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Some Fixed Point and Common Fixed Point Theorem in Banach Space

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***Abstract**: In this paper we prove fixed point and common fixed point Theorem in non-expansive and existence mappings using Banach space for new rational expression. Which generalize the well known result.*
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***Keywords**: Banach space, Common fixed point, fixed point, rational expression.*

I. Introduction

Fixed point has drawn the attentions of the authors working in non-linear analysis, the study of non-expansive mapping and the existence of fixed point. The non-expansive mappings include contraction as well as contractive mappings. Browder [1] was the first mathematician to study non-expansive mappings; he applied these results for proving the existence of solutions of certain integral equations. Browder [1], Gohde [3] and Kirk [11] have independently proved a fixed point theorem for non-expansive mappings defined on a closed bounded and convex subset of a uniformly convex Banach space and in the spaces with richer generalizations of non-expansive mappings, prominent being Datson [12], Emmanuele [2], Goebel [4], Goebel and Zlotkienwicz [5], Iseki [6], Sharma & Rajput [9], Singh and Chatterjee [8]. They have derived valuable results with non-contraction mapping in Banach space. It is well known that a Banach space is a linear space which is also in a special way a complete metric space. The combination of algebraic and metric structures opens up the possibility of studying linear transformation of one Banach space into another which has the additional property of being continuous. A normed linear space is a linear space N in which to each vector z, there corresponds a real number denoted by $\|x\|$ and called the norm of x in such a manner that

(i) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$
(ii) $\|x + y\| \leq \|x\| + \|y\|$
(iii) $\|\alpha x\| = |\alpha|\|x\|$ β is real

The non negative real number $\|x\|$ is to be thought of as the length of vector x. If we regard $\|x\|$ as a real function defined on N. It is easy to verify that the normed function is called norm on N. It is easy to verify that the normed linear space N is a metric space w.r.to the metric d defined by $d(x, y) = \|x - y\|$. A Banach space is a complete normed linear space. Let X be a Banach space on the complex field numbers E, and E be a non empty convex subset of X. A map T : C→ C is said to be an mean nonexpansive mapping. A mapping T on a subset E of a Banach space X is called a nonexpansive mapping if $\|Tx - Ty\| \leq x - y$ for all x, y ∈ E.

Our object in this chapter is to prove some fixed and common fixed point theorems in non-expansive and existence mappings using Banach space.

Definition: Let K be a nonempty subset of a real normed space E. Let T be a self-mapping of K. Then T is said to be non expansive if $\|Tx - Ty\| \leq \|x - y\|$ for all x, y ∈ K.

II. Main Result

Theorem 2.1: Let T be a mapping of a Banach spaces into itself. If T satisfies the following conditions:

1. $T^2 = I$, where I is identity mapping.

2.

$$\|Tx - Ty\| \geq \alpha\left[\frac{\|x - Tx\|^2\|y - Ty\| + \|x - Tx\|\|y - x\|^2}{2\|x - Tx\|\|y - Ty\|}\right] + \beta\left[\frac{\|x - Tx\|^2\|x - Ty\| + \|y - x\|^2\|x - Tx\|}{\|x - Tx\|\|y - Ty\| + \|x - Tx\|\|y - x\|}\right] + \gamma\left[\frac{\|x - Tx\| + \|y - Ty\|}{2}\right] + \delta\left[\frac{\|x - Ty\| + \|y - Tx\|}{2}\right] + \eta\|x - y\| + \xi\left[\frac{\|x - Tx\|\|y - Ty\|}{\|x - y\|}\right]$$

For every $x \neq y$ and $20\alpha + 4\beta + 8\gamma + 8\delta + 4\eta + 16\xi > 8$.Then T has a unique fixed point in X.

