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Markov Operator: Applications to Iterated Function Systems of Generalized Cantor Sets

Md. Jahurul Islam[1], Md. Shahidul Islam[2]
[1,2]*Department of Mathematics, University of Dhaka, Dhaka, Bangladesh*

Abstract : *In this paper, we discuss iterated function systems with probabilities of generalized Cantor sets (IFSGCS) and show that these functions are non-expansiveness and asymptotically stable if the Markov operator has the corresponding property.*
Keywords - *Cantor set, Markov operator, iterated function system, non-expansiveness, asymptotically stable.*
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I. Introduction

A Fractal, as defined by B. Mandelbrot, "is a shape made of parts similar to the whole in some way" [2]. Fractal is a geometric object that possesses the two properties: self-similar and non-integer dimensions. So a fractal is an object or quantity which displays self-similarity. The Cantor set is the prototypical fractal [1]. Cantor sets were discovered by the German Mathematician George Cantor in the late 19th to early 20th centuries (1845-1918). He introduced fractal which has come to be known as the Cantor set, or Cantor dust.
We studied Cantor set and found generalized Cantor sets and show its dynamical behaviors and fractal dimensions [3]. Then we studied generalized Cantor sets in measure space and found that these special types of sets are Borel set as well as Borel measurable whose Lebesgue measure is zero [4]. Also we find the iterated function systems with probabilities of generalized Cantor sets and show their invariant measures [5].
If $\{w_1, w_2, \ldots, w_N\}$ is a finite family of strict contracting transformations, we may consider the Barnsley-Hutchinson multiplication [6, 7] given by the formula

$$F(A) = \bigcup_{i=1}^{N} w_i(A). \tag{1.1}$$

Fractals are strongly related to Markov operator acting on the space of all Borel measures [8]. If $w_k : X \to X$ are continuous transformations and $p_k : X \to [0,1]$ are continuous such that $\sum_{k=1}^{N} p_k(x) = 1,$ then we may define the Markov operator

$$P\mu(A) = \sum_{k=1}^{N} \int_{w_k^{-1}(A)} p_k(x)\, \mu(dx) \text{ for } \mu \in \ . \tag{1.2}$$

where denotes the family of all probability Borel measure X into Borel measure on X.

In this paper, we study the iterated function systems of generalized Cantor sets using Markov operator. We define the transition operator P_w for the iterated function systems of generalized Cantor sets, which is a Markov operator. We show that the iterated function systems of generalized Cantor sets is asymptotically stable if the Markov operator P_w is asymptotically stable and also is non-expansive if the Markov operator P_w is non-expansive with respect to the metric $\varphi(\rho(x, y))$. Our proof is based on the result of Lasota and Yorke [9].
The organization of the paper is as follows. Section 2 contains some definitions and notation from the theory of Markov operators acting on the space of measure. In Section 3 we discuss the iterated function systems of generalized Cantor sets with probabilities and prove the transition operator P_w is a Markov operator. In Section 4 we show that the iterated function systems of generalized Cantor sets are non-expansiveness and asymptotically stable if the Markov operator P_w has the corresponding property.

II. Notation and Preliminaries

Definition 2.1. A non empty set $\Gamma \subset \mathbf{R}$ is called a Cantor set if
(a) Γ is closed and bounded. (b) Γ contains no intervals.
(c) Every point in Γ is an accumulation point of Γ.

Definition 2.2. A measure μ defined on a σ-algebra of subsets of a set X is called finite if $\mu(X)$ is a finite real number (rather than ∞). The measure μ is called σ-finite if X is the countable union of measurable sets with finite measure. A set in a measure space is said to have σ-finite measure if it is a countable union of sets with finite measure.

Definition 2.3. A Borel set is any set in topological space that can be formed from open sets (or, equivalently, from closed sets) through the operations of countable union, countable intersection and relative complement. Borel sets are named after Emile Borel.

Definition 2.4. Let X be a locally compact Hausdorff space, let Σ be the smallest σ-algebra that contains the open sets (or, equivalently, the closed sets) of X; this is known as the σ-algebra of Borel sets. Any measure μ defined on the σ-algebra of Borel sets is called a Borel measure.

Definition 2.5. Let (X,ρ) be a metric space. A function $f: X \to X$ is a contraction mapping, or contraction on (X,ρ), with the property that there is some nonnegative real number $0 \le \beta < 1$ such that for all *x* and *y* in X,

$$\rho(f(x), f(y)) \le \beta\, \rho(x,y).$$

The smallest such value of β is called the Lipschitz constant of f. The Contractive maps are sometimes called Lipschitzian maps. If the above condition is instead satisfied for $\beta \le 1$, then the mapping is said to be a non-expansive map.

Let (X,ρ) be a separable complete metric space. We assume that every closed ball in X

$$B(r,x) = \{y \in X : \rho(x,y) \le r\}$$

is a compact set. We denote by $\mathbf{B}(X)$ the σ-algebra of Borel subsets of X. By we denote the family of Borel measure (nonnegative, σ-additive) on X such that $\mu(B) < \infty$ for every ball B. By $_1$ we denote the subsets of such that $\mu(X) = 1$ for $\mu \in {}_1$. The elements of $_1$ will be distributions. Further by $C(X)$ we denote the space of bounded continuous functions $F: X \to \mathbf{R}$ with the supremum norm. As usual we denote by $C_0(X)$ the subspace $C(X)$ of which contains functions with compact supports. The indicator function of a set $A \subset X$ will be a denoted by 1_A.

A linear functional $\varphi: C_0 \to \mathbf{R}$ is called positive if $\varphi(f) \ge 0$ for $f \ge 0$. According to the Riesz theorem for every linear positive functional $\varphi: C_0 \to \mathbf{R}$ there is a unique measure $\mu \in$ such that

$$\varphi(f) = \int_X f\, d\mu =: < f, \mu > \text{ for } f \in C_0.$$

An operator $P:$ T will be called a Markov operator if it satisfies the following two conditions.
(i) Positive linearity: $P(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 P\mu_1 + \lambda_2 P\mu_2$ for $\lambda_1, \lambda_2 \ge 0; \mu_1, \mu_2 \in$
(ii) Preservation of the norm: $P\mu(X) = \mu(X)$ for $\mu \in$.

