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On Jordan (σ,τ)-Higher Homomorphisms of ΓM-Module

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Abstract: Let M be a Γ-ring and X be a left ΓM-module, in this paper proved that every Jordan (σ,τ)-higher homomorphism from a Γ-ring M into a prime left ΓM - module X is either (σ,τ)-higher homomorphism or (σ,τ)-higher anti homomorphism.
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I. Introduction

Let M and Γ be two additive a belian groups, suppose that there is a mapping from $M\times\Gamma\times M \longrightarrow M$
(the image of (a,α,b) being denoted by $a\alpha b$, $a, b \in M$ and $\alpha \in \Gamma$) satisfying for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$
(i) $(a + b)\alpha c = a\alpha c + b\alpha c$
$a(\alpha + \beta)c = a\alpha c + a\beta c$
$a\alpha(b + c) = a\alpha b + a\alpha c$
(ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$.
Then M is called Γ-ring . This definition is due to Barnes [1].
Let M be a Γ-ring and X be an additive a belian group. X is a left ΓM-module if there exists a mapping
$M\times\Gamma\times X \longrightarrow X$ (sending $(m,\alpha,x) \longrightarrow m\,\alpha\, x$), such that
(i) $(m_1 + m_2)\alpha x = m_1\alpha x + m_2\alpha x$
(ii) $m\alpha(x_1 + x_2) = m\alpha x_1 + m\alpha x_2$
(iii) $(m_1\,\alpha\, m_2)\beta x = m_1\alpha(m_2\beta x)$
For all $m, m_1, m_2 \in M$ and $x, x_1, x_2 \in X$ and $\alpha, \beta \in \Gamma$, [4].
A Γ-ring M is commutative if $a\alpha b = b\alpha a$, [6] .
X is prime if $a\Gamma X\Gamma b = (0)$ implies $a = 0$ or $b = 0$,for all $x\in X$ and X is semiprime if $a\Gamma X\Gamma a = (0)$ implies $a = 0$,for all $x\in X$.
X is called a 2-torsion free if $2x=0$ implies $x=0$ for all $x \in X$ [4].
Let X be a 2-torsion free semiprime ΓM- module X and suppose that $a, b \in$ ΓM- module X if
$a\Gamma X\Gamma b + b\Gamma X\Gamma a = 0$ for all $x\in X$, then $a\Gamma X\Gamma b = b\Gamma X\Gamma a = 0$.
Let M be Γ-ring , a mapping $*: M \longrightarrow X$ is called an involution if for all $a, b\in M$ and $\alpha \in \Gamma$
(i) $a^{**} = a$.
(ii) $(a + b)^* = a^* + b^*$
(iii) $(a\alpha b)^* = b^*\alpha\, a^*$, [5].
Let θ be an additive mapping of a ring R into a ring R', θ is called a homomorphism if $\theta(a\, b) = \theta(a)\, \theta(b)$.
And θ is called a Jordan homomorphism if for all $a, b \in R$
$\theta(a\, b + b\, a) = \theta(a)\, \theta(b) + \theta(b)\, \theta(a)$ for all $a , b \in R$, [2] .
Let θ be an additive mapping of a Γ-ring M into a Γ-ring M', θ is called homomorphism if
$\theta(a\alpha b) = \theta(a)\alpha\theta(b)$, for all $a, b \in M$ and $\alpha \in \Gamma$. [1].
Let θ be an additive mapping of Γ-ring M into a Γ-ring M', θ is called Jordan homomorphism if
$\theta(a\alpha b + b\alpha a) = \theta(a)\alpha\theta(b) + \theta(b)\alpha\theta(a)$, for all $a, b \in M$ and $\alpha \in \Gamma$. [7].
Let θ be an additive mapping of a ring R into a ring R' and σ, τ be two endomorphism of R. θ is called (σ,τ)-homomorphism if
$\theta(ab) = \theta(\sigma(a))\, \theta(\tau(b))$, for all $a,b\in R$.
And θ is called Jordan (σ,τ)-homomorphism if
$\theta(ab + ba) = \theta(\sigma(a))\, \theta(\tau(b)) + \theta(\sigma(b))\, \theta(\tau(a))$, for all $a, b \in R$, [3].
Let θ be an additive mapping of a Γ-ring M into a Γ-ring M' and σ, τ be two endomorphism of M . θ is called (σ,τ)-homomorphism if
$\theta(a\alpha b) = \theta(\sigma(a))\alpha\theta(\tau(b))$, for all $a, b \in M$ and $\alpha \in \Gamma$.
And θ is called Jordan (σ,τ)-homomorphism if
$\theta(a\alpha b + b\alpha a) = \theta(\sigma(a))\alpha\theta(\tau(b)) + \theta(\sigma(b))\alpha\theta(\tau(a))$, for all $a, b \in M$ and $\alpha \in \Gamma$, [3] .

Now, in this paper presented the definitions of (σ,τ)-higher homomorphism, Jordan (σ,τ)-higher homomorphism , Jordan triple (σ,τ)-higher homomorphism on a left ΓM-module and prove that every Jordan (σ,τ)-higher homomorphism from a Γ-ring M into2-torsion free ΓM - module X , such that $a\alpha b\beta c = a\beta b\alpha c$, for all a, b, c ∈ M and α, β ∈Γ, $\sigma^{i^2} = \sigma^i$, $\tau^{i^2} = \tau^i$, $\sigma^i\tau^i = \sigma^i\tau^{n-i}$ and $\sigma^i\tau^i = \tau^i\sigma^i$ then θ is a Jordan triple (σ,τ) -higher homomorphism.

II. Jordan (σ,τ)-Higher Homomorphism of ΓM - Module

Definition (2.1): Let $\theta = (\phi_i)_{i\in N}$ be a family of additive mappings of a Γ-ring M into a left ΓM- module X and σ,τ be two endomorphisms of M. θ is called a **(σ,τ)-higher homomorphism** if

$$\phi_n(a\alpha b) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\tau^i(b))$$

for all a, b ∈ M , α ∈ Γ and n ∈ N .

Example (2.2): Let $\theta = (\theta_i)_{i\in N}$ be a (σ,τ)-higher homomorphism of a ring R into a ring R'. Let $M = M_{1\times 2}(R)$ and $\Gamma = \left\{\begin{pmatrix} n \\ 0 \end{pmatrix}, n \in Z\right\}$.Then M is a Γ-ring.

Let $\phi = (\phi_i)_{i\in N}$ be a family of additive mappings from Γ-ring M into a left ΓM - module X defined by:
$\phi_n((a \quad b)) = (\theta_n(a) \quad \theta_n(b))$, for all $(a \quad b)$ ∈ M.
Let σ_1^n, τ_1^n be two endomorphisms of M, such that
$\sigma_1^n((a \quad b)) = ((\sigma^n(a) \quad \sigma^n(b))$, $\tau_1^n((a \quad b)) = ((\tau^n(a) \quad \tau^n(b))$.
Then ϕ_n is a (σ,τ)-higher homomorphism.

Definition (2.3): Let $\theta = (\phi_i)_{i\in N}$ be a family of additive mappings of a Γ-ring M into a left ΓM - module X and σ,τ be two endomorphisms of a Γ-ring M. θ is called **Jordan (σ,τ)-higher homomorphism** if

$$\phi_n(a\alpha b + b\alpha a) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\tau^i(b)) + \sum_{i=1}^{n} \phi_i(\sigma^i(b))\alpha\phi_i(\tau^i(a))$$

for all a, b ∈ M , α ∈ Γ and for n ∈ N .