Proof : Suppose x is any point in Banach space. Taking $y = \frac{1}{2}(T + I)x$, $z = T(y)$, $v = 2y - z$, we have

$\|z - x\| = \|Ty - T^2x\| = \|Ty - T(Tx)\|$

$$\|z-x\| \geq \alpha\left[\frac{\|y-Ty\|^2\|Tx-T(Tx)\|+\|y-Ty\|\|Tx-y\|^2}{2\|y-Ty\|\|Tx-T(Tx)\|}\right] + \beta\left[\frac{\|y-Ty\|^2\|y-T(Tx)\|+\|Tx-y\|^2\|y-Ty\|}{\|y-Ty\|\|Tx-T(Tx)\|+\|y-Ty\|\|Tx-y\|}\right]+\gamma\left[\frac{\|y-Ty\|+\|Tx-T(Tx)\|}{2}\right] + \delta\left[\frac{\|y-T(Tx)\|+\|Tx-Ty\|}{2}\right]+\eta\|y-Tx\|+\xi\left[\frac{\|y-Ty\|\|Tx-T(Tx)\|}{\|y-Tx\|}\right]$$

$$\|z-x\| \geq \alpha\left[\frac{\|y-Ty\|^2\|Tx-x\|+\|y-Ty\|\|Tx-y\|^2}{2\|y-Ty\|\|Tx-x\|}\right]+\beta\left[\frac{\|y-Ty\|^2\|y-x\|+\|Tx-y\|^2\|y-Ty\|}{\|y-Ty\|\|Tx-x\|+\|y-Ty\|\|Tx-y\|}\right] + \gamma\left[\frac{\|y-Ty\|+\|Tx-x\|}{2}\right]+\delta\left[\frac{\|y-x\|+\|Tx-y\|+\|y-Ty\|}{2}\right]+\eta\|y-Tx\| + \xi\left[\frac{\|y-Ty\|\|Tx-x\|}{\|y-Tx\|}\right]$$

$$\|z-x\| \geq \alpha\left[\frac{\|y-Ty\|^2\|Tx-x\|+\|y-Ty\|\left\|Tx-\frac{1}{2}(T+I)x\right\|^2}{2\|y-Ty\|\|Tx-x\|}\right] + \beta\left[\frac{\|y-Ty\|^2\left\|\frac{1}{2}(T+I)x-x\right\|+\left\|Tx-\frac{1}{2}(T+I)x\right\|^2\|y-Ty\|}{\|y-Ty\|\|Tx-x\|+\|y-Ty\|\left\|Tx-\frac{1}{2}(T+I)x\right\|}\right] + \gamma\left[\frac{\|y-Ty\|+\|Tx-x\|}{2}\right]+\delta\left[\frac{\left\|\frac{1}{2}(T+I)x-x\right\|+\left\|Tx-\frac{1}{2}(T+I)x\right\|+\|y-Ty\|}{2}\right] + \eta\left\|\frac{1}{2}(T+I)x-Tx\right\|+\xi\left[\frac{\|y-Ty\|\|Tx-x\|}{\left\|\frac{1}{2}(T+I)x-Tx\right\|}\right]$$

$$\|z-x\| \geq \alpha\left[\frac{\|y-Ty\|^2\|Tx-x\|+\|y-Ty\|\frac{1}{4}\|Tx-x\|^2}{2\|y-Ty\|\|Tx-x\|}\right] + \beta\left[\frac{\|y-Ty\|^2\frac{1}{2}\|Tx-x\|+\frac{1}{4}\|Tx-x\|^2\|y-Ty\|}{\|y-Ty\|\|Tx-x\|+\|y-Ty\|\frac{1}{2}\|Tx-x\|}\right]+\gamma\left[\frac{\|y-Ty\|+\|Tx-x\|}{2}\right] + \delta\left[\frac{\frac{1}{2}\|Tx-x\|+\frac{1}{2}\|Tx-x\|+\|y-Ty\|}{2}\right]+\eta\frac{1}{2}\|x-Tx\|+\xi\left[\frac{\|y-Ty\|\|Tx-x\|}{\frac{1}{2}\|x-Tx\|}\right]$$