A Markov operator P is called a Feller operator if there is a linear operator $U: C_0(X) \to C(X)$ (dual to P) such that

$$< Uf, \mu > = < f, P\mu > \quad \text{for } f \in C_0,\ \mu \in \ . \tag{2.1}$$

Observe that the range of the operator U is contained in $C(X)$ but not necessarily in $C_0(X)$. We may extend U to all bounded measurable (or nonnegative measurable) function by setting

$$Uf(x) = < Uf, \delta_x > = < f, P\delta_x > \tag{2.2}$$

where $\delta_x \in {}_1$ is a point (Dirac) measure supported at x. For $f \ge 0$ the function Uf is nonnegative but may be unbounded or even admit infinite values for unbounded f.

Every Markov operator P can be easily extended to the space of signed measures

$$_{sig} = \{\mu_1 - \mu_2 : \mu_1, \mu_2 \in \quad \}$$

Namely for every $\nu \in \quad _{sig}$ we define

$$P\nu = P\mu_1 - P\mu_2 \text{ where } \nu = \mu_1 - \mu_2 : \mu_1, \mu_2 \in \quad .$$

It is easy to verify that this definition of $P\nu$ does not depend on the choice of $\mu_1, \mu_2 \in \quad$. In the space $\quad _{sig}$ we define the Fortet–Mourier norm

$$\| \mu \| = \sup_F \{ | < f, \nu > | : f \in F \}, \tag{2.3}$$

where $< f, \mu > = \int_X f(x)\mu(dx)$ and $F = \{ f \in C(X) \}$ is the set of all f such that $\| f \|_c \leq 1$ and $| f(x) - f(y) | \leq \rho(x, y)$ for $x, y \in X.$

It is easy to verify that the value (2.3) will not change if F is replaced by $F_0 = F \cap C_0.$ For $\mu \in \quad$, we have $\| \mu \| = \mu(X).$ The space $\quad _1$ with the distance $\| \mu_1 - \mu_2 \|$ is a complete metric space and the convergence

$$\lim_{n \to \infty} \| \mu_n - \mu \| = 0 \quad \text{for } \mu_n, \mu \in \quad _1$$

is equivalent to the condition

$$\lim_{n \to \infty} < f, \mu_n > = < f, \mu > \tag{2.4}$$

for all $f \in C(X),$ or equivalently for all $f \in C_0(X).$

Let P be a Markov operator. A measure $\mu \in \quad$ is called stationary or invariant if $P\mu = \mu.$ A Markov operator P is called asymptotically stable if there exists a stationary distribution μ_* such that

$$\lim_{n \to \infty} \| P^n \mu - \mu_* \| = 0 \text{ for } \mu \in \quad _1. \tag{2.5}$$

Clearly a distribution μ_* satisfying (2.5) is unique. However, in general, condition (2.5) does not imply that μ_* is stationary. This implication holds for Feller operators.

A Markov operator P is called non-expansive if

$$\| P\mu_1 - P\mu_2 \| = \| \mu_1 - \mu_2 \| \text{ for } \mu_1, \mu_2 \in \quad _1. \tag{2.6}$$

Lemma 2.1. Let P be a Feller operator. Assume that there exists a linear positive functional $\varphi : C(X) \to \mathbf{R}$ such that $\varphi(1_X) = 1_X$ and

$$\varphi(U(h)) = \varphi(h) \quad \text{for } h \in C_0(X)$$

where U is dual to $P.$ Further let $\mu_* \in \quad$ be the unique (Riesz theorem) measure satisfying $\varphi(h) = < h, \mu_* >$ for $h \in C_0(X).$ Then $\mu_* \in \quad$ and $P\mu_* = \mu_*.$

Proof: The proof can be found in [9].

Lemma 2.2. A non-expansive Markov operator is a Feller operator.

Proof: The proof can be found in [9].

III. Iterated Function Systems of Generalized Cantor Sets with Probabilities

Let (X, ρ) be a complete separable metric space. An iterated function system is given by a family of contracting transformations

$$S_i : X \to X, i \in I \quad \text{where the index set } I \text{ is finite.}$$

If, in addition, there is given a family of continuous functions

$$p_i : X \to [0,1], i \in I$$

satisfying $\sum_{i=1}^{N} p_i(x) = 1$ for every $x \in X,$ then the family $\{(S_i, p_i) : i \in I\}$ is called an iterated function system with probabilities.

3.1. Iterated Function Systems (IFS) of Cantor middle $\frac{1}{3}$ set with probabilities are

$$w_1(x) = \frac{x}{3}, \quad p_1 = \frac{1}{2},$$
$$w_2(x) = \frac{x}{3} + \frac{2}{3}, \quad p_2 = \frac{1}{2}, \tag{3.1}$$

where p_1 and p_2 are probabilities which control the evolution distribution of $w_1(x)$ and $w_2(x).$ According to the theory of density evolution [10], the density for $f(x)$ mapping satisfying the density evolution equation

$$f_{n+1}(x) = Pf_n(x), n = 0,1,2,\ldots$$

with $Pf(x) = p_1 \frac{d\mu \circ w_1^{-1}}{d\mu}(x) f(w_1^{-1}(x)) + p_2 \frac{d\mu \circ w_2^{-1}}{d\mu}(x) f(w_2^{-1}(x)),$ which is called Markov operator [11]. That is, $Pf(x) = \frac{3}{2} f(w_1^{-1}(x)) + \frac{3}{2} f(w_2^{-1}(x)).$ Now we assume the probability density over the initial interval $[0, 1]$ is $f_0(x) = \begin{cases} 1, & x \in [0, 1], \\ 0, & \text{otherwise}, \end{cases}$

then what will happen for $f_0(x)$ under the Markov operator? According to Barnsley-Hutchinson operator (1.1), the attractor of equation (3.1) is the unit interval. i.e., $A_\infty = [0, 1].$

Now for a subset $A \subset \left[0, \frac{1}{3}\right],$ we have $w_1^{-1}(A) \subset [0, 1],$ $w_2^{-1}(A) \subset [-2, -1],$ then $f(w_2^{-1}(A)) = 0.$ In the same way, for a subset $A \subset \left[\frac{2}{3}, 1\right],$ there is $w_1^{-1}(A) \subset [2, 3],$ $w_2^{-1}(A) \subset [0, 1]$ and $f(w_1^{-1}(A)) = 0.$ Thus after the first step $f_0(x)$ becomes

$$f_1(x) = \begin{cases} \frac{3}{2}, & x \in \left[0, \frac{1}{3}\right] \\ \frac{3}{2}, & x \in \left[\frac{2}{3}, 1\right] \\ 0, & \text{otherwise} \end{cases}$$

under the Markov operator. Similarly, Markov operator acting on $f_1(x),$ and so on. This is shown in Figure 3.1.