Remark(2.4): Clearly every (σ,τ)-higher homomorphism is Jordan (σ,τ)-higher homomorphism but the converse is not true in general , as shown by the following example .

Example (2.5): Let S be any Γ-ring with nontrivial involution ∗ and Γ be the set of all integer numbers.
Let M = S ⊕ S, such that a ∈ Z(S), $s_1\alpha\, a\, \alpha s_2 = 0$, $s_1 \neq s_2$ and $a^2 = a$, for all s_1, s_2 ∈ S.
Let $\theta = (\phi_i)_{i\in N}$ be a family of additive mappings of a Γ-ring M into a left ΓM - module X defined by:

$$\phi_n(s,t) = \begin{cases} ((2-n)a\alpha s, (n-1)t^*) & , n = 1,2 \\ 0 & , n \geq 3 \end{cases}$$

for all (s,t) ∈ M.
Let σ^n, τ^n be two endomorphisms of M , such that $\sigma^n((s,t)) = (ns,t)$, $\tau^n((s,t)) = (n^2s,t)$. Then θ is Jordan (σ,τ)-higher homomorphism but not (σ,τ)-higher homomorphism.

Definition (2.6): Let $\theta = (\phi_i)_{i\in N}$ be a family of additive mappings of a Γ-ring M into a left ΓM - module X and σ,τ be two endomorphisms of M . θ is called a **Jordan triple (σ,τ)-higher homomorphism** if

$$\phi_n(a\alpha b\beta a) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\sigma^i\tau^{n-i}(b))\beta\phi_i(\tau^i(a))$$

for all a, b ∈ M, α,β ∈ Γ and n ∈ N

Definition (2.7): Let $\theta = (\phi_i)_{i\in N}$ be a family of additive mappings of a Γ-ring M into a left ΓM - module X and σ,τ be two endomorphisms of M .θ is called a **(σ,τ)-higher anti homomorphism** if

$$\phi_n(a\alpha b) = \sum_{i=1}^{n} \phi_i(\sigma^i(b))\alpha\phi_i(\tau^i(a))$$

for all a, b ∈ M , α ∈ Γ and n ∈ N .

Lemma (2.8): Let $\theta = (\phi_i)_{i\in N}$ be a Jordan triple (σ,τ)-higher homomorphism of a Γ-ring M into a left ΓM - module X , then for all *a, b, c* ∈ M, α, β ∈ Γ and n ∈ N

(i) if $\sigma^{i^2} = \sigma^i,\ \tau^{i^2} = \tau^i,\ \sigma^i\tau^i = \sigma^i\tau^{n-i}$ and $\sigma^i\tau^i = \tau^i\sigma^i$

$$\phi_n(a\alpha b\beta a + a\beta b\alpha a) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\sigma^i\tau^{n-i}(b))\beta\phi_i(\tau^i(a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(a))\beta\phi_i(\sigma^i\tau^{n-i}(b))\alpha\phi_i(\tau^i(a))$$

(ii) $$\phi_n(a\alpha b\beta c + c\alpha b\beta a) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\sigma^i\tau^{n-i}(b))\beta\phi_i(\tau^i(c)) + \sum_{i=1}^{n} \phi_i(\sigma^i(c))\alpha\phi_i(\sigma^i\tau^{n-i}(b))\beta\phi_i(\tau^i(a))$$

(iii) In particular, if M is commutative and ΓM - module X is a 2-torsion free , then

$$\phi_n(a\alpha b\beta c) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\sigma^i\tau^{n-i}(b))\beta\phi_i(\tau^i(c))$$

(iv) $$\phi_n(a\alpha b\alpha c + c\alpha b\alpha a) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\sigma^i\tau^{n-i}(b))\alpha\phi_i(\tau^i(c)) + \sum_{i=1}^{n} \phi_i(\sigma^i(c))\alpha\phi_i(\sigma^i\tau^{n-i}(b))\alpha\phi_i(\tau^i(a))$$

Proof:

(i) Replace *a*β*b* + *b*β*a* for *b* in Definition (2.3) , we get :

$$\phi_n(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\tau^i(a\beta b + b\beta a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(a\beta b + b\beta a))\alpha\phi_i(\tau^i(a))$$

$$= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\tau^i(a)\beta\tau^i(b) + \tau^i(b)\beta\tau^i(a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(a)\beta\sigma^i(b) + \sigma^i(b)\beta\sigma^i(a))\alpha\phi_i(\tau^i(a))$$

$$= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\left(\sum_{j=1}^{i} \phi_j(\sigma^j\tau^j(a))\beta\phi_j(\tau^{j^2}(b)) + \sum_{j=1}^{i}\phi_j(\sigma^j\tau^j(b))\beta\phi_j(\tau^{j^2}(a))\right) + \sum_{i=1}^{n}\left(\sum_{j=1}^{i} \phi_j(\sigma^{j^2}(a))\beta\phi_j(\tau^j\sigma^j(b)) + \sum_{j=1}^{i} \phi_j(\sigma^{j^2}(b))\beta\phi_j(\tau^j\sigma^j(a))\right)\alpha\phi_i\tau^i(a))$$

$$= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\sigma^i\tau^i(a))\beta\phi_i(\tau^{i^2}(b)) + \sum_{i=1}^{n}\phi_i(\sigma^i(a))\alpha\phi_i(\sigma^i\tau^i(b)\beta\phi_i(\tau^{i^2}(a)) + \sum_{i=1}^{n} \phi_i(\sigma^{i^2}(a))\beta\phi_i(\tau^i\sigma^i(b))\alpha\phi_i(\tau^i(a)) + \sum_{i=1}^{n}\phi_i(\sigma^{i^2}(b))\beta\phi_i(\tau^i\sigma^i(a)\alpha\phi_i(\tau^i(a))$$

Since $\sigma^{i^2} = \sigma^i,\ \tau^{i^2} = \tau^i,\ \sigma^i\tau^i = \sigma^i\tau^{n-i}$ and $\sigma^i\tau^i = \tau^i\sigma^i$

$$= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\sigma^i\tau^{n-i}(a))\beta\phi_i(\tau^i(b)) + \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\sigma^i\tau^{n-i}(b))\beta\phi_i(\tau^i(a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(a))\beta\phi_i(\sigma^i\tau^{n-i}(b))\alpha\phi_i(\tau^i(a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(b))\beta\phi_i(\sigma^i\tau^{n-i}(a))\alpha\phi_i(\tau^i(a))$$

…(1)

On the other hand:

$$\phi_n(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a) = \phi_n(a\alpha a\beta b + a\alpha b\beta a + a\beta b\alpha a + b\beta a\alpha a)$$
$$= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\sigma^i\tau^{n-i}(a))\beta\phi_i(\tau^i(b)) + \sum_{i=1}^{n} \phi_i(\sigma^i(b))\beta\phi_i(\sigma^i\tau^{n-i}(a))\alpha\phi_i(\tau^i(a)) + \phi_n(a\alpha b\beta a + a\beta b\alpha a) \qquad \ldots(2)$$

Comparing (1) and (2), we get:

$$\phi_n(a\alpha b\beta a + a\beta b\alpha a) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\sigma^i\tau^{n-i}(b))\beta\phi_i(\tau^i(a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(a))\beta\phi_i(\sigma^i\tau^{n-i}(b))\alpha\phi_i(\tau^i(a))$$

