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On the Diophantine equation $5(x^2+y^2)-9xy=35z^2$

M.A.Gopalan[1], Nirmala Rebecca Paul[2], P. Suganya[3]

[1]*Professor, Department of Mathematics, Shrimati Inddira Gandhi College, Trichy, Tamil Nadu, India*
[2]*Department of Mathematics, Lady Doak College, Madurai, Tamil Nadu, India*
[3]*Department of Mathematics, Lady Doak College, Madurai, Tamil Nadu, India*

Abstract: *The ternary quadratic Diophantine equation* $5(x^2+y^2)-9xy=35z^2$ *representing cone is analyzed for its non-zero distinct integer points on it.*

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I. Introduction

The ternary quadratic Diophantine equations offer an unlimited field for research by reason of their variety [1,2,3]. In particular, one may refer [4-12] for finding points in integers on some specific three dimensional surfaces. This communication concern with yet another ternary quadratic Diophantine equation $5(x^2+y^2)-9xy=35z^2$ representing cone for determining its infinitely many integer solutions.

II. Method of Analysis

Consider the equation

$$5(x^2+y^2)-9xy=35z^2 \qquad (1)$$

The transformed equation of (1) after using the linear transformations

$$x=u+v, y=u-v(u\neq v\neq 0) \qquad (2)$$

is $u^2+19v^2=35z^2$ (3)

The above equation is solved through different methods and employing (2), different sets of distinct integer solutions to (1) are obtained which are illustrated below:

Method: 1

Write 35 as $35=(4+i\sqrt{19})(4-i\sqrt{19})$ (4)

Assume $z=a^2+19b^2$ (5)

where a and b are non zero distinct integers

Using (4) & (5) in (3) and employing the method of factorization, define

$$u+i\sqrt{19}v=(4+i\sqrt{19})(a+i\sqrt{19}b)^2$$

from which, on equating the real and imaginary parts

$$u=4(a^2-19b^2)-38ab$$

$$v=(a^2-19b^2)+8ab$$

Substituting the above values of u and v in (2), the values of x and y are given by

$$x=5(a^2-19b^2)-30ab \qquad (6)$$

$$y=3(a^2-19b^2)-46ab \qquad (7)$$

Thus, (5), (6) and (7) represent non zero distinct integer solutions to (1) in two parameters.

Note: In addition to (4), one may write 35 as $35=\frac{(11+i\sqrt{19})(11-i\sqrt{19})}{4}$

For this choice, the corresponding integer solutions to (1) are given by

$$x=6(a^2-19b^2)-8ab$$

$$y=5(a^2-19b^2)-30ab$$

$$z=a^2+19b^2$$

Method: 2

Consider (3) as $u^2-16z^2=19(z^2-v^2)$ (8)

Write (8) in the form of ratio as

$$\frac{u+4z}{z-v}=\frac{19(z+v)}{u-4z}=\frac{a}{b}, b>0$$

Which is equivalent to the system of double equations

$$(a-4b)z-av-bu=0$$

$$(-4a-19b)z-19bv+au=0$$

Applying the method of cross multiplication to the above equations, we have

$$u=4(a^2-19b^2)+38ab$$

$$v=(a^2-19b^2)-8ab$$

$$z=a^2+19b^2 \quad (9)$$

Substituting the above values of u and v in (2), the values of x and y are given by

$$x=5(a^2-19b^2)+30ab$$

$$y=3(a^2-19b^2)+46ab \quad (10)$$

Thus, (9) and (10) represent non zero distinct integer solutions to (1) in two parameters.

Note: (8) can also be expressed in the form of ratio in three different ways as follows:

(i) $$\frac{u+4z}{19(z-v)}=\frac{(z+v)}{u-4z}=\frac{a}{b}, b>0$$

(ii) $$\frac{u+4z}{19(z+v)}=\frac{(z-v)}{u-4z}=\frac{a}{b}, b>0$$

(iii) $$\frac{u+4z}{z+v}=\frac{19(z-v)}{u-4z}=\frac{a}{b}, b>0$$

Repeating the analysis as above, we get three different sets of integer solutions to (1) and they are presented below:

Solutions of (i):

$$x=95a^2-5b^2+30ab$$

$$y=57a^2-3b^2+46ab$$

$$z=19a^2+b^2$$

Solutions of (ii):

$$x=-57a^2+3b^2-46ab$$

$$y=-95a^2+5b^2-30ab$$

$$z=-a^2-19b^2$$

Solutions of (iii):

$$x=-3a^2+57b^2-46ab$$

$$y=-5a^2+95b^2-30ab$$

$$z=-a^2-19b^2$$

Method: 3

Write (3) as $19v^2=35z^2-u^2$ (11)

Write 19 as $19=(\sqrt{35}+4)(\sqrt{35}-4)$ (12)

Assume $v=35a^2-b^2$ (13)

Where a and b are non zero distinct integers

Using (12) & (13) in (11) and employing the method of factorization, define

$$\sqrt{35}z+u=(\sqrt{35}+4)(\sqrt{35}a+b)^2$$

Equating the rational and irrational parts, we get

$$u=4(35a^2+b^2)+70ab$$
$$z=(35a^2+b^2)+8ab \qquad (14)$$

Substituting the above values of u and v in (2), the values of x and y are obtained as

$$x=175a^2+3b^2+70ab$$
$$y=105a^2+5b^2+70ab \qquad (15)$$

Thus, (14) and (15) represent the integer solutions of (1).