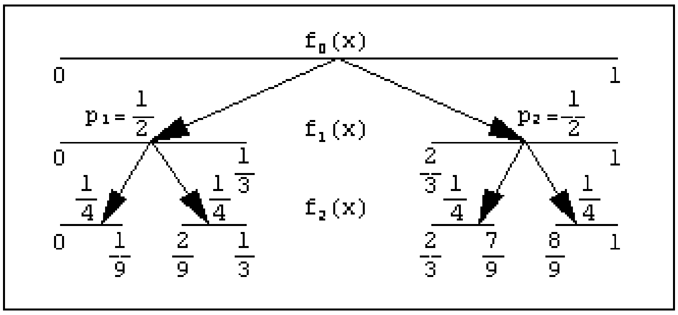

Figure 3.1. Transform from p_1 and p_2 over unit interval.

Thus the IFS of Cantor middle $\frac{1}{3}$ set with probabilities is $\{(w_k, p_k) : k = 1,2\}.$

3.2. Iterated Function Systems (IFS) of Cantor middle $\frac{1}{5}$ set with probabilities are

$$w_1(x) = \frac{x}{5}, \quad p_1 = \frac{1}{3},$$

$$w_2(x) = \frac{x}{5} + \frac{2}{5}, \quad p_2 = \frac{1}{3},$$
$$w_3(x) = \frac{x}{5} + \frac{4}{5}, \quad p_3 = \frac{1}{3}, \tag{3.2}$$

where p_1, p_2 and p_3 are probabilities which control the evolution distribution of $w_1(x)$, $w_2(x)$ and $w_3(x)$. Now we assume the probability density over the initial interval $[0, 1]$ is

$$f_0(x) = \begin{cases} 1, & x \in [0, 1], \\ 0, & \text{otherwise}, \end{cases}$$

then what will happen for $f_0(x)$ under the Markov operator? According to Barnsley-Hutchinson operator (1.1), the attractor of equation (3.2) is the unit interval. i.e., $A_\infty = [0, 1]$.

Now for a subset $A \subset \left[0, \frac{1}{5}\right]$, we have $w_1^{-1}(A) \subset [0, 1]$, $w_2^{-1}(A) \subset [-2, -1]$, $w_3^{-1}(A) \subset [-4, -3]$, then $f(w_2^{-1}(A)) = 0$ and $f(w_3^{-1}(A)) = 0$. In the same way, for a subset $A \subset \left[\frac{2}{5}, \frac{3}{5}\right]$, we have $w_1^{-1}(A) \subset [2, 3]$, $w_2^{-1}(A) \subset [0, 1]$, $w_3^{-1}(A) \subset [-2, -1]$, then $f(w_1^{-1}(A)) = 0$ and $f(w_3^{-1}(A)) = 0$. For a subset $A \subset \left[\frac{4}{5}, 1\right]$, we have $w_1^{-1}(A) \subset [4, 5]$, $w_2^{-1}(A) \subset [2, 3]$ and $w_3^{-1}(A) \subset [0, 1]$, then $f(w_1^{-1}(A)) = 0$ and $f(w_2^{-1}(A)) = 0$. Thus after the first step, $f_0(x)$ becomes

$$f_1(x) = \begin{cases} \frac{5}{3}, & x \in [0, \frac{1}{5}] \\ \frac{5}{3}, & x \in [\frac{2}{5}, \frac{3}{5}] \\ \frac{5}{3}, & x \in [\frac{4}{5}, 1] \\ 0, & \text{otherwise} \end{cases}$$

under the Markov operator. Similarly, Markov operator acting on $f_1(x)$, and so on. This is shown in Figure 3.2.

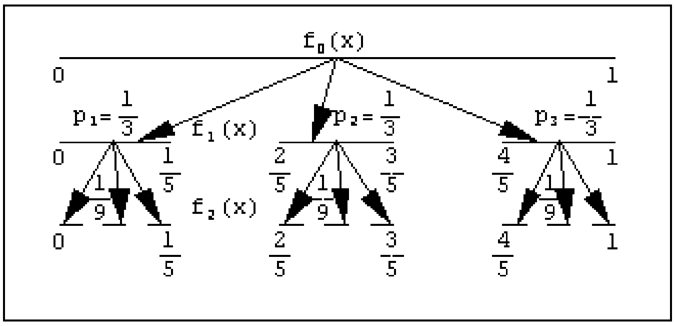

Figure 3.2. Transform from p_1, p_2 and p_3 over unit interval.

Thus the IFS of Cantor middle $\frac{1}{5}$ set with probabilities is $\{(w_k, p_k) : k = 1, 2, 3\}$.

3.3. Iterated Function Systems (IFS) of Cantor middle $\frac{1}{2N-1}$ set with probabilities are

$$w_1(x) = \frac{x}{2N-1}, \quad p_1 = \frac{1}{N}$$
$$w_2(x) = \frac{x}{2N-1} + \frac{2}{2N-1}, \quad p_2 = \frac{1}{N}$$

$$w_3(x) = \frac{x}{2N-1} + \frac{4}{2N-1}, \quad p_3 = \frac{1}{N}$$
$$\vdots$$
$$w_N(x) = \frac{x}{2N-1} + \frac{2(N-1)}{2N-1}, \quad p_N = \frac{1}{N} \text{ for } 2 \le N < \infty. \tag{3.3}$$

where $p_1, p_2, ..., p_N$ are probabilities which control the evolution distribution of $w_1(x), w_2(x), ..., w_N(x)$. Now we assume the probability density over the initial interval $[0, 1]$ is

$$f_0(x) = \begin{cases} 1, & x \in [0, 1], \\ 0, & \text{otherwise}, \end{cases}$$

then what will happen for $f_0(x)$ under the Markov operator? According to Barnsley-Hutchinson operator (1.1), the attractor of equation (3.3) is the unit interval, i.e., $A_\infty = [0, 1]$.