(ii) Replace *a* + *c* for *a* in Definition (2.6) , we get :

$$\phi_n((a+c)\alpha b\beta(a+c)) = \sum_{i=1}^{n} \phi_i(\sigma^i(a+c))\alpha\phi_i(\sigma^i\tau^{n-i}(b))\beta\phi_i(\tau^i(a+c))$$
$$= \sum_{i=1}^{n}\phi_i(\sigma^i(a) + \sigma^i(c))\alpha\phi_i(\sigma^i\tau^{n-i}(b))\beta\phi_i(\tau^i(a) + \tau^i(c))$$

$$= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\,\alpha\,\phi_i(\sigma^i\tau^{n-i}(b))\beta\,\phi_i(\tau^i(a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(a))\,\alpha\,\phi_i(\sigma^i\tau^{n-i}(b))\beta\,\phi_i(\tau^i(c)) +$$
$$\sum_{i=1}^{n} \phi_i(\sigma^i(c))\,\alpha\,\phi_i(\sigma^i\tau^{n-i}(b))\beta\phi_i(\tau^i(a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(c))\,\alpha\,\phi_i(\sigma^i\tau^{n-i}(b))\beta\phi_i(\tau^i(c))$$

…(1)

On the other hand :

$\phi_n((a+c)\alpha b\beta(a+c)) = \phi_n(a\alpha b\beta a + a\alpha b\beta c + c\alpha b\beta a + c\alpha b\beta c)$

$$= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\sigma^i\tau^{n-i}(b))\beta\,\phi_i(\tau^i(a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(c))\alpha\,\phi_i(\sigma^i\tau^{n-i}(b))\beta\phi_i(\tau^i(c)) +$$
$$\phi_n(a\alpha b\beta c + c\alpha b\beta a) \qquad \ldots(2)$$

Comparing (1) and (2), we get:

$$\phi_n(a\alpha b\beta c + c\alpha b\beta a) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\sigma^i\tau^{n-i}(b))\beta\,\phi_i(\tau^i(c)) +$$
$$\sum_{i=1}^{n} \phi_i(\sigma^i(c))\alpha\,\phi_i(\sigma^i\tau^{n-i}(b))\beta\,\phi_i(\tau^i(a))$$

(iii) By (ii) and since M be a commutative and ΓM - module X is a 2-torsion free

$$\phi_n(a\,\alpha b\;\beta c + a\;\alpha b\;\beta c) = 2\phi_n(a\;\alpha b\;\beta c) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\,\phi_i(\sigma^i\tau^{n-i}(b))\beta\phi_i(\tau^i(c))$$

(iv) Replace β for α in (ii), we get:

$$\phi_n(a\alpha b\alpha c + c\alpha b\alpha a) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\sigma^i\tau^{n-i}(b))\alpha\,\phi_i(\tau^i(c)) +$$
$$\sum_{i=1}^{n} \phi_i(\sigma^i(c))\alpha\,\phi_i(\sigma^i\tau^{n-i}(b))\alpha\,\phi_i(\tau^i(a))$$

Definition (2.9): Let $\theta = (\phi_i)_{i\in N}$ be a Jordan (σ,τ)-higher homomorphism from a Γ-ring M into a left ΓM - module X , then for all a, $b \in$ M, $\alpha \in \Gamma$ and n ∈ N, we define

$$G_n(a,b)_\alpha = \phi_n(a\alpha b) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\tau^i(b))$$

Lemma (2.10): Let $\theta = (\phi_i)_{i\in N}$ be a Jordan (σ,τ)-higher homomorphism from a Γ-ring M into a left ΓM - module X , then for all a, b, $c \in$ M, α, β ∈ Γ and n ∈ N:

(i) $G_n(a,b)_\alpha = -\,G_n(b,a)_\alpha$
(ii) $G_n(a + b,c)_\alpha = G_n(a,c)_\alpha + G_n(b,c)_\alpha$
(iii) $G_n(a,b + c)_\alpha = G_n(a,b)_\alpha + G_n(a,c)_\alpha$
(iv) $G_n(a,b)_{\alpha+\beta} = G_n(a,b)_\alpha + G_n(a,b)_\beta$

Proof:

(i) By Definition (2.3)

$$\phi_n(a\alpha b + b\,\alpha a) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\tau^i(b)) + \sum_{i=1}^{n} \phi_i(\sigma^i(b))\alpha\phi_i(\tau^i(a))$$

$$\phi_n(a\alpha b) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\,\alpha\,\phi_i(\tau^i(b)) = -\,(\phi_n(b\alpha a) - \sum_{i=1}^{n} \phi_i(\sigma^i(b))\,\alpha\,\phi_i(\tau^i(a)))$$

$G_n(a,b)_\alpha = -\,G_n(b,a)_\alpha$

(ii) $$G_n(a+b,c)_\alpha = \phi_n((a+b)\alpha c) - \sum_{i=1}^{n} \phi_i(\sigma^i(a+b))\alpha\phi_i(\tau^i(c))$$

$$= \phi_n(a\,\alpha c + b\alpha c) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\tau^i(c)) - \sum_{i=1}^{n} \phi_i(\sigma^i(b))\alpha\phi_i(\tau^i(c))$$

$$= \phi_n(a\,\alpha c) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\tau^i(c)) + \phi_n(b\alpha c) - \sum_{i=1}^{n} \phi_i(\sigma^i(b))\alpha\phi_i(\tau^i(c))$$

$= G_n(a,c)_\alpha + G_n(b,c)_\alpha$

(iii) $$G_n(a,b+c)_\alpha = \phi_n(a\,\alpha(b+c)) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\tau^i(b+c))$$

$$= \phi_n(a\alpha b + a\alpha c) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\,\alpha\,\phi_i(\tau^i(b)) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\,\alpha\,\phi_i(\tau^i(c))$$

$= \phi_n(a\alpha b) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\,\alpha\,\phi_i(\tau^i(b)) + \phi_n(a\alpha c) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\,\alpha\,\phi_i(\tau^i(c))$

$= G_n(a,b)_\alpha + G_n(a,c)_\alpha$

(iv) $G_n(a,b)_{\alpha+\beta} = \phi_n(a(\alpha+\beta)b) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))(\alpha+\beta)\phi_i(\tau^i(b))$

$= \phi_n(a\alpha b) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\tau^i(b)) + \phi_n(a\beta b) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\beta\phi_i(\tau^i(b))$

$= G_n(a,b)_\alpha + G_n(a,b)_\beta$

Remark (2.11): Note that $\theta = (\phi_i)_{i\in N}$ is a (σ,τ)-higher homomorphism from a Γ-ring M into a left ΓM - module X if and only if $G_n(a,b)_\alpha = 0$ for all $a, b \in$ M, $\alpha \in \Gamma$ and $n \in N$.

Lemma (2.12): Let $\theta = (\phi_i)_{i\in N}$ be a Jordan (σ,τ)-higher homomorphism of a Γ-ring M into a left ΓM - module X , such that $\sigma^{n^2} = \sigma^n$, $\tau^n\sigma^n = \sigma^n$, $\sigma^i\tau^{n-i} = \tau^i\sigma^i$, $\sigma^2 = \sigma$, $\tau^2 = \tau$ and $\sigma^i\tau^i = \tau^i\sigma^i$ for all $i \in N$, then for all $a, b, m \in M$, $\alpha, \beta \in \Gamma$ and $n \in N$

(i) $G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(a))_\alpha +$
$G_n(\sigma^n(b),\sigma^n(a))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(a),\tau^n(b))_\alpha = 0$

(ii) $G_n(\sigma^n(a),\sigma^n(b))_\alpha\alpha\phi_n(\sigma^n(m))\alpha G_n(\tau^n(b),\tau^n(a))_\alpha +$
$G_n(\sigma^n(b),\sigma^n(a))_\alpha\alpha\phi_n(\sigma^n(m))\alpha G_n(\tau^n(a),\tau^n(b))_\alpha = 0$