$$\|z-x\| \geq \alpha\left[\frac{\|y-Ty\|\|Tx-x\|\left(\|y-Ty\|+\frac{1}{4}\|Tx-x\|\right)}{2\|y-Ty\|\|Tx-x\|}\right] + \beta\left[\frac{\|y-Ty\|\|Tx-x\|\left(\frac{1}{2}\|y-Ty\|+\frac{1}{4}\|x-Tx\|\right)}{\frac{3}{2}\|y-Ty\|\|Tx-x\|}\right]+\gamma\left[\frac{\|y-Ty\|+\|Tx-x\|}{2}\right] + \delta\left[\frac{\|Tx-x\|+\|y-Ty\|}{2}\right]+\frac{\eta}{2}\|x-Tx\|+2\xi\|y-Ty\|$$

$$\|z-x\| \geq \frac{\alpha}{2}\|y-Ty\|+\frac{\alpha}{8}\|Tx-x\|+\frac{\beta}{3}\|y-Ty\|+\frac{\beta}{6}\|Tx-x\|+\frac{\gamma}{2}\|y-Ty\|+\frac{\gamma}{2}\|Tx-x\|+\frac{\delta}{2}\|Tx-x\| + \frac{\delta}{2}\|Ty-y\|+\frac{\eta}{2}\|Tx-x\|+2\xi\|y-Ty\|$$

$$\|z-x\| \geq \left(\frac{\alpha}{2}+\frac{\beta}{3}+\frac{\gamma}{2}+\frac{\delta}{2}+2\xi\right)\|y-Ty\|+\left(\frac{\alpha}{8}+\frac{\beta}{6}+\frac{\gamma}{2}+\frac{\delta}{2}+\frac{\eta}{2}\right)\|Tx-x\| \tag{2.1}$$

$$\|v-x\| = \|2y-z-x\| = \left\|2\frac{1}{2}(T+I)x-Ty-x\right\| = \|Tx-Ty\|$$

$$\|v-x\| \ge \alpha\left[\frac{\|y-Ty\|^2\|x-Tx\| + \|y-Ty\|\|x-y\|^2}{2\|y-Ty\|\|x-Tx\|}\right] + \beta\left[\frac{\|y-Ty\|^2\|y-Tx\| + \|x-y\|^2\|y-Ty\|}{\|y-Ty\|\|x-Tx\| + \|y-Ty\|\|x-y\|}\right]$$
$$+ \gamma\left[\frac{\|y-Ty\| + \|x-Tx\|}{2}\right] + \delta\left[\frac{\|y-Tx\| + \|x-Ty\|}{2}\right] + \eta\|y-x\| + \xi\left[\frac{\|y-Ty\|\|x-Tx\|}{\|y-x\|}\right]$$

$$\|v-x\| \ge \alpha\left[\frac{\|y-Ty\|^2\|x-Tx\| + \|y-Ty\|\left\|x-\frac{1}{2}(T+I)x\right\|^2}{2\|y-Ty\|\|x-Tx\|}\right]$$
$$+ \beta\left[\frac{\|y-Ty\|^2\left\|\frac{1}{2}(T+I)x-Tx\right\| + \left\|x-\frac{1}{2}(T+I)x\right\|^2\|y-Ty\|}{\|y-Ty\|\|x-Tx\| + \|y-Ty\|\left\|x-\frac{1}{2}(T+I)x\right\|}\right]$$
$$+ \gamma\left[\frac{\|y-Ty\| + \|x-Tx\|}{2}\right] + \delta\left[\frac{\left\|\frac{1}{2}(T+I)x-Tx\right\| + \left\|x-\frac{1}{2}(T+I)x\right\| + \|y-Ty\|}{2}\right]$$
$$+ \eta\left\|\frac{1}{2}(T+I)x-x\right\| + \xi\left[\frac{\|y-Ty\|\|x-Tx\|}{\left\|\frac{1}{2}(T+I)x-x\right\|}\right]$$