Now for a subset $A \subset \left[0, \frac{1}{2N-1}\right]$, we have $w_1^{-1}(A) \subset [0, 1]$, $w_2^{-1}(A) \subset [-2, -1]$, $w_3^{-1}(A) \subset [-4, -3], \ldots, w_N^{-1}(A) \subset [-(2N-2), -(2N-3)]$, then $f(w_2^{-1}(A)) = 0$, $f(w_3^{-1}(A)) = 0, \ldots, f(w_n^{-1}(A)) = 0$. In the same way, for a subset $A \subset \left[\frac{2}{2N-1}, \frac{3}{2N-1}\right]$, we have $w_1^{-1}(A) \subset [2, 3]$, $w_2^{-1}(A) \subset [0, 1]$, $w_3^{-1}(A) \subset [-2, -1], ..., w_N^{-1}(A) \subset [-(2N-4), -(2N-5)]$, then $f(w_1^{-1}(A)) = 0$, $f(w_3^{-1}(A)) = 0, ..., f(w_N^{-1}(A)) = 0$. For a subset $A \subset \left[\frac{2N-2}{2N-1}, 1\right]$, we have $w_1^{-1}(A) \subset [(2N-2), (2N-1)]$, $w_2^{-1}(A) \subset [(2N-4), (2N-3)], \ldots, w_{N-1}^{-1}(A) \subset [2, 3]$, and $w_N^{-1}(A) \subset [0, 1]$, then $f(w_1^{-1}(A)) = 0$, $f(w_2^{-1}(A)) = 0, ..., f(w_{N-1}^{-1}(A)) = 0$. Thus after the first step, $f_0(x)$ becomes

$$f_1(x) = \begin{cases} \frac{2N-1}{N}, & x \in [0, \frac{1}{2N-1}] \\ \frac{2N-1}{N}, & x \in [\frac{2}{2N-1}, \frac{3}{2N-1}] \\ \vdots & \\ \frac{2N-1}{N}, & x \in [\frac{2(N-1)}{2N-1}, 1] \\ 0, & \text{otherwise} \end{cases} \tag{3.4}$$

under the Markov operator. Similarly, Markov operator acting on $f_1(x)$, and so on. This is shown in Figure 3.3.

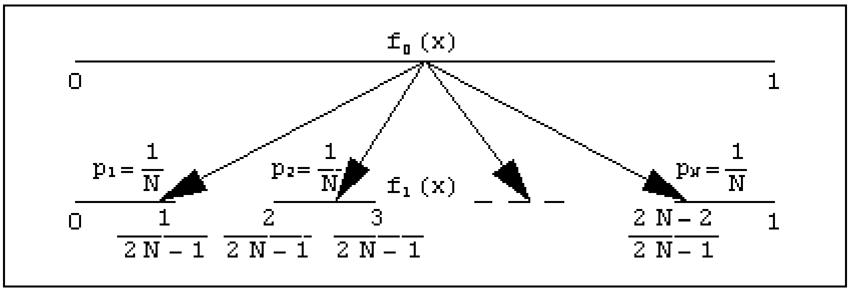

Figure 3.3. Similar transform from $p_1, p_2, \ldots, p_N$ over unit interval.

Thus the IFS of Cantor middle $\frac{1}{2N-1}$ set with probabilities is $\{(w_k, p_k): k=1,2,\ldots,N\}$.

We may summarize the above functions in the following statement:

Iterated Function Systems of Generalized Cantor Sets with Probabilities: Let $X=[0,1]$. Let (X,ρ) be a complete separable metric space. If $w_k : X \to X$ is defined by

$$w_k(x)=\frac{x}{2N-1}+\frac{2(k-1)}{2N-1},\, p_k=\frac{1}{N},$$

where $p_k(x)$ are probabilities such that $\sum_{k=1}^{N} p_k(x)=1$ for every $x\in X$, which control the evolution distribution of $w_k(x)$ with contracting factor or Lipschitz constant $L_k=\frac{1}{2N-1}$ for $2\le N<\infty$ and $1\le k\le N$. Then the family $(w,p)_N=\{(w_k,p_k): k=1,2,\ldots,N\}$ is called *iterated function systems of generalized Cantor sets with probabilities*.

For the iterated function systems of generalized Cantor sets $(w,p)_N$, we define the transition operator P_w : T by the formula

$$P_w\mu(A)=\sum_{k=1}^{N}\int_{w_k^{-1}(A)} p_k\,d\mu \quad \text{for } A\in\mathbf{B}(X) \text{ and } \mu\in \;. \tag{3.5}$$

Theorem 3.1. If P_w satisfies the following two conditions

(i) Positive linearity: $P_w(\lambda_1\mu_1+\lambda_2\mu_2)=\lambda_1P_w\mu_1+\lambda_2P_w\mu_2$ for $\lambda_1,\lambda_2\ge 0; \mu_1,\mu_2\in$

(ii) Preservation of the norm: $P_w\mu(A)=\mu(A)$ for $\mu\in$,

then P_w is a Markov operator for IFSGCS $(w,p)_N$.

Proof: Let $X=[0,1]$. Let (X,ρ) be a complete separable metric space. The iterated function systems $w_k : X\to X$ is defined by $w_k(x)=\frac{x}{2N-1}+\frac{2(k-1)}{2N-1},\, p_k=\frac{1}{N}$, for $2\le N<\infty$ and $1\le k\le N$. Let $A=[0,\frac{1}{2N-1}]\cup[\frac{2}{2N-1},\frac{3}{2N-1}]\cup\cdots\cup[\frac{2(N-1)}{2N-1},1]\subset X$.