(iii) $G_n(\sigma^n(a),\sigma^n(b))_\beta\alpha\phi_n(\sigma^n(m))\alpha G_n(\tau^n(b),\tau^n(a))_\beta +$
$G_n(\sigma^n(b),\sigma^n(a))_\beta\alpha\phi_n(\sigma^n(m))\alpha G_n(\tau^n(a),\tau^n(b))_\beta = 0.$

Proof:

(i) We prove by using the induction, if n = 1

Let $w = a\alpha b\beta m\beta b\alpha a + b\alpha a\beta m\beta a\alpha b$, since θ is a Jordan (σ,τ)-homomorphism

$\theta(w) = \theta(a\alpha(b\beta m\beta b)\alpha a + b\alpha(a\beta m\beta a)\alpha b)$
$= \theta(\sigma(a))\alpha\theta(\sigma\tau(b\beta m\beta b))\alpha\theta(\tau(a)) + \theta(\sigma(b))\alpha\theta(\sigma\tau(a\beta m\beta a))\alpha\theta(\tau(b))$
$= \theta(\sigma(a))\alpha\theta(\sigma(\sigma\tau(b)))\beta\theta(\sigma\tau(\sigma\tau(m)))\beta\theta(\tau(\sigma\tau(b)))\alpha\theta(\tau(a)) +$
$\theta(\sigma(b))\alpha\theta(\sigma(\sigma\tau(a)))\beta\theta(\sigma\tau(\sigma\tau(m)))\beta\theta(\tau(\sigma\tau(a)))\alpha\theta(\tau(b))$ …(1)

On the other hand

$\theta(w) = \theta((a\alpha b)\beta m\beta(b\alpha a) + (b\alpha a)\beta m\beta(a\alpha b))$
$= \theta(\sigma(a\alpha b))\beta\theta(\sigma\tau(m))\beta\theta(\tau(b\alpha a)) + \theta(\sigma(b\alpha a))\beta\theta(\sigma\tau(m))\beta\theta(\tau(a\alpha b))$
$= \theta(\sigma(a\alpha b))\beta\theta(\sigma\tau(m))\beta(\theta(\sigma\tau(a))\alpha\theta(\tau^2(b)) + \theta(\sigma\tau(b))\alpha\theta(\tau^2(a)) -$
$\theta(\tau(a\alpha b))) + (-\theta(\sigma(a\alpha b)) + \theta(\sigma^2(a))\alpha\theta(\tau\sigma(b)) + \theta(\sigma^2(b))\alpha$
$\theta(\tau\sigma(a)))\beta\theta(\sigma\tau(m))\beta\theta(\tau(a\alpha b))$
$= -\theta(\sigma(a\alpha b))\beta\theta(\sigma\tau(m))\beta(\theta(\tau(a\alpha b)) - \theta(\sigma\tau(a))\alpha\theta(\tau^2(b))) -$
$\theta(\sigma(a\alpha b))\beta\theta(\sigma\tau(m))\beta(\theta(\tau(a\alpha b)) - \theta(\sigma\tau(b))\alpha\theta(\tau^2(a))) +$
$\theta(\sigma^2(a))\alpha\theta(\tau\sigma(b))\beta\theta(\sigma\tau(m))\beta\theta(\tau(a\alpha b)) + \theta(\sigma^2(b))\alpha\theta(\tau\sigma(a))\beta\theta(\sigma\tau(m))\beta\theta(\tau(a\alpha b))$ …(2)

Compare (1), (2) and since $\sigma\tau = \tau\sigma$

$0 = -\theta(\sigma(a\alpha b))\beta\theta(\sigma\tau(m))\beta G(\tau(a),\tau(b))_\alpha - \theta(\sigma(a\alpha b))\beta\theta(\sigma\tau(m))\beta G(\tau(b),\tau(a))_\alpha +$
$\theta(\sigma^2(a))\alpha\theta(\tau\sigma(b))\beta\theta(\sigma\tau(m))\beta\theta(\tau(a\alpha b)) + \theta(\sigma^2(b))\alpha\theta(\tau\sigma(a))\beta\theta(\sigma\tau(m))\beta$
$\theta(\tau(a\alpha b)) - \theta(\sigma(a))\alpha\theta(\sigma^2\tau(b))\beta\theta(\sigma^2\tau^2(m))\beta\theta(\sigma\tau^2(b))\alpha\theta(\tau(a)) -$
$\theta(\sigma(b))\alpha\theta(\sigma^2\tau(a))\beta\theta(\sigma^2\tau^2(m))\beta\theta(\sigma\tau^2(a))\alpha\theta(\tau(b))$

Since $\sigma^2 = \sigma$ and $\tau^2 = \tau$

$0 = -\theta(\sigma(a\alpha b))\beta\theta(\sigma\tau(m))\beta G(\tau(a),\tau(b))_\alpha - \theta(\sigma(a\alpha b))\beta\theta(\sigma\tau(m))\beta G(\tau(b),\tau(a))_\alpha +$
$\theta(\sigma(a))\alpha\theta(\tau\sigma(b))\beta\theta(\sigma\tau(m))\beta\theta(\tau(a\alpha b)) + \theta(\sigma(b))\alpha\theta(\tau\sigma(a))\beta\theta(\sigma\tau(m))\beta$
$\theta(\tau(a\alpha b)) - \theta(\sigma(a))\alpha\theta(\tau\sigma(b))\beta\theta(\sigma\tau(m))\beta\theta(\sigma\tau(b))\alpha\theta(\tau(a)) -$
$\theta(\sigma(b))\alpha\theta(\tau\sigma(a))\beta\theta(\sigma\tau(m))\beta\theta(\sigma\tau(a))\alpha\theta(\tau(b))$

$0 = -\theta(\sigma(a\alpha b))\beta\theta(\sigma\tau(m))\beta G(\tau(a),\tau(b))_\alpha - \theta(\sigma(a\alpha b))\beta\theta(\sigma\tau(m))\beta G(\tau(b),\tau(a))_\alpha +$
$\theta(\sigma(a))\alpha\theta(\tau\sigma(b))\beta\theta(\sigma\tau(m))\beta(\theta(\tau(a\alpha b)) - \theta(\sigma\tau(b))\alpha\theta(\tau(a))) +$
$\theta(\sigma(b))\alpha\theta(\tau\sigma(a))\beta\theta(\sigma\tau(m))\beta(\theta(\tau(a\alpha b)) - \theta(\sigma\tau(a))\alpha\theta(\tau(b)))$

$0 = -\theta(\sigma(a\alpha b))\beta\theta(\sigma\tau(m))\beta G(\tau(a),\tau(b))_\alpha - \theta(\sigma(a\alpha b))\beta\theta(\sigma\tau(m))\beta G(\tau(b),\tau(a))_\alpha +$

$\theta(\sigma(a))\alpha\theta(\tau\sigma(b))\beta\theta(\sigma\tau(m))\beta G(\tau(b),\tau(a))_\alpha + \theta(\sigma(b))\alpha\theta(\tau\sigma(a))\beta$
$\theta(\sigma\tau(m))\beta G(\tau(a),\tau(b))_\alpha$

$0 = -(\theta(\sigma(a\alpha b)) - \theta(\sigma(b))\alpha\theta(\tau\sigma(a)))\beta\theta(\sigma\tau(m))\beta G(\tau(a),\tau(b))_\alpha -$
$(\theta(\sigma(a\alpha b)) - \theta(\sigma(a))\alpha\theta(\tau\sigma(b)))\beta\theta(\sigma\tau(m))\beta G(\tau(b),\tau(a))_\alpha$

Thus, we have:

$G(\sigma(a),\sigma(b))_\alpha\beta\theta(\sigma\tau(m))\beta G(\tau(b),\tau(a))_\alpha + G(\sigma(b),\sigma(a))_\alpha\beta\theta(\sigma\tau(m))\beta G(\tau(a),\tau(b))_\alpha = 0$

Now, we can assume that:

$$G_s(\sigma^s(a),\sigma^s(b))_\alpha\beta\phi_s(\sigma^s(m))\beta G_s(\tau^s(b),\tau^s(a))_\alpha +$$
$$G_s(\sigma^s(b),\sigma^s(a))_\alpha\beta\phi_s(\sigma^s(m))\beta G_s(\tau^s(a),\tau^s(b))_\alpha = 0$$

for all a, b, m $\in$ M, and s, n $\in$ N, s < n.

