensor of second order.

(11.11):-Finsler space of scalar curvature. A Finslerspace of scalar curvature K is characterized by

$R_{i\,0\,j} = KL^2 h_{ij}$, where R_{ijk} are components of (v) h'-torsion tensor of CΓ.

(11.12):-Finsler space with $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ metric.A Finsler metric $L(x, y)$ is called an (α, β)metric, when L is positively homogeneous function$L(\alpha, \beta)$ of first degree in two variables $\alpha(x, y) = \left\{a_{ij}(x) x^i y^j\right\}^{1/2}$ and $\beta(x, y) = b_i(x) y^i$

XII.Various special Finsler spaces with special $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ metric have their particular names which are given below.

(12.1):-Randers space. The$(\alpha, \beta) -$ metric $L = \alpha + \beta$is called a Randers metric and Finsler space equipped with this metric is called a Randers space.

(12.2):- Kropinaspace. The$(\alpha, \beta) -$ metric $L = \alpha^2/\beta$ iscalled a Kropina metric and Finsler space equipped with this metric is called a Kropina space.

(12.3):-Generalized m- Kropinaspace. The$(\alpha, \beta) -$ metric, $L = \alpha^{m+1}\beta^{-m}\,(m \neq 0, -1)$is called a generalized m-Kropina metricandFinsler space equipped with this metric is called a generalized m-Kropina space.

(12.4):- Matsumoto space. An n-dimensional $(\alpha, \beta) - metric L = \frac{\alpha^2}{\alpha - \beta}$is called a slope metric or Matsumoto metric and a Finsler space equipped with this metric is called a Matsumoto space.

(12.5):-Finsler space with cubic metric. A Finsler metric $L(x,y)$ is called a cubic metric when $L(x,y) = \left(a_{ijk}(x)y^i y^j y^k\right)^{1/3}$ and .the space with such metric is called Finsler space with cubic metric.

(12.6):-Finsler space with m^{th} root metric. A Finsler metric L(x, y) is called a m^{th}root metric if $L(x,y) = \left(a_{i_1 i_2 \ldots i_m}(x)y^{i_1}y^{i_2} - - - y^{i_m}\right)^{1/m}$ and the space with such metric is called Finsler space with m^{th} root metric.

(12.7):-Finsler space with one form metric. A Finsler metric L(x, y) is called one form metric if L is positively homogeneous function $L(a^{\alpha})$where $a^{\alpha} = a_i^{\alpha} y^i$linearly independent differential 1-form and the space are is called the one-form Finsler space.

XIII. Application of Curvature:-

1. If we choose a co-ordinate system geared to the surface .Let x^1, x^2vary and x^3, x^4be constant on the surface .The only contra variant components of C and D are then the 1 and 2 components. Weare interested then only in the Riemann components with indices 1 and 2, but there is only one such independent component, the non-vanishing Riemann Components of this type are related by symmetries which can be expressed as

$$R_{ijkl} \propto \left(g_{ik}g_{il} - g_{il}g_{jk}\right)i,j,k,l = 1,2,$$

It follows immediately that in this coordinate system K is Independent of C and D .Since K is clearly co-ordinate independent the desired result is proved.

Theorem 13.1:- Let C and D be two linearly independent vectors tangent at appoint to a two dimensional surface, in a space of dimension≥ 2. The Riemannian curvature of the 2-surface at the point is defined as

$$K = \frac{R_{\alpha\beta\gamma\delta}\, C^{\alpha}C^{\beta}D^{\alpha}D^{\beta}}{(a_{\alpha\beta}\,g_{\gamma\delta} - g_{\alpha\delta}\,g_{\beta\gamma})C^{\alpha}C^{\beta}D^{\alpha}D^{\beta}}$$

Show that K is unchanged if C and D are replaced by linear combinational of C and D.

2. Since the metric is always covariantly constant

$$R_{\alpha\beta\gamma\delta\,;\lambda} = K_{\lambda}\left(g_{\alpha\gamma}\,g_{\beta\delta} - g_{\alpha\delta}\,g_{\beta\gamma}\right)$$

Now substituting in the Bianchi identities

$$0 = R_{\alpha\beta\gamma\delta\,;\lambda} + R_{\alpha\beta\lambda\gamma\,;\delta} + R_{\alpha\beta\delta\lambda\,;\gamma}$$

and contract on α, βandγ, δ to find $K_{,\lambda} = 0$ i.e. K is Constant.

Theorem 13.2:- If the Riemann curvature is isotropic, the Riemann curvature tensor can be written as

$$R_{\alpha\beta\gamma\delta} = K(g_{\alpha\gamma}\,g_{\beta\delta} - g_{\alpha\delta}\,g_{\beta\gamma})$$

Use the Bianchi identities to show (Schurz's theorem) that K must be a constant.

3. The metric has form

$$ds^2 = -dr^2 + g_{ij}\,gx^i gx^j$$

where$g_{ij} = a^2(r)\lambda_{ij}(x^k)$.

The normal vector to the r=constant surfaces is$\nabla_i = \frac{\partial}{\partial x}$, Thus

$K_{ij} = -e_j.\nabla_i n = n.\nabla_i e_j = \Gamma_{nji} = -\frac{1}{2}g_{ij,n} = \frac{-a_{,r}}{a}g_{ij}$.

Theorem(13.3):- What is the extrinsic curvature of the r = constant slice of the metric,

$$ds^2 = -dr^2 + a^2(r)\left[y_{ij}(x^k)dx^i dx^j\right]$$

4. Since there is no favored direction in the spherical surface the extrinsic curvature tensor must be $K_{ij} \propto \delta_{ij}$is an orthonormal basis; this implies that any vector is an eigenvector. From the definition of K as the rate of change of n, the eigenvector must be $-$

$$\frac{1}{sphere's\ radius} = -\frac{1}{a}\ .$$

We can give the mathematical development of these intuitive results by going to the usual spherical co-ordinates r, θ, ϕ. These are clearly Gaussian normal co-ordinates and

$$ds^2 = g_{ik}dx^i dx^k = r^2(d\theta^2 + sin^2\theta d\phi^2)$$

Now$K_{ij} = -\frac{1}{2}g_{ij,n} = -\frac{1}{2}g_{ij,r}\ so$

$$K_{\hat{\theta}\hat{\theta}} = \frac{1}{g_{\theta\theta}} K_{\theta\theta} = r^{\frac{1}{2}}\left(-\frac{1}{2}r^2\right), r = -\frac{1}{a}$$
$$K_{\hat{\phi}\hat{\phi}} = \frac{1}{g_{\phi\phi}} K_{\phi\phi} = \frac{1}{r^2 sin^2\theta}\left(-\frac{1}{2}r^2 sin^2\theta\right), r = -\frac{1}{a}.$$

Theorem (13.4):- The Eigen values and Eigen vectors of the extrinsic curvature tensor are called curvature and directions for the following surfaces embedded in a 3-dimensions Euclidean space

$$sphere: \quad x^2 + y^2 + z^2 = a^{2.}$$

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