(i) By (3.5) we have

$$P_w(\lambda_1\mu_1+\lambda_2\mu_2)(A)=\sum_{k=1}^{N}\int_{w_k^{-1}(A)} p_k(\lambda_1 d\mu_1+\lambda_2 d\mu_2)$$

$$=\sum_{k=1}^{N}\int_0^1 p_k(\lambda_1 d\mu_1+\lambda_2 d\mu_2)=\sum_{k=1}^{N}\int_0^1 p_k\lambda_1 d\mu_1+\sum_{k=1}^{N}\int_0^1 p_k\lambda_2 d\mu_2=\lambda_1+\lambda_2$$

and $\lambda_1P_w\mu_1+\lambda_2P_w\mu_2=\lambda_1\sum_{k=1}^{N}\int_{w_k^{-1}(A)} p_k d\mu_1+\lambda_2\sum_{k=1}^{N}\int_{w_k^{-1}(A)} p_k d\mu_2$

$$=\lambda_1\sum_{k=1}^{N}\int_0^1 p_k d\mu_1+\lambda_2\sum_{k=1}^{N}\int_0^1 p_k d\mu_2=\lambda_1+\lambda_2$$

i.e., $P_w(\lambda_1\mu_1+\lambda_2\mu_2)=\lambda_1P_w\mu_1+\lambda_2P_w\mu_2$ for $\lambda_1,\lambda_2\ge 0; \mu_1,\mu_2\in$.

(ii) By (3.5) we have

$$P_w\mu(A)=\sum_{k=1}^{N}\int_{w_k^{-1}(A)} p_k d\mu=\sum_{k=1}^{N}\int_0^1 p_k d\mu=1$$

and $\mu(A) = \mu[0, \frac{1}{2N-1}] + \mu[\frac{2}{2N-1}, \frac{3}{2N-1}] + \cdots + \mu[\frac{2(N-1)}{2N-1}, 1]$

$$= \int_0^{\frac{1}{2N-1}} \frac{2N-1}{N} d\mu + \int_{\frac{2}{2N-1}}^{\frac{3}{2N-1}} \frac{2N-1}{N} d\mu + \cdots + \int_{\frac{2(N-1)}{2N-1}}^{1} \frac{2N-1}{N} d\mu = 1 \quad \text{[Using function (3.4)]}$$

i.e., $P_w\mu(A) = \mu(A)$ for $\mu \in$.

Thus P_w is a Markov operator for IFSGCS $(w, p)_N$. □

By the second condition of theorem 3.1 we can easily show that $P_w\mu_w = \mu_w$ for $\mu_w \in {}_1$. That is, P_w has a stationary or invariant measure μ_w. Thus we say that $\mu_w \in {}_1$ is a invariant measure for $(w, p)_N$.

Following Lasota and Yorke [9] we have a sequence of transformations

$$w_k : X \to X,\ \ k = 1, 2, \ldots, N$$

and a probabilities vector $\{p_1(x), p_2(x), \ldots, p_N(x)\},\ p_k(x) \ge 0,\ \sum_{k=1}^{N} p_k(x) = 1$ for $x \in X$, $k = 1, 2, \ldots, N.$ If an initial point x_0 is chosen, then we randomly select from the set $\{1, 2, 3, \ldots, N\}$ an integer such a way that probability of choosing k is $p_k(x_0),\ k = 1, 2, \ldots, N.$ When a number k_0 is drawn we define $x_1 = w_{k_0}(x_0).$ Having x_1 we select k_1 according to the distribution $p_k(x_1),\ k = 1, 2, \ldots, N$ and we define $x_2 = w_{k_1}(x_1)$ and so on. Denoting by $\mu_n, n = 0, 1, \ldots$ the distribution of x_n. i.e., $\mu_n(A) = \text{prob}(x_n \in A)$ for every non-negative integer n. We define P_w as transition operator such that $\mu_{n+1} = P_w\mu_n$, where μ_n is the sequence of measures.

The above procedure can be easily formalized. Let $\mu_0 = \delta_x$ be the Dirac measure supported at a point $x \in X$. According to the definition of the dual vector U we have

$$Uf(x) = \langle Uf, \delta_x \rangle = \langle f, P_w\delta_x \rangle = \langle f, \mu_1 \rangle$$

This means that $Uf(x)$ is mathematical expectation of $f(x_1)$ if $x_0 = x$ is fixed. On the other hand, according to our description, the expectation of $f(x_1)$ is equal to

$$\sum_{k=1}^{N} p_k(x) f(w_k(x)).$$

Since x was arbitrary this gives

$$Uf(x) = \sum_{k=1}^{N} p_k(x) f(w_k(x)). \tag{3.6}$$

We admits this formula as the precise formal definition of our process and we define P_w as the Markov operator corresponding to U given by (3.6). Therefore P_w is the unique operator satisfying

$$\langle f, P_w\mu \rangle = \langle Uf, \mu \rangle = \sum_{k=1}^{N} \int_X p_k (f \circ w_k) d\mu \tag{3.7}$$

and it must be of the form

$$P_w\mu(A) = \sum_{k=1}^{N} \int_{w_k^{-1}(A)} p_k d\mu \tag{3.8}$$

For such P_w, equation (3.7) holds for every bounded Borel measurable f and $\mu \in$. Equation (3.8) is the desired formal definition of Markov operator P_w. Since the transformations $w_k : X \to X$, and the functions $p_k : X \to \mathbf{R}$ for $k = 1, 2, \ldots, N$ are continuous, P_w given by (3.8) is a Feller operator.

Now we will study asymptotic behavior of P_w under some additional assumptions concerning p_k and w_k. We will say that the iterated function systems of generalized Cantor sets $(w,p)_N = \{(w_k, p_k) : k = 1, 2, \ldots, N\}$ is non-expansive, has an invariant density or is asymptotically stable if the Markov operator (3.8) has the corresponding property.

We say that the iterated function systems of generalized Cantor sets $(w,p)_N$ is asymptotically stable if P_w is asymptotically stable. Now we will formulate assumptions that ensure the non-expansiveness and asymptotic stability of iterated function systems of generalized Cantor sets $(w,p)_N = \{(w_k, p_k) : k = 1, 2, \ldots, N\}$.

IV. Non-expansiveness and Asymptotic Stability of Iterated Function Systems of GCS

4.1 Non-expansiveness of Iterated Function Systems of Generalized Cantor Sets

Lemma 4.1.1. The IFSGCS $(w,p)_N$ $i.e., w_k(x) = \frac{x}{2N-1} + \frac{2(k-1)}{2N-1}$ is uniform continuous for $x, y \in X$, $(2 \le N < \infty)$ and $1 \le k \le N$.