Let $w = a\alpha b\beta m\beta b\alpha a + b\alpha a\beta m\beta a\alpha b$

Since θ is a Jordan (σ,τ)-higher homomorphism , then

$\phi_n(w) = \phi_n(a\alpha(b\beta m\beta b)\alpha a + b\alpha(a\beta m\beta a)\alpha b)$

$$= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\sigma^i\tau^{n-i}(b\beta m\beta b))\alpha\phi_i(\tau^i(a)) +$$
$$\sum_{i=1}^{n} \phi_i(\sigma^i(b))\alpha\phi_i(\sigma^i\tau^{n-i}(a\beta m\beta a))\alpha\phi_i(\tau^i(b))$$
$$= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\left(\sum_{j=1}^{i}\phi_j(\sigma^j(\sigma^j\tau^{n-j}(b)))\beta\phi_j(\sigma^j\tau^{n-j}(\sigma^j\tau^{n-j}(m)))\beta\phi_j(\tau^j(\sigma^j\tau^{n-j}(b)))\right)\alpha\phi_i(\tau^i(a)) +$$
$$\sum_{i=1}^{n} \phi_i(\sigma^i(b))\alpha\left(\sum_{j=1}^{i}\phi_j(\sigma^j(\sigma^j\tau^{n-j}(a)))\beta\phi_j(\sigma^j\tau^{n-j}(\sigma^j\tau^{n-j}(m)))\beta\phi_j(\tau^j(\sigma^j\tau^{n-j}(a)))\right)\alpha\phi_i(\tau^i(b))$$
$$= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\sigma^i(\sigma^i\tau^{n-i}(b)))\beta\phi_i(\sigma^i\tau^{n-i}(\sigma^i\tau^{n-i}(m)))\beta\phi_i(\tau^i(\sigma^i\tau^{n-i}(b)))\alpha\phi_i(\tau^i(a)) +$$
$$\sum_{i=1}^{n} \phi_i(\sigma^i(b))\alpha\phi_i(\sigma^i(\sigma^i\tau^{n-i}(a)))\beta\phi_i(\sigma^i\tau^{n-i}(\sigma^i\tau^{n-i}(m)))\beta\phi_i(\tau^i(\sigma^i\tau^{n-i}(a)))\alpha\phi_i(\tau^i(b))$$
$$= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\sigma^i(\sigma^i\tau^{n-i}(b)))\beta\phi_i(\sigma^i\tau^{n-i}(\sigma^i\tau^{n-i}(m)))\beta\sum_{j=1}^{i}\phi_j(\tau^j(\sigma^j\tau^{n-j}(b)))\alpha\phi_j(\tau^j(a)) +$$
$$\sum_{i=1}^{n} \phi_i(\sigma^i(b))\alpha\phi_i(\sigma^i(\sigma^i\tau^{n-i}(a)))\beta\phi_i(\sigma^i\tau^{n-i}(\sigma^i\tau^{n-i}(m)))\beta\sum_{j=1}^{i}\phi_j(\tau^j(\sigma^j\tau^{n-j}(a)))\alpha\phi_j(\tau^j(b))$$
$$= \phi_n(\sigma^n(a))\alpha\phi_n(\sigma^n(\sigma^n(b)))\beta\phi_n(\sigma^n(\sigma^n(m)))\beta\sum_{j=1}^{i}\phi_j(\tau^j(\sigma^j\tau^{n-j}(b)))\alpha\phi_j(\tau^j(a)) +$$
$$\sum_{i=1}^{n-1} \phi_i(\sigma^i(a))\alpha\phi_i(\sigma^i(\sigma^i\tau^{n-i}(b)))\beta\phi_i(\sigma^i\tau^{n-i}(\sigma^i\tau^{n-i}(m)))\beta\sum_{j=1}^{i}\phi_j(\tau^j(\sigma^j\tau^{n-j}(b)))\alpha\phi_j(\tau^j(a)) +$$
$$\phi_n(\sigma^n(b))\alpha\phi_n(\sigma^n(\sigma^n(a)))\beta\phi_n(\sigma^n(\sigma^n(m)))\beta\sum_{j=1}^{i}\phi_j(\tau^j(\sigma^j\tau^{n-j}(a)))\alpha\phi_j(\tau^j(b)) +$$
$$\sum_{i=1}^{n-1} \phi_i(\sigma^i(b))\alpha\phi_i(\sigma^i(\sigma^i\tau^{n-i}(a)))\beta\phi_i(\sigma^i\tau^{n-i}(\sigma^i\tau^{n-i}(m)))\beta\sum_{j=1}^{i}\phi_j(\tau^j(\sigma^j\tau^{n-j}(a)))\alpha\phi_j(\tau^j(b))$$

…(1)

On the other hand:

$\phi_n(w) = \phi_n((a\alpha b)\beta m\beta(b\alpha a) + (b\alpha a)\beta m\beta(a\alpha b))$

$$= \sum_{i=1}^{n} \phi_i(\sigma^i(a\alpha b))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta\phi_i(\tau^i(b\alpha a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(b\alpha a))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta\phi_i(\tau^i(a\alpha b))$$
$$= \sum_{i=1}^{n} \phi_i(\sigma^i(a\alpha b))\beta\phi_i((\sigma^i\tau^{n-i}(m))\beta\left(\sum_{j=1}^{i}\phi_j(\sigma^j\tau^j(a))\alpha\phi_j(\tau^{j^2}(b)) + \sum_{j=1}^{i}\phi_j(\sigma^j\tau^j(b))\alpha\phi_j(\tau^{j^2}(a))\right) - \phi_i(\tau^i(a\alpha b)) +$$
$$\sum_{i=1}^{n}\left(\sum_{j=1}^{i}\phi_j(\sigma^{j^2}(a))\alpha\phi_j(\tau^j\sigma^j(b)) + \sum_{j=1}^{i}\phi_j(\sigma^{j^2}(b))\alpha\phi_j(\tau^j\sigma^j(a)) - \right)\phi_i(\sigma^i(a\alpha b))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta\phi_i(\tau^i(a\alpha b))$$