Method: 3

Introducing the linear transformations

$$z=\alpha\pm19\beta, v=\alpha\pm35\beta, u=4U \qquad (16)$$

in (3), it leads to $\alpha^2=U^2+665\beta^2$ (17)

which is satisfied by $\beta=2pq, U=665p^2-q^2, \alpha=665p^2+q^2$

Substituting the above values of α,β,U in (16) and (2), the corresponding non-zero integer solutions to (1) are given by

$$x=3325p^2-3q^2\pm70pq$$
$$y=1995p^2-5q^2\mp70pq$$
$$z=665p^2+q^2\pm38pq$$

It is worth to mention here that, (17) may be expressed as the system of double equations as shown in the table below:

Table 1: system of equations

system	1	2	3	4	5	6	7	8	9	10	11	12
$\alpha+U$	β^2	$5\beta^2$	$7\beta^2$	$19\beta^2$	$35\beta^2$	$95\beta^2$	$133\beta^2$	$665\beta^2$	35β	95β	133β	665β
$\alpha-U$	665	133	95	35	19	7	5	1	19β	7β	5β	β

Solving each of the above system for α,β,U and using (16) and (2), the corresponding non-zero integer solutions satisfying (1) are exhibited in the table below:

Table 2: integer solutions

system	x	y	z
1	$10k^2+80k-960$ $10k^2-60k-1030$	$6k^2-64k-1696$ $6k^2+76k-1626$	$2k^2+40k+352$ $2k^2-36k+314$
2	$50k^2+120k-152$ $50k^2-20k-222$	$30k^2-40k-360$ $30k^2+100k-290$	$10k^2+48k+88$ $10k^2-28k+50$
3	$70k^2+140k-90$ $70k^2-160$	$42k^2-28k-262$ $42k^2+112k-192$	$14k^2+52k+70$ $14k^2-24k+32$
4	$190k^2+260k+30$ $190k^2+120k-40$	$114k^2+44k-94$ $114k^2+184k-24$	$38k^2+76k+46$ $38k^2+8$
5	$350k^2+420k+30$ $350k^2+280k-40$	$210k^2+140k-94$ $210k^2+280k-24$	$70k^2+108k+46$ $70k^2+32k+8$
6	$950k^2+1020k+262$ $950k^2+880+192$	$570k^2+500k+90$ $570k^2+640k+160$	$190k^2+228k+70$ $190k^2+152k+32$

7	$1330k^2+1400k+360$ $1330k^2+1260k+290$	$798k^2+728k+152$ $798k^2+868k+222$	$266k^2+304k+88$ $266k^2+228k+50$
8	$6650k^2+6720k+1696$ $6650k^2+6580k+1626$	$3990k^2+3920k+960$ $3990k^2+4060k+1030$	$1330k^2+1368k+352$ $1330k^2+1292k+314$
9	$94\beta, 24\beta$	$-30\beta, 40\beta$	$46\beta, 8\beta$
10	$262\beta, 292\beta$	$90\beta, 160\beta$	$70\beta, 32\beta$
11	$360\beta, 290\beta$	$152\beta, 222\beta$	$88\beta, 50\beta$
12	$1696\beta, 1626\beta$	$960\beta, 1030\beta$	$352\beta, 314\beta$

Method: 5

Consider (3) as $u^2+19v^2=35z^2*1$ (18)

Write 1 as $1=\frac{(5+i3\sqrt{19})(5-i3\sqrt{19})}{14^2}$ (19)

Using (4) , (5) and (19) in (18) and employing the method of factorization, define

$$u+i\sqrt{19}v=(4+i\sqrt{19})(a+i\sqrt{19}b)^2\frac{(5+i3\sqrt{19})}{14}$$

Equating the real and imaginary parts, we have

$$u=\frac{1}{14}[-37(a^2-19b^2)-646ab]$$

$$v=\frac{1}{14}[17(a^2-19b^2)-74ab]$$

Substituting the above values of u and v in (2), the values of x and y are given by

$$x=\frac{1}{7}[10(a^2-19b^2)+360ab]$$

$$y=\frac{1}{7}[27(a^2-19b^2)+286ab] \quad (20)$$

Replacing a by 7A and b by 7B in (20) and (5), the corresponding non-zero integer solutions to (1) are given by

$$x=-[70(A^2-19B^2)+2520AB]$$

$$y=-[189(A^2-19B^2)+2002AB]$$

$$z=49(A^2+19B^2)$$

Note: In addition to (19), one may write 1 as $1=\frac{(3+i5\sqrt{19})(3-i5\sqrt{19})}{484}$

For this choice, a different set of solutions to (1) are obtained.