Proof. Choose $\varepsilon > 0$. Let $\delta = (2N-1)\varepsilon$. Choose $x_0, x \in X$. Assume that $|x - x_0| < \delta$. Then

$$|w_k(x) - w_k(x_0)| = \left|\frac{x}{2N-1} - \frac{x_0}{2N-1}\right| = \frac{1}{2N-1}|x - x_0| < \frac{1}{2N-1}.\delta = \varepsilon$$

$$i.e., |w_k(x) - w_k(x_0)| < \varepsilon.$$

Thus the IFSGCS $(w,p)_N$ is uniform continuous. □

Lemma 4.1.2. The IFSGCS $(w,p)_N$ satisfies the Dini function if there is a function $\omega : [0,\infty] \to [0,\infty]$ is a modulus of continuity for w_k $i.e., |w_k(x) - w_k(y)| \le \omega(|x-y|)$ for $x, y \in X$.

Proof. Assume that $\omega : [0,\infty] \to [0,\infty]$ is defined by $\omega(t) = kt$, where k is a Lipschitz constant.

Now $|w_k(x) - w_k(y)| = \left|\frac{x}{2N-1} + \frac{2(k-1)}{2N-1} - \frac{x}{2N-1} - \frac{2(k-1)}{2N-1}\right|$

$= \frac{1}{2N-1}|x-y| = L_k|x-y| = \omega(|x-y|)$, where $L_k = \frac{1}{2N-1}$ is a Lipschitz constant for $(2 \le N < \infty)$ and $1 \le k \le N$.

$$i.e., |w_k(x) - w_k(y)| \le \omega(|x-y|) \text{ for } x, y \in X.$$

Thus ω is a Dini function of the IFSGCS $(w,p)_N$. □

Lemma 4.1.3. If the IFSGCS $(w,p)_N$ satisfies the inequality

$\sum_{k=1}^{N} p_k(x)\rho(w_k(x), w_k(y)) \le r(\rho(x,y))$ for $x, y \in X$, where $r < 1$ is a non-negative constant, then $(w,p)_N$ is contraction transformation with contracting factor or Lipschitz constant $L_k = \frac{1}{2N-1}$ for $(2 \le N < \infty)$ and $1 \le k \le N$.

Proof. The IFSGCS $(w,p)_N$ is $w_k(x) = \frac{x}{2N-1} + \frac{2(k-1)}{2N-1}, p_k = \frac{1}{N}$, for $x, y \in X$, where $p_k(x)$ are probabilities such that $\sum_{k=1}^{N} p_k(x) = 1$ for every $x \in X$.

Now $\sum_{k=1}^{N} p_k(x)\rho(w_k(x), w_k(y)) = \sum_{k=1}^{N} p_k(x)\|w_k(x) - w_k(y)\|$

$$= \sum_{k=1}^{N} p_k(x)\left\|\left(\frac{x}{2N-1} + \frac{2(k-1)}{2N-1}\right) - \left(\frac{y}{2N-1} + \frac{2(k-1)}{2N-1}\right)\right\|$$

$$=\sum_{k=1}^{N} p_k(x)\left\| \frac{x}{2N-1}-\frac{y}{2N-1}\right\| = \sum_{k=1}^{N} p_k(x)\frac{1}{2N-1}\| x-y\| = L_k\cdot \rho(x,y)$$

That is, $\sum_{k=1}^{N} p_k(x)\rho(w_k(x),w_k(y)) \le r(\rho(x,y))$ for $x,y\in X$, say $r = L_k = \dfrac{1}{2N-1}$.

Thus $(w,p)_N$ is contraction transformation with contracting factor or Lipschitz constant $L_k = \dfrac{1}{2N-1}$ for $(2\le N<\infty)$ and $1\le k\le N.$ □

Since there exists a Dini function of the IFSGCS $(w,p)_N$, there exists a continuous non-decreasing and concave function $\varphi:[0,\infty]\to[0,\infty]$ such that $\varphi(0)=0, \varphi(\infty)=\infty$ and the Markov operator P_w corresponding to $(w,p)_N$ is non-expansive with respect to the metric $\varphi(\rho(x,y)) = \rho_\varphi(x,y)$ for $x,y\in X$, that is, we will calculate the value of $\| P_w(\mu_1-\mu_2)\|$ for operator (3.8).

$$\| P_w(\mu_1-\mu_2)\| = \| P_w\mu_1 - P_w\mu_2\| = \sup_{F_\varphi} |< f, P_w\mu_1 - P_w\mu_2 >| = \sup_{F_\varphi} |< Uf, \mu_1-\mu_2 >$$

$$= \sup_{F_\varphi} |< \sum_{k=1}^{N} p_k (f\circ w_k), \mu_1-\mu_2 >| = \sup_{F_\varphi} |< \sum_{k=1}^{N} p_k f(w_k), \mu_1-\mu_2 >|$$

$$= \sup_{F_\varphi} |< 1, \mu_1-\mu_2 > = \sup_{F_\varphi} |< f, \mu_1-\mu_2 > = \| \mu_1-\mu_2\|$$

$$i.e., \| P_w(\mu_1-\mu_2)\| = \| \mu_1-\mu_2\|$$

Thus P_w is non-expansive with respect to the metric $\varphi\circ\rho.$

Since the IFSGCS $(w,p)_N$ satisfies the Lemma 4.1.3 and the Markov operator P_w corresponding to $(w,p)_N$ is non-expansive with respect to the metric $\varphi\circ\rho,$ the iterated function systems of generalized Cantor sets $(w,p)_N$ is non-expansive with respect to the metric $\varphi\circ\rho.$

Theorem 4.1.1. Let P_w be a non-expansive Markov operator. Assume that for every $\varepsilon>0$ there is a Borel set A with *diam* $A\le\varepsilon,$ a real number $\alpha>0$ and an integer n such that

$$\lim_{n\to\infty}\inf\ P_w^n\mu(A)\ge\alpha \quad \text{for} \quad \mu\in\ _1. \tag{4.1}$$

Then P_w is asymptotically stable.