$$= \sum_{i=1}^{n} \phi_i(\sigma^i(a\alpha b))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta\sum_{j=1}^{i}\phi_j(\sigma^j\tau^j(a))\alpha\phi_j(\tau^{j^2}(b)) + \sum_{i=1}^{n} \phi_i(\sigma^i(a\alpha b))\beta\phi_i((\sigma^i\tau^{n-i}(m))\beta$$
$$\sum_{j=1}^{i}\phi_j(\sigma^j\tau^j(b))\alpha\phi_j(\tau^{j^2}(a)) - \sum_{i=1}^{n} \phi_i(\sigma^i(a\alpha b))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta\phi_i(\tau^i(a\alpha b)) + \sum_{i=1}^{n} \phi_i(\sigma^{i^2}(a))\alpha\phi_i(\tau^i\sigma^i(b))\beta$$
$$\phi_i(\sigma^i\tau^{n-i}(m))\beta\phi_i(\tau^i(a\alpha b)) + \sum_{i=1}^{n} \phi_i(\sigma^{i^2}(b))\alpha\phi_i(\tau^i\sigma^i(a))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta\phi_i(\tau^i(a\alpha b))$$
$$-\sum_{i=1}^{n} \phi_i(\sigma^i(a\alpha b))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta\phi_i(\tau^i(a\alpha b))$$
$$= -\sum_{i=1}^{n} \phi_i(\sigma^i(a\alpha b))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta(\phi_i(\tau^i(a\alpha b)) - \sum_{j=1}^{i}\phi_j(\sigma^j\tau^j(a))\alpha\phi_j(\tau^{j^2}(b))) -$$
$$\sum_{i=1}^{n} \phi_i(\sigma^i(a\alpha b))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta(\phi_i(\tau^i(a\alpha b)) - \sum_{j=1}^{i}\phi_j(\sigma^j\tau^j(b))\alpha\phi_j(\tau^{j^2}(a))) + \sum_{i=1}^{n} \phi_i(\sigma^{i^2}(a))\alpha\phi_i(\tau^i\sigma^i(b))$$
$$\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta\phi_i(\tau^i(a\alpha b)) + \sum_{i=1}^{n} \phi_i(\sigma^{i^2}(b))\alpha\phi_i(\tau^i\sigma^i(a))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta\phi_i(\tau^i(a\alpha b))$$
$$= -\phi_n(\sigma^n(a\alpha b))\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(a),\tau^n(b))_\alpha - \sum_{i=1}^{n-1} \phi_i(\sigma^i(a\alpha b))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta G_i(\tau^i(a),\tau^i(b))_\alpha -$$

$$\phi_n(\sigma^n(a\alpha b))\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(a))_\alpha - \sum_{i=1}^{n-1}\phi_i(\sigma^i(a\alpha b))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta G_i(\tau^i(a),\tau^i(b))_\alpha +$$
$$\phi_n(\sigma^{n^2}(a))\alpha\phi_n(\tau^n\sigma^n(b))\beta\phi_n(\sigma^n(m))\beta\phi_n(\tau^n(a\alpha b)) + \sum_{i=1}^{n-1}\phi_i(\sigma^{i^2}(a))\alpha\phi_i(\tau^i\sigma^i(b))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta\phi_i(\tau^i(a\alpha b)) +$$
$$\phi_n(\sigma^{n^2}(b))\alpha\phi_n(\tau^n\sigma^n(a))\beta\phi_n(\sigma^n(m))\beta\phi_n(\tau^n(a\alpha b)) + \sum_{i=1}^{n-1}\phi_i(\sigma^{i^2}(b))\alpha\phi_i(\tau^i\sigma^i(a))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta\phi_i(\tau^i(a\alpha b))\ldots(2)$$

Compare (1), (2) and since $\sigma^{n^2}=\sigma^n$, $\tau^n\sigma^n=\sigma^n$, $\sigma^i\tau^{n-i}=\tau^i\sigma^i$, $\sigma^i\tau^i=\tau^i\sigma^i$

$$0=-\phi_n(\sigma^n(a\alpha b))\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(a),\tau^n(b))_\alpha - \phi_n(\sigma^n(a\alpha b))\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(a))_\alpha +$$
$$\phi_n(\sigma^n(a))\alpha\phi_n(\sigma^n(b))\beta\phi_n(\sigma^n(m))\beta(\phi_n(\tau^n(a\alpha b)) - \sum_{i=1}^{n}\phi_i(\tau^i(\sigma^i\tau^{n-i}(b)))\alpha\phi_i(\tau^i(a))) +$$
$$\phi_n(\sigma^n(b))\alpha\phi_n(\sigma^n(a))\beta\phi_n(\sigma^n(m))\beta(\phi_n(\tau^n(a\alpha b)) - \sum_{i=1}^{n}\phi_i(\tau^i(\sigma^i\tau^{n-i}(a)))\alpha\phi_i(\tau^i(b))) -$$
$$\sum_{i=1}^{n-1}\phi_i(\sigma^i(a\alpha b))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta G_i(\tau^i(a),\tau^i(b))_\alpha - \sum_{i=1}^{n-1}\phi_i(\sigma^i(a\alpha b))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta$$
$$G_i(\tau^i(b),\tau^i(a))_\alpha + \sum_{i=1}^{n-1}\phi_i(\sigma^i(a))\alpha\phi_i(\tau^i\sigma^i(b))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta(\phi_i(\tau^i(a\alpha b)) -$$
$$\sum_{j=1}^{i}\phi_j(\tau^j(\sigma^j\tau^{n-j}(b)))\alpha\phi_j(\tau^j(a))) + \sum_{i=1}^{n-1}\phi_i(\sigma^i(b))\alpha\phi_i(\tau^i\sigma^i(a))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta$$
$$(\phi_i(\tau^i(a\alpha b)) - \sum_{j=1}^{i}\phi_j(\tau^j(\sigma^j\tau^{n-j}(a)))\alpha\phi_j(\tau^j(b)))$$

$$0=-\phi_n(\sigma^n(a\alpha b))\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(a),\tau^n(b))_\alpha - \phi_n(\sigma^n(a\alpha b))\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(a))_\alpha +$$
$$\phi_n(\sigma^n(a))\alpha\phi_n(\sigma^n(b))\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(a))_\alpha +$$

$$\phi_n(\sigma^n(b))\alpha\phi_n(\sigma^n(a))\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(a),\tau^n(b))_\alpha -$$
$$\sum_{i=1}^{n-1}\phi_i(\sigma^i(a\alpha b))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta G_i(\tau^i(a),\tau^i(b))_\alpha -$$
$$\sum_{i=1}^{n-1}\phi_i(\sigma^i(a\alpha b))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta G_i(\tau^i(b),\tau^i(a))_\alpha +$$
$$\sum_{i=1}^{n-1}\phi_i(\sigma^i(a))\alpha\phi_i(\tau^i\sigma^i(b))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta G_i(\tau^i(b),\tau^i(a))_\alpha +$$
$$\sum_{i=1}^{n-1}\phi_i(\sigma^i(b))\alpha\phi_i(\tau^i\sigma^i(a))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta G_i(\tau^i(a),\tau^i(b))_\alpha$$

$$0=-G_n(\sigma^n(b),\sigma^n(a))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(a),\tau^n(b))_\alpha -$$
$$G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(a))_\alpha -$$
$$\sum_{i=1}^{n-1}G_i(\sigma^i(b),\sigma^i(a))_\alpha\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta G_i(\tau^i(a),\tau^i(b))_\alpha -$$
$$\sum_{i=1}^{n-1}G_i(\sigma^i(a),\sigma^i(b))_\alpha\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta G_i(\tau^i(b),\tau^i(a))_\alpha$$

$$0=-(G_n(\sigma^n(b),\sigma^n(a))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(a),\tau^n(b))_\alpha +$$
$$G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(a))_\alpha) -$$
$$(\sum_{i=1}^{n-1}G_i(\sigma^i(b),\sigma^i(a))_\alpha\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta G_i(\tau^i(a),\tau^i(b))_\alpha +$$
$$\sum_{i=1}^{n-1}G_i(\sigma^i(a),\sigma^i(b))_\alpha\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta G_i(\tau^i(b),\tau^i(a))_\alpha)$$

By our hypothesis, we have:

$$G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(a))_\alpha +$$
$$G_n(\sigma^n(b),\sigma^n(a))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(a),\tau^n(b))_\alpha = 0$$

(ii) Replace β by α in (i) proceeding in the same way as in the proof of (i) by the similar arguments, we get (ii).