$$\|v-x\| \ge \alpha\left[\frac{\|y-Ty\|^2\|x-Tx\| + \|y-Ty\|\frac{1}{4}\|x-Tx\|^2}{2\|y-Ty\|\|x-Tx\|}\right]$$
$$+ \beta\left[\frac{\|y-Ty\|^2\frac{1}{2}\|x-Tx\| + \frac{1}{4}\|x-Tx\|^2\|y-Ty\|}{\|y-Ty\|\|x-Tx\| + \|y-Ty\|\frac{1}{2}\|x-Tx\|}\right] + \gamma\left[\frac{\|y-Ty\| + \|x-Tx\|}{2}\right]$$
$$+ \delta\left[\frac{\frac{1}{2}\|x-Tx\| + \frac{1}{2}\|x-Tx\| + \|y-Ty\|}{2}\right] + \eta\frac{1}{2}\|Tx-x\| + \xi\left[\frac{\|y-Ty\|\|x-Tx\|}{\frac{1}{2}\|Tx-x\|}\right]$$

$$\|v-x\| \ge \alpha\left[\frac{\|y-Ty\|\|x-Tx\|\left(\|y-Ty\| + \frac{1}{4}\|x-Tx\|\right)}{2\|y-Ty\|\|x-Tx\|}\right]$$
$$+ \beta\left[\frac{\|y-Ty\|\|x-Tx\|\left(\frac{1}{2}\|y-Ty\| + \frac{1}{4}\|x-Tx\|\right)}{\frac{3}{2}\|y-Ty\|\|x-Tx\|}\right] + \gamma\left[\frac{\|y-Ty\| + \|x-Tx\|}{2}\right]$$
$$+ \delta\left[\frac{\|x-Tx\| + \|y-Ty\|}{2}\right] + \frac{\eta}{2}\|Tx-x\| + 2\xi\|y-Ty\|$$

$$\|v-x\| \ge \frac{\alpha}{2}\|y-Ty\| + \frac{\alpha}{8}\|x-Tx\| + \frac{\beta}{3}\|y-Ty\| + \frac{\beta}{6}\|x-Tx\| + \frac{\gamma}{2}\|y-Ty\| + \frac{\gamma}{2}\|x-Tx\| + \frac{\delta}{2}\|x-Tx\|$$
$$+ \frac{\delta}{2}\|Ty-y\| + \frac{\eta}{2}\|x-Tx\| + 2\xi\|y-Ty\|$$

$$\|v-x\| \ge \left(\frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{\delta}{2} + 2\xi\right)\|y-Ty\| + \left(\frac{\alpha}{8} + \frac{\beta}{6} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2}\right)\|x-Tx\| \quad (2.2)$$

Now, by (2.1) and (2.2), we get

$$\|z-v\| \ge \|z-x\| + \|x-v\|$$

$$\|z-v\| \ge 2\left(\frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{\delta}{2} + 2\xi\right)\|y-Ty\| + 2\left(\frac{\alpha}{8} + \frac{\beta}{6} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2}\right)\|x-Tx\| \quad (2.3)$$

Also,

$$\|z-v\| = \|Ty-(2y-z)\| = \|Ty-2y+Ty\| = 2\|Ty-y\| \quad (2.4)$$

Therefore by (2.3) and (2.4),we have

$$2\|Ty-y\| \ge 2\left(\frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{\delta}{2} + 2\xi\right)\|y-Ty\| + 2\left(\frac{\alpha}{8} + \frac{\beta}{6} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2}\right)\|x-Tx\|$$

$$2\|Ty-y\| - 2\left(\frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{\delta}{2} + 2\xi\right)\|y-Ty\| \ge 2\left(\frac{\alpha}{8} + \frac{\beta}{6} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2}\right)\|x-Tx\|$$

$$\|x-Tx\| \le \frac{1-\left(\frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{\delta}{2} + 2\xi\right)}{\left(\frac{\alpha}{8} + \frac{\beta}{6} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2}\right)}\|y-Ty\|$$