Proof: Since a non-expansive Markov operator is a Feller operator, P_w is a Feller operator. Then P_w has an invariant distribution μ_*. To complete the proof of asymptotic stability it remains to verify condition

$$\lim_{n\to\infty} < f,\mu_n > = < f,\mu > \quad \text{for all } f\in C(X).$$

When an invariant distribution exists the above condition is equivalent to a more symmetric relation

$$\lim_{n\to\infty} \| P_w^n(\mu_1-\mu_2)\| = 0 \quad \text{for } \mu_1,\mu_1\in\ _1. \tag{4.2}$$

Let $\mu_1,\mu_2\in\ _1$ and $\varepsilon>0.$ Choose $A\subset X$ and $\alpha, 0<\alpha<1.$ Following Lasota and Yorke [7] we will define by an induction argument a sequences of integers n_k and four sequences of distributions $(\mu_i^k),(\nu_i^k), k=0,1,2,\ldots, i=1,2.$ If $k=0$ we define $n_0=0$ and $\nu_i^0=\mu_i^0=\mu_i.$ If $k\ge1$ is fixed and $n_{k-1},\mu_i^{k-1},\nu_i^{k-1}$ are given we choose according (4.1) a number n_k such that

$$P_w^{n_k}\mu_i^{k-1}(A)\ge\sigma \quad \text{for } i=1,2.$$

and we define

$$\nu_i^k(B) = \frac{P_w^{n_k}\mu_i^{k-1}(B\cap A)}{P_w^{n_k}\mu_i^{k-1}(A)} \tag{4.3}$$

$$\mu_i^k(B)=\frac{1}{1-\alpha}\{P_w^{n_k}\,\mu_i^{k-1}(B)-\alpha\nu_i^{k-1}(B)\}.$$

Since $P_w^{n_k}\,\mu_i^{k-1}(A)\geq\sigma,$ we have

$$P_w^{n_k}\,\mu_i^{k-1}(B)=P_w^{n_k}\,\mu_i^{k-1}(B\cap A)=P_w^{n_k}\,\mu_i^{k-1}(A)\nu_i^k(B)\geq\alpha\nu_i^k(B).$$

Observe that $\nu_i^k(X\setminus A)=0$ and consequently

$$\|\nu_1^k-\nu_2^k\|=\sup_{f\in F}|\int_X f\,d\nu_1^k-\int_X f\,d\nu_2^k|=\sup_{f\in F}|\int_A f\,d\nu_1^k-\int_A f\,d\nu_2^k|\leq diam\,A\leq\varepsilon. \tag{4.4}$$

Using equation (4.3) it is easy to verify by an induction argument that

$P_w^{n_1+\ldots+n_k}\mu_i=\sigma P_w^{n_2+\ldots+n_k}\nu_i^1+\sigma(1-\sigma)P_w^{n_2+\ldots+n_k}\nu_i^2+\ldots+\sigma(1-\sigma)^{k-1}\nu_i^k+(1-\sigma)^k\mu_i^k$ for $k\geq 1.$

Since P_w is non-expansive this implies

$$\|P_w^{n_1+\ldots+n_k}(\mu_1-\mu_2)\|\leq\sigma\|\nu_1^1-\nu_2^1\|+\sigma(1-\sigma)\|\nu_1^2-\nu_2^2\|$$
$$+\ldots+\sigma(1-\sigma)^{k-1}\|\nu_1^k-\nu_2^k\|+(1-\sigma)^k\|\mu_1^k-\mu_2^k\|.$$

From this, condition (4.4) and the obvious inequality $\|\mu_1^k-\mu_2^k\|\leq 2$ it follows

$$\|P_w^{n_1+\ldots+n_k}(\mu_1-\mu_2)\|\leq\varepsilon+2(1-\sigma)^k$$

Again, using the non-expansiveness of P_w^n we obtain

$$\|P_w^n(\mu_1-\mu_2)\|\leq\varepsilon+2(1-\sigma)^n \text{ for } n\geq n_1+\ldots+n_k.$$

Since $\varepsilon>0$ is arbitrary and k does not depend on μ_1,μ_2 we have

$$\|P_w^n\mu_1-P_w^n\mu_2\|\leq\varepsilon \text{ for } n\geq n_0 \text{ and every two measures } \mu_1,\mu_1\in{}_1.$$

So, we are given

$$\|P_w^n\mu-P_w^m\mu\|\leq\varepsilon \text{ for } n,m\geq n_0 \text{ and every } \mu\in{}_1.$$

Really, if $n>m$ we have

$$P_w^n\mu=P_w^m(P_w^{n-m}\mu)$$

and because $m\geq n_0$

$$\|P_w^m(\mu-P_w^{n-m}\mu)\|\leq\varepsilon.$$

Since ${}_1$ is a complete metric space, the sequence $(P_w^n\mu:n\in\mathbf{N})$ convergences to some $\mu_*\in{}_1$. Obviously $P_w\mu_*=\mu_*$ and

$$\lim_{n\to\infty}\|P_w^n\mu-\mu_*\|=\lim_{n\to\infty}\|P_w^n(\mu-\mu_*)\|=0 \text{ for every } \mu\in{}_1.$$

This completes the proof. □

4.2 Asymptotic Stability of Iterated Function Systems of Generalized Cantor Sets

Theorem 4.2.1. Let $(w,p)_N=\{(w_k,p_k):k=1,2,\ldots,N\}$ be iterated function systems of generalized Cantor set. If $(w,p)_N$ satisfies the following conditions

(i) there is a Dini function of $(w,p)_N$

(ii) $\inf_{x\in X} p_k(x)>0$ for every $k\in\{1,2,\ldots,N\}$

(iii) the transformations $w_k:X\to X$ are Lipschitzian for every $k\in\{1,2,\ldots,N\}$ and there exists a non-negative integer λ_w such that $\sum_{k=1}^N p_k(x)L_k\leq\lambda_w<1$ for $x\in X,$

then the IFSGCS $(w,p)_N$ is asymptotically stable.

Proof: (i) By Lemma 4.1.2, we say that the IFSGCS $(w,p)_N$ has a Dini function.

(ii) Since $p_k(x)=\dfrac{1}{N}$ for $2 \le N < \infty$ and $1 \le k \le N,$ clearly $\inf\limits_{x \in X} p_k(x) > 0$ for every $k \in \{1,2,\ldots,N\}$.