(iii) Interchanging α and β in (i), we get (iii).

Lemma (2.13): Let $\theta = (\phi_i)_{i\in N}$ be a Jordan (σ,τ)-higher homomorphism of a Γ-ring M into a 2- torsion free prime left ΓM - module X , then for all *a*, *b*, m ∈ M, α, β∈Γ and n ∈ N

(i) $G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(a))_\alpha =$
$G_n(\sigma^n(b),\sigma^n(a))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(a),\tau^n(b))_\alpha = 0$

(ii) $G_n(\sigma^n(a),\sigma^n(b))_\alpha\alpha\phi_n(\sigma^n(m))\alpha G_n(\tau^n(b),\tau^n(a))_\alpha =$
$G_n(\sigma^n(b),\sigma^n(a))_\alpha\alpha\phi_n(\sigma^n(m))\alpha G_n(\tau^n(a),\tau^n(b))_\alpha = 0$

(iii) $G_n(\sigma^n(a),\sigma^n(b))_\beta\alpha\phi_n(\sigma^n(m))\alpha G_n(\tau^n(b),\tau^n(a))_\beta =$
$G_n(\sigma^n(b),\sigma^n(a))_\beta\alpha\phi_n(\sigma^n(m))\alpha G_n(\tau^n(a),\tau^n(b))_\beta = 0$

Proof:

(i) By Lemma (2.12) (i), we have:

$$G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(a))_\alpha +$$
$$G_n(\sigma^n(b),\sigma^n(a))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(a),\tau^n(b))_\alpha = 0$$

And by Lemma (Let X be a 2-torsion free semiprime left ΓM- module X and suppose that $a, b \in$ ΓM- module X if $a\Gamma X\Gamma b + b\Gamma X\Gamma a = 0$ for all $x \in X$, then $a\Gamma X\Gamma b = b\Gamma X\Gamma a = 0$), we get:

$$G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(a))_\alpha =$$
$$G_n(\sigma^n(b),\sigma^n(a))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(a),\tau^n(b))_\alpha = 0$$

(ii) Replace β for α in (i), we get (ii).

(iii) Interchanging α and β in (i), we get (iii).

Theorem (2.14): Let $\theta = (\phi_i)_{i\in N}$ be a Jordan (σ,τ)-higher homomorphism of a Γ-ring M into a prime left ΓM - module X , then for all a, b, c, d, m ∈ M, α, β∈Γ and n ∈ N

(i) $G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(d),\tau^n(c))_\alpha = 0$

(ii) $G_n(\sigma^n(a),\sigma^n(b))_\alpha\alpha\phi_n(\sigma^n(m))\alpha G_n(\tau^n(d),\tau^n(c))_\alpha = 0$

(iii) $G_n(\sigma^n(a),\sigma^n(b))_\alpha\alpha\phi_n(\sigma^n(m))\alpha G_n(\tau^n(d),\tau^n(c))_\beta = 0$

Proof:

(i) Replacing $a + c$ for a in Lemma (2.13) (i), we get:

$$G_n(\sigma^n(a+c),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(a+c))_\alpha = 0$$
$$G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(a))_\alpha +$$
$$G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(c))_\alpha +$$
$$G_n(\sigma^n(c),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(a))_\alpha +$$
$$G_n(\sigma^n(c),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(c))_\alpha = 0$$

By Lemma (2.13)(i), we get:

$$G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(c))_\alpha +$$
$$G_n(\sigma^n(c),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(a))_\alpha = 0$$

Therefore, we get:

$$G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(c))_\alpha\beta\phi_n(\sigma^n(m))\beta$$
$$G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(c))_\alpha = 0$$
$$= -G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(c))_\alpha\beta\phi_n(\sigma^n(m))\beta$$
$$G_n(\sigma^n(c),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(a))_\alpha = 0$$

Hence, by the primness of left ΓM - module X :

$$G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(c))_\alpha = 0 \qquad \ldots(1)$$

Now, replacing $b + d$ for b in Lemma (2.13) (i), we get:

$$G_n(\sigma^n(a),\sigma^n(b+d))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b+d),\tau^n(a))_\alpha = 0$$
$$G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(a))_\alpha +$$
$$G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(d),\tau^n(a))_\alpha +$$
$$G_n(\sigma^n(a),\sigma^n(d))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(a))_\alpha +$$
$$G_n(\sigma^n(a),\sigma^n(d))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(d),\tau^n(a))_\alpha = 0$$

By Lemma (2.13)(i), we get:

$$G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(d),\tau^n(a))_\alpha +$$
$$G_n(\sigma^n(a),\sigma^n(d))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(a))_\alpha = 0$$

Therefore, we get:

$$G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(d),\tau^n(a))_\alpha\beta\phi_n(\sigma^n(m))\beta$$
$$G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(d),\tau^n(a))_\alpha = 0$$
$$= -G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(d),\tau^n(a))_\alpha\beta\phi_n(\sigma^n(m))\beta$$
$$G_n(\sigma^n(a),\sigma^n(d))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(a))_\alpha = 0$$

Since left ΓM - module X is prime , then :

$$G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(d),\tau^n(a))_\alpha = 0 \qquad \ldots(2)$$

Thus , $G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b+d),\tau^n(a+c))_\alpha = 0$

$G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(a))_\alpha +$
$G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(b),\tau^n(c))_\alpha +$
$G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(d),\tau^n(a))_\alpha +$
$G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(d),\tau^n(c))_\alpha = 0$

By (1), (2) and Lemma (2.13)(i), we get:

$G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(d),\tau^n(c))_\alpha = 0$.

(ii) Replace β for α in (i), we get (ii).

(iii) Replacing α + β for α in (ii), we get:

$G_n(\sigma^n(a),\sigma^n(b))_{\alpha+\beta}\alpha\phi_n(\sigma^n(m))\alpha G_n(\tau^n(d),\tau^n(c))_{\alpha+\beta} = 0$
$G_n(\sigma^n(a),\sigma^n(b))_\alpha\alpha\phi_n(\sigma^n(m))\alpha G_n(\tau^n(d),\tau^n(c))_\alpha +$
$G_n(\sigma^n(a),\sigma^n(b))_\alpha\alpha\phi_n(\sigma^n(m))\alpha G_n(\tau^n(d),\tau^n(c))_\beta +$
$G_n(\sigma^n(a),\sigma^n(b))_\beta\alpha\phi_n(\sigma^n(m))\alpha G_n(\tau^n(d),\tau^n(c))_\alpha +$
$G_n(\sigma^n(a),\sigma^n(b))_\beta\alpha\phi_n(\sigma^n(m))\alpha G_n(\tau^n(d),\tau^n(c))_\beta = 0$

By (i) and (ii), we get:

$G_n(\sigma^n(a),\sigma^n(b))_\alpha\alpha\phi_n(\sigma^n(m))\alpha G_n(\tau^n(d),\tau^n(c))_\beta +$
$G_n(\sigma^n(a),\sigma^n(b))_\beta\alpha\phi_n(\sigma^n(m))\alpha G_n(\tau^n(d),\tau^n(c))_\alpha = 0$

Therefore, we have:

$G_n(\sigma^n(a),\sigma^n(b))_\alpha\alpha\phi_n(\sigma^n(m))\alpha G_n(\tau^n(d),\tau^n(c))_\beta\alpha\phi_n(\sigma^n(m))\alpha$
$G_n(\sigma^n(a),\sigma^n(b))_\alpha\alpha\phi_n(\sigma^n(m))\alpha G_n(\tau^n(d),\tau^n(c))_\beta = 0$

$= - G_n(\sigma^n(a),\sigma^n(b))_\alpha\alpha\phi_n(\sigma^n(m))\alpha G_n(\tau^n(d),\tau^n(c))_\beta\alpha\phi_n(\sigma^n(m))\alpha$

$G_n(\sigma^n(a),\sigma^n(b))_\beta\alpha\phi_n(\sigma^n(m))\alpha G_n(\tau^n(d),\tau^n(c))_\alpha = 0$

Since left ΓM - module X is prime , then:

$G_n(\sigma^n(a),\sigma^n(b))_\alpha\alpha\phi_n(\sigma^n(m))\alpha G_n(\tau^n(d),\tau^n(c))_\beta = 0$.