$\|x - Tx\| \le m\|y - Ty\|$ As $20\alpha + 4\beta + 8\gamma + 8\delta + 4\eta + 16\xi > 8$.

where $m = \dfrac{1 - \left(\frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{\delta}{2} + 2\xi\right)}{\left(\frac{\alpha}{8} + \frac{\beta}{6} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2}\right)} < 1$

Let $R = \frac{1}{2}(T + I)$ then for every $x \in X$,

$$\|R^2x - Rx\| = \|R(Rx) - Rx\| = \|Ry - y\| = \left\|\frac{1}{2}(T + I)y - y\right\| = \frac{1}{2}\|y - Ty\| < \frac{m}{2}\|x - Tx\|$$

By the definition of R we claim that $\{R^n(x)\}$ is a Cauchy sequence in X , $\{R^n(x)\}$ is converges to so element x_0 in X. So $\lim_{n\to\infty}\{R^n(x)\} = x_0$ So $\{R(x_0)\} = x_0$. Hence $T(x_0) = x_0$. So x_0 is a fixed point of T.

Uniqueness: If possible let $y_0 \neq x_0$ is another fixed point of T .Then

$$\begin{aligned}\|x_0 - y_0\| &= \|Tx_0 - Ty_0\| \\ &\ge \alpha\left[\frac{\|x_0 - Tx_0\|^2\|y_0 - Ty_0\| + \|x_0 - Tx_0\|\|y_0 - x_0\|^2}{2\|x_0 - Tx_0\|\|y_0 - Ty_0\|}\right] \\ &\quad + \beta\left[\frac{\|x_0 - Tx_0\|^2\|x_0 - Ty_0\| + \|y_0 - x_0\|^2\|x_0 - Tx_0\|}{\|x_0 - Tx_0\|\|y_0 - Ty_0\| + \|x_0 - Tx_0\|\|y_0 - x_0\|}\right] + \gamma\left[\frac{\|x_0 - Tx_0\| + \|y_0 - Ty_0\|}{2}\right] \\ &\quad + \delta\left[\frac{\|x_0 - Ty_0\| + \|y_0 - Tx_0\|}{2}\right] + \eta\|x_0 - y_0\| + \xi\left[\frac{\|x_0 - Tx_0\|\|y_0 - Ty_0\|}{\|x_0 - y_0\|}\right]\end{aligned}$$

Putting $T(x_0) = x_0$ and $T(y_0) = y_0$.

$$\begin{aligned}\|x_0 - y_0\| &\ge \alpha\left[\frac{\|x_0 - x_0\|^2\|y_0 - y_0\| + \|x_0 - x_0\|\|y_0 - x_0\|^2}{2\|x_0 - x_0\|\|y_0 - y_0\|}\right] \\ &\quad + \beta\left[\frac{\|x_0 - x_0\|^2\|x_0 - y_0\| + \|y_0 - x_0\|^2\|x_0 - x_0\|}{\|x_0 - x_0\|\|y_0 - y_0\| + \|x_0 - x_0\|\|y_0 - x_0\|}\right] + \gamma\left[\frac{\|x_0 - x_0\| + \|y_0 - y_0\|}{2}\right] \\ &\quad + \delta\left[\frac{\|x_0 - y_0\| + \|y_0 - x_0\|}{2}\right] + \eta\|x_0 - y_0\| + \xi\left[\frac{\|x_0 - x_0\|\|y_0 - y_0\|}{\|x_0 - y_0\|}\right]\end{aligned}$$

$\|x_0 - y_0\| \ge (\delta + \eta)\|x_0 - y_0\|$ Therefore, $\|x_0 - y_0\| = 0$. Which is contradiction, so $x_0 = y_0$. Hence x_0 is a unique fixed point in T.

Theorem 2.2: Let T and G be two expansion mapping of a Banach spaces X into itself. If T and G satisfies the following conditions:

1. $T^2 = I$ and $G^2 = I$, where I is identity mapping.