III. Generation of solutions

Illustration 1:

Let (x_0, y_0, z_0) be the given initial solution of (1). $x_1=10x_0-3h, y_1=10y_0, z_1=10z_0+h$ (21)

be the second solution of (1) where h is any non-zero integer to be determined.

Substituting (21) in (1) and simplifying, we have $h=30x_0-27y_0+70z_0$

Therefore, the second solution (x_1, y_1, z_1) of (1) expressed in the matrix form is

$$(x_1, y_1, z_1)^t = M(x_0, y_0, z_0)^t \text{ Where} = M=\begin{pmatrix} -80 & 81 & -210 \\ 0 & 10 & 0 \\ 30 & -27 & 80 \end{pmatrix}$$

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The repletion of the above process leads to the general solution of (1) represented as follows:

$$(x_{2n-1}, y_{2n-1}, z_{2n-1})^t = 10^{2(n-1)} M(x_0, y_0, z_0)^t$$

$$(x_{2n}, y_{2n}, z_{2n})^t = 10^{2n} M(x_0, y_0, z_0)^t$$

Illustration 2:

Let (u_0, v_0, z_0) be the given initial solution of (3).

Let $u_1 = 6h - u_0$, $v_1 = v_0$, $z_1 = z_0 + h$ (22)

be the second solution of (3) where h is any non-zero integer to be determined.

Substituting (22) in (3) and simplifying, we get $h = 12u_0 + 70z_0$

Therefore, the second solution (x_1, y_1, z_1) of (3) expressed in the matrix form is

$$(u_1, z_1)^t = M(u_0, z_0)^t \ , \ v_1 = v_0 \ \text{Where=} M = \begin{pmatrix} 71 & 420 \\ 12 & 71 \end{pmatrix}$$

Repeating the above process, we have, in general

$$(u_n, z_n)^t = M^n (u_0, z_0)^t \ , \ v_n = v_0$$

It is known that $M^n = \frac{\alpha^n}{\alpha - \beta}(M - \beta I) + \frac{\beta^n}{\beta - \alpha}(M - \alpha I)$

where α, β are the Eigen values of M and I is a 2x2 unit matrix. For our problem, we have, after simplification,

$$M^n = \begin{pmatrix} \frac{\alpha^n + \beta^n}{2} & \frac{\sqrt{35}(\alpha^n - \beta^n)}{2} \\ \left(\frac{\alpha^n - \beta^n}{2\sqrt{35}}\right) & \frac{\alpha^n + \beta^n}{2} \end{pmatrix}$$

in which α, β are the Eigen values of M given by $\alpha = 71 + 12\sqrt{35}, \beta = 71 - 12\sqrt{35}$

In view of (2), the general solution (x_n, y_n, z_n) of (1) is given by

$$x_n = \left(\frac{\alpha^n + \beta^n}{2}\right) u_0 + \frac{\sqrt{35}}{2}(\alpha^n - \beta^n) z_0 + v_0$$

$$y_n = \left(\frac{\alpha^n + \beta^n}{2}\right) u_0 + \frac{\sqrt{35}}{2}(\alpha^n - \beta^n) z_0 - v_0$$

$$z_n = \frac{1}{2\sqrt{35}}(\alpha^n - \beta^n) u_0 + \frac{1}{2}(\alpha^n + \beta^n) z_0$$

Illustration3:

Let $u_1 = 8u_0$, $v_1 = 8v_0 + h$, $z_1 = h - 8z_0$ be the second solution of (3).

Following the analysis presented above, the corresponding integer solutions to (1) are given by

$$x_n = 8^n u_0 + \left(\frac{\alpha^n + \beta^n}{2}\right) v_0 + \frac{\sqrt{35}}{2\sqrt{19}}(\alpha^n - \beta^n) z_0$$

$$y_n = 8^n u_0 - \left(\frac{\alpha^n + \beta^n}{2}\right) v_0 - \frac{\sqrt{35}}{2\sqrt{19}}(\alpha^n - \beta^n) z_0$$

$$z_n = \frac{\sqrt{19}}{2\sqrt{35}}(\alpha^n - \beta^n)v_0 + \frac{1}{2}(\alpha^n + \beta^n)z_0$$

where $\alpha = 27+\sqrt{665}, \beta = 27-\sqrt{665}$

IV. Conclusion

To conclude, one may search for other patterns of general solutions to ternary quadratic Diophantine equation in the title and obtain their corresponding properties.

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