III. The Main Result

Theorem(3.1): Every Jordan (σ,τ)-higher homomorphism from a Γ-ring M into a prime left ΓM - module X is either (σ,τ)-higher homomorphism or (σ,τ)-higher anti homomorphism.

Proof: Let $\theta = (\phi_i)_{i\in N}$ be a Jordan (σ,τ)-higher homomorphism of a Γ-ring M into a prime left ΓM - module X , then by Lemma (2.14)(i)**:**

$G_n(\sigma^n(a),\sigma^n(b))_\alpha\beta\phi_n(\sigma^n(m))\beta G_n(\tau^n(d),\tau^n(c))_\alpha = 0$

Since left ΓM - module X is prime , therefore either $G_n(\sigma^n(a),\sigma^n(b))_\alpha = 0$ or

$G_n(\tau^n(d),\tau^n(c))_\alpha = 0$, for all *a*, *b*, *c*, *d* ∈ M, α,∈Γ and n ∈ N.

If $G_n(\tau^n(d),\tau^n(c))_\alpha \neq 0$, for all *c*, *d* ∈ M, α∈Γ and n ∈ N then

$G_n(\sigma^n(a),\sigma^n(b))_\alpha = 0$, for all *a*, *b* ∈ M, α∈Γ and n ∈ N, hence, we get

θ is a (σ,τ)-higher homomorphism .

But if $G_n(\tau^n(d),\tau^n(c))_\alpha = 0$, for all *c*, *d* ∈ M, α∈Γ and n ∈ N ,then

θ is a (σ,τ)-higher anti homomorphism.

Proposition (3.2): Let $\theta = (\phi_i)_{i\in N}$ be a Jordan (σ,τ)-higher homomorphism from a Γ-ring M into a 2-torsion free left ΓM - module X , such that *a*αbβc = *a*βbαc, for all *a*, b, c ∈ M and α ,β ∈ Γ, $\sigma^{i^2} = \sigma^i$, $\tau^{i^2} = \tau^i$, $\sigma^i\tau^i = \sigma^i\tau^{n-i}$ and $\sigma^i\tau^i = \tau^i\sigma^i$, then θ is a Jordan triple (σ,τ) -higher homomorphism.

Proof: Replace *b* by *a*β*b* + *b*β*a* in Definition (2.3) , we get :

$\phi_n(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a)$

$= \sum_{i=1}^{n}\phi_i(\sigma^i(a))\alpha\phi_i(\tau^i(a\beta b + b\beta a)) + \sum_{i=1}^{n}\phi_i(\sigma^i(a\beta b + b\beta a))\alpha\phi_i(\tau^i(a))$

$= \sum_{i=1}^{n}\phi_i(\sigma^i(a))\alpha\phi_i(\tau^i(a)\beta\tau^i(b) + \tau^i(b)\beta\tau^i(a)) +$

$\sum_{i=1}^{n}\phi_i(\sigma^i(a)\beta\sigma^i(b) + \sigma^i(b)\beta\sigma^i(a))\alpha\phi_i(\tau^i(a))$

$$= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\left(\sum_{j=1}^{i}\phi_j(\sigma^j\tau^j(a))\beta\phi_j(\tau^{j^2}(b)) + \sum_{j=1}^{i}\phi_j(\sigma^j\tau^j(b))\beta\phi_j(\tau^{j^2}(a))\right) +$$
$$\sum_{i=1}^{n}\left(\sum_{j=1}^{i}\phi_j(\sigma^{j^2}(a))\beta\phi_j(\tau^j\sigma^j(b)) + \sum_{j=1}^{i}\phi_j(\sigma^{j^2}(b))\beta\phi_j(\tau^j\sigma^j(a))\right)\alpha\phi_i\tau^i(a))$$
$$= \sum_{i=1}^{n}\phi_i(\sigma^i(a))\alpha\phi_i(\sigma^i\tau^i(a))\beta\phi_i(\tau^{i^2}(b)) + \sum_{i=1}^{n}\phi_i(\sigma^i(a))\alpha\phi_i(\sigma^i\tau^i(b))\beta\phi_i(\tau^{i^2}(a)) +$$
$$\sum_{i=1}^{n}\phi_i(\sigma^{i^2}(a))\beta\phi_i(\tau^i\sigma^i(b))\alpha\phi_i(\tau^i(a)) + \sum_{i=1}^{n}\phi_i(\sigma^{i^2}(b))\beta\phi_i(\sigma^i\tau^i(a))\alpha\phi_i(\tau^i(a))$$

Since X is left ΓM - module , $\sigma^{i^2} = \sigma^i$, $\tau^{i^2} = \tau^i$, $\sigma^i\tau^i = \sigma^i\tau^{n-i}$ and $\sigma^i\tau^i = \tau^i\sigma^i$, we get

$$= \sum_{i=1}^{n}\phi_i(\sigma^i(a))\alpha\phi_i(\sigma^i\tau^{n-i}(a))\beta\phi_i(\tau^i(b)) + 2\sum_{i=1}^{n}\phi_i(\sigma^i(a))\alpha\phi_i(\sigma^i\tau^{n-i}(b))\beta\phi_i(\tau^i(a)) +$$
$$\sum_{i=1}^{n}\phi_i(\sigma^i(b))\beta\phi_i(\sigma^i\tau^{n-i}(a))\alpha\phi_i(\tau^i(a)) \qquad \ldots(1)$$

On the other hand:

$\phi_n(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a) = \phi_n(a\alpha a\beta b + a\alpha b\beta a + a\beta b\alpha a + b\beta a\alpha a)$

Since $a\alpha b\beta c = a\beta b\alpha c$, for all $a, b, c \in M$ and $\alpha,\beta \in \Gamma$

$= \phi_n(a\alpha a\beta b + b\beta a\alpha a) + 2\phi_n(a\alpha b\beta a)$

$$= \sum_{i=1}^{n}\phi_i(\sigma^i(a))\alpha\phi_i(\sigma^i\tau^{n-i}(a))\beta\phi_i(\tau^i(b)) + \sum_{i=1}^{n}\phi_i(\sigma^i(b))\beta\phi_i(\sigma^i\tau^{n-i}(a))\alpha\phi_i(\tau^i(a)) +$$
$$2\phi_n(a\alpha b\beta a) \qquad \ldots(2)$$

Compare (1) and (2), we get:

$2\phi_n(a\alpha b\beta a) = 2\sum_{i=1}^{n}\phi_i(\sigma^i(a))\,\alpha\,\phi_i(\sigma^i\tau^{n-i}(b))\,\beta\,\phi_i(\tau^i(a))$.

Since X is a 2-torsion free left ΓM - module , we obtain that θ is a Jordan triple (σ,τ)-higher homomorphism .

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