2. T ad G commute.

3.

$$\begin{aligned}\|Tx - Ty\| &\ge \alpha\left[\frac{\|Gx - Tx\|^2\|Gy - Ty\| + \|Gx - Tx\|\|Gy - Gx\|^2}{2\|Gx - Tx\|\|Gy - Ty\|}\right] \\ &\quad + \beta\left[\frac{\|Gx - Tx\|^2\|Gx - Ty\| + \|Gy - Gx\|^2\|Gx - Tx\|}{\|Gx - Tx\|\|Gy - Ty\| + \|Gx - Tx\|\|Gy - Gx\|}\right] + \gamma\left[\frac{\|Gx - Tx\| + \|Gy - Ty\|}{2}\right] \\ &\quad + \delta\left[\frac{\|Gx - Ty\| + \|Gy - Tx\|}{2}\right] + \eta\|Gx - Gy\| + \xi\left[\frac{\|Gx - Tx\|\|Gy - Ty\|}{\|Gx - Gy\|}\right]\end{aligned}$$

For every $x, y \in X$ and $\alpha, \beta, \gamma, \delta, \eta, \xi \in [0,1]$ with $x \neq y$ and $\|Gx - Gy\| \neq 0$ and $\delta + \eta > 1$. Then there exists a unique commo fixed point of T and G such that $Tx_0 = x_0$ and $Gx_0 = x_0$.

Proof : Suppose that x is any point in Banach space X it is clear that $(TG)^2 = I$.

$$\begin{aligned}&\|T(G^2x) - T(G^2y)\| \\ &\ge \alpha\left[\frac{\|G(G^2x) - T(G^2x)\|^2\|G(G^2y) - T(G^2y)\| + \|G(G^2x) - T(G^2x)\|\|G(G^2y) - G(G^2x)\|^2}{2\|G(G^2x) - T(G^2x)\|\|G(G^2y) - T(G^2y)\|}\right] \\ &\quad + \beta\left[\frac{\|G(G^2x) - T(G^2x)\|^2\|G(G^2x) - T(G^2y)\| + \|G(G^2y) - G(G^2x)\|^2\|G(G^2x) - T(G^2x)\|}{\|G(G^2x) - Tx\|\|G(G^2y) - T(G^2y)\| + \|G(G^2x) - T(G^2x)\|\|G(G^2y) - G(G^2x)\|}\right] \\ &\quad + \gamma\left[\frac{\|G(G^2x) - T(G^2x)\| + \|G(G^2y) - T(G^2y)\|}{2}\right] \\ &\quad + \delta\left[\frac{\|G(G^2x) - T(G^2y)\| + \|G(G^2y) - T(G^2x)\|}{2}\right] + \eta\|G(G^2x) - G(G^2y)\| \\ &\quad + \xi\left[\frac{\|G(G^2x) - T(G^2x)\|\|G(G^2y) - T(G^2y)\|}{\|G(G^2x) - G(G^2y)\|}\right]\end{aligned}$$

$$\|TG(Gx)-TG(Gy)\| \geq \alpha\left[\frac{\|Gx-TG(Gx)\|^2\|Gy-TG(Gy)\|+\|Gx-TG(Gx)\|\|Gy-Gx\|^2}{2\|Gx-TG(Gx)\|\|Gy-TG(Gy)\|}\right] + \beta\left[\frac{\|Gx-TG(Gx)\|^2\|Gx-TG(Gy)\|+\|Gy-Gx\|^2\|Gx-TG(Gx)\|}{\|Gx-TG(Gx)\|\|Gy-TG(Gy)\|+\|Gx-TG(Gx)\|\|Gy-Gx\|}\right] + \gamma\left[\frac{\|Gx-TG(Gx)\|+\|Gy-TG(Gy)\|}{2}\right]+\delta\left[\frac{\|Gx-TG(Gy)\|+\|Gy-TG(Gx)\|}{2}\right] + \eta\|Gx-Gy\|+\xi\left[\frac{\|Gx-TG(Gx)\|\|Gy-TG(Gy)\|}{\|Gx-Gy\|}\right]$$