(iii) Since $w_k(x)=\dfrac{x}{2N-1}+\dfrac{2(k-1)}{2N-1}, p_k=\dfrac{1}{N},$ is a Lipschitzian with Lipschitz's constant $L_k=\dfrac{1}{2N-1}$ for $x, y \in X$, $2 \le N < \infty$ and $1 \le k \le N,$ then

$$\sum_{k=1}^{N} p_k(x)L_k = p_1(x)L_1 + p_2(x)L_2 + \cdots + p_N(x)L_N$$

$$= \frac{1}{N}.\frac{1}{2N-1}+\frac{1}{N}.\frac{1}{2N-1}+\cdots+\frac{1}{N}.\frac{1}{2N-1}=\frac{1}{2N-1} \text{ for } 2 \le N < \infty.$$

and $\lambda_w = \sup\limits_{x \in X} \sum_{k=1}^{N} p_k(x)L_k = \dfrac{1}{3}$ for $N=2.$ Thus $\sum_{k=1}^{N} p_k(x)L_k \le \lambda_w < 1$ for $x \in X,$

Since the IFSGCS $(w,p)_N$ satisfies the above three conditions, the iterated function systems of generalized Cantor sets $(w,p)_N$ is asymptotically stable.

This completes the proof. □

We say that a Markov operator $P_w:$ T satisfies the Prokhorov condition if there exists a compact set and a number β such that

$$\lim_{n\to\infty} \inf P_w{}^n \mu(Y) \ge \beta \quad \text{for } \mu \in {}_1. \tag{4.5}$$

This condition is clearly satisfied if X is a compact space or if P_w is an asymptotically stable operator.

Proposition 4.2.2. Let $(w,p)_N = \{(w_k, p_k): k=1,2,\ldots,N\}$ be an iterated function systems of generalized Cantor sets such that w_1 is bounded and $\inf\ p_1 > 0.$ Then $(w,p)_N = \{(w_k, p_k): k=1,2,\ldots,N\}$ has a stationary distribution and satisfies the Prokhorov condition $\lim\limits_{n\to\infty} \inf P_w{}^n \mu(Y) \ge \beta$ for $\mu \in {}_1$, where Y is a compact set and a number $\beta.$

Proof. We know $P_w\mu(A)=\sum_{k=1}^{N} \int_{w_k^{-1}(A)} p_k d\mu$ corresponding to $(w,p)_N$.

Let $Y=[0,\frac{1}{3}]\cup[\frac{2}{3},1] \supset w_1(X)$ be a compact set. For every $\mu \in {}_1$ we have

$$P_w\mu(Y)=\sum_{k=1}^{N} \int_{w_k^{-1}(Y)} p_k d\mu = 1$$

That is, $P_w^n\mu(Y)=\sum_{k_1=1}^{N}\cdots\sum_{k_n=1}^{N} \int_{w_{k_1}^{-1}\circ\ldots\circ w_{k_n}^{-1}(Y)} p_{k_n}(w_{k_{n-1}} \circ \ldots \circ w_{k_1}(x)) \cdot \ldots \cdot p_{k_1}(x) d\mu = 1$

and $\mu(w_1^{-1}(Y)) \inf\{\ p_1\} = \mu([0,1]) \inf\{\ p_1\} = \dfrac{1}{2} = \beta$ (say).

Thus $\lim\limits_{n\to\infty} \inf P_w{}^n \mu(Y) \ge \beta$ for $\mu \in {}_1$, where Y is a compact set and a number $\beta.$ □

Theorem 4.2.2. Suppose iterated function systems of generalized Cantor sets $(w,p)_N = \{(w_k, p_k): k=1,2,\ldots,N\}$ are essentially non-expansive and satisfies the Prokhorov condition. Also suppose that w_1 satisfies the inequality

$$\sum_{k=1}^{N} p_k(x)\rho(w_1(x), w_1(y)) \le r\rho(x,y) \text{ for } x, y \in X, \tag{4.6}$$

where $r < 1$ is a non-negative constant, and has an attracting fixed point x_*, then

$$\lim_{n\to\infty} \rho(w_1^n(x), x_*) = 0 \qquad \text{for } x, \in X$$

If in addition $\inf\{ p_1\} > 0,$ then $(w, p)_N = \{(w_k, p_k) : k = 1, 2, \ldots, N\}$ is asymptotically stable.

Proof. Following Lasota and Yorke [7] consider the dynamical system $(w_1, \frac{1}{2})$ given by only one transformation w_1 and the probability $\frac{1}{2}.$ Condition (4.6) implies that $(w_1, \frac{1}{2})$ is non-expansive. The Markov operator P_w corresponding to $(w_1, \frac{1}{2})$ is given by formula

$$P_w \mu(A) = \sum_{k=1}^{N} \int_{w_k^{-1}(A)} p_k \, d\mu = \mu(w_1^{-1}(A)) \quad \text{for } A = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] \subset X,$$

and has the property that a point measure $\mu = \delta_x$ is transformed into the point measure $P_w \mu = \delta_{w_1(x)}$. For every $x_0 \in X$ the sequence $x_n = w_1^n(x_0)$ converges to attracting fixed point x_* and consequently for every $x_0 \in X$ the sequence of measures $P_w^n \delta_{x_0} = \delta_{x_n}$ converges weakly to $\delta_{x_*} = P_w \delta_{x_*}$. Since the family of Dirac measures is linearly dense in $_1$ (in the Fortet Mourier metric) and the operators $\{P_w^n\}$ are uniformly continuous, we have

$$\lim_{n\to\infty} \| P_w^n \mu - \delta_{x_*} \| = 0 \quad \text{for } \mu \in {}_1.$$

Thus the system $(w_1, \frac{1}{2})$ is asymptotically stable. This completes the proof. □

V. Conclusion

We discuss iterated function systems with probabilities of generalized Cantor sets (IFSGCS) and show that these functions are non-expansiveness and asymptotically stable if the Markov operator has the corresponding property. We would like to study the iterated function systems of two dimensional fractals such as the Sierpiński triangle or gasket, carpet and the Box fractal and also three dimensional fractals such as the Tetrahedron, the Menger sponge and the Octahedron in Markov operator.

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