Taking Gx = p , Gy = q and TG = R, whrer $p \neq q$, we get

$$\|Rp-Rq\| \geq \alpha\left[\frac{\|p-Rp\|^2\|q-Rq\|+\|p-Rp\|\|q-p\|^2}{2\|p-Rp\|\|q-Rq\|}\right]+\beta\left[\frac{\|p-Rp\|^2\|p-Rq\|+\|q-p\|^2\|p-Rp\|}{\|p-Rp\|\|q-Rq\|+\|p-Rp\|\|q-p\|}\right] + \gamma\left[\frac{\|p-Rp\|+\|q-Rq\|}{2}\right]+\delta\left[\frac{\|p-Rq\|+\|q-Rp\|}{2}\right]+\eta\|p-q\| + \xi\left[\frac{\|p-Rp\|\|q-Rq\|}{\|p-q\|}\right]$$

It is clear by theorem (3.1) that R = TG has at least one fixed point say x_0 in K that is $R(x_0) = TG(x_0) = x_0$ and so $T(TG)(x_0) = T(x_0)$ or $T^2(Gx_0) = T(x_0)$ Implies that $G(x_0) = T(x_0)$.

Now $\|Tx_0-x_0\| = \|Tx_0-T^2x_0\| = \|Tx_0-T(Tx_0)\|$

$$\geq \alpha\left[\frac{\|G(x_0)-T(x_0)\|^2\|G(Tx_0)-T(Tx_0)\|+\|G(x_0)-T(x_0)\|\|G(Tx_0)-G(x_0)\|^2}{2\|G(x_0)-T(x_0)\|\|G(Tx_0)-T(Tx_0)\|}\right] + \beta\left[\frac{\|G(x_0)-T(x_0)\|^2\|G(x_0)-T(Tx_0)\|+\|G(Tx_0)-G(x_0)\|^2\|G(x_0)-T(x_0)\|}{\|G(x_0)-T(x_0)\|\|G(Tx_0)-T(Tx_0)\|+\|G(x_0)-T(x_0)\|\|G(Tx_0)-G(x_0)\|}\right] + \gamma\left[\frac{\|G(x_0)-T(x_0)\|+\|G(Tx_0)-T(Tx_0)\|}{2}\right]+\delta\left[\frac{\|G(x_0)-T(Tx_0)\|+\|G(Tx_0)-T(x_0)\|}{2}\right] + \eta\|G(x_0)-G(Tx_0)\|+\xi\left[\frac{\|G(x_0)-T(x_0)\|\|G(Tx_0)-T(Tx_0)\|}{\|G(x_0)-G(Tx_0)\|}\right]$$

Putting $Gx_0 = Tx_0$,

$$\|T(x_0)-x_0\| \geq \alpha\left[\frac{\|T(x_0)-T(x_0)\|^2\|x_0-x_0\|+\|T(x_0)-T(x_0)\|\|x_0-T(x_0)\|^2}{2\|T(x_0)-T(x_0)\|\|x_0-x_0\|}\right] + \beta\left[\frac{\|T(x_0)-T(x_0)\|^2\|T(x_0)-x_0\|+\|x_0-T(x_0)\|^2\|T(x_0)-T(x_0)\|}{\|T(x_0)-T(x_0)\|\|x_0-x_0\|+\|T(x_0)-T(x_0)\|\|x_0-T(x_0)\|}\right] + \gamma\left[\frac{\|T(x_0)-T(x_0)\|+\|x_0-x_0\|}{2}\right]+\delta\left[\frac{\|T(x_0)-x_0\|+\|x_0-T(x_0)\|}{2}\right]+\eta\|T(x_0)-x_0\| + \xi\left[\frac{\|T(x_0)-T(x_0)\|\|x_0-x_0\|}{\|T(x_0)-x_0\|}\right]$$

$$\|T(x_0)-x_0\| \geq (\delta+\eta)\|T(x_0)-x_0\|$$

So $Tx_0 = x_0$ $(\delta+\eta > 1)$ that is x_0 is the fixed point of T. But $Tx_0 = Gx_0$ so $Gx_0 = x_0$. Hence x_0 is the fixed point of T and G.

Similarly above theorem the uniqueness part is obvious.

Theorem 2.3: Let T and G be two non expansion mapping of a Banach spaces X into itself. If T and G satisfies the following conditions:

1. $T^2 = TG = G^2 = I$, where I is identity mapping.

2. T ad G commute.

3.

$$\|Tx-Ty\| \leq \alpha\left[\frac{\|Gx-Tx\|^2\|Gy-Ty\|+\|Gx-Tx\|\|Gy-Gx\|^2}{2\|Gx-Tx\|\|Gy-Ty\|}\right] + \beta\left[\frac{\|Gx-Tx\|^2\|Gx-Ty\|+\|Gy-Gx\|^2\|Gx-Tx\|}{\|Gx-Tx\|\|Gy-Ty\|+\|Gx-Tx\|\|Gy-Gx\|}\right]+\gamma\left[\frac{\|Gx-Tx\|+\|Gy-Ty\|}{2}\right] + \delta\left[\frac{\|Gx-Ty\|+\|Gy-Tx\|}{2}\right]+\eta\|Gx-Gy\|+\xi\left[\frac{\|Gx-Tx\|\|Gy-Ty\|}{\|Gx-Gy\|}\right]$$

For every $x, y \in X$ and $\alpha,\beta,\gamma,\delta,\eta,\xi \geq 0$. where $20\alpha+4\beta+8\gamma+8\delta+4\eta+16\xi < 8$. Then T and G have common fixed point.

Proof : Suppose x is any point in Banach space. Taking $y = \frac{1}{2}(T + I)x$, $z = T(y)$, $v = 2y - z$, we have

$$\|z - x\| = \|Ty - T^2x\| = \|Ty - T(Tx)\|$$

$$\|z - x\| \le \alpha\left[\frac{\|y - Ty\|^2\|Tx - T(Tx)\| + \|y - Ty\|\|Tx - y\|^2}{2\|y - Ty\|\|Tx - T(Tx)\|}\right] + \beta\left[\frac{\|y - Ty\|^2\|y - T(Tx)\| + \|Tx - y\|^2\|y - Ty\|}{\|y - Ty\|\|Tx - T(Tx)\| + \|y - Ty\|\|Tx - y\|}\right] + \gamma\left[\frac{\|y - Ty\| + \|Tx - T(Tx)\|}{2}\right] + \delta\left[\frac{\|y - T(Tx)\| + \|Tx - Ty\|}{2}\right] + \eta\|y - Tx\| + \xi\left[\frac{\|y - Ty\|\|Tx - T(Tx)\|}{\|y - Tx\|}\right]$$

$$\|z - x\| \le \alpha\left[\frac{\|y - Ty\|^2\|Tx - x\| + \|y - Ty\|\|Tx - y\|^2}{2\|y - Ty\|\|Tx - x\|}\right] + \beta\left[\frac{\|y - Ty\|^2\|y - x\| + \|Tx - y\|^2\|y - Ty\|}{\|y - Ty\|\|Tx - x\| + \|y - Ty\|\|Tx - y\|}\right] + \gamma\left[\frac{\|y - Ty\| + \|Tx - x\|}{2}\right] + \delta\left[\frac{\|y - x\| + \|Tx - y\| + \|y - Ty\|}{2}\right] + \eta\|y - Tx\| + \xi\left[\frac{\|y - Ty\|\|Tx - x\|}{\|y - Tx\|}\right]$$

$$\|z - x\| \le \alpha\left[\frac{\|y - Ty\|^2\|Tx - x\| + \|y - Ty\|\left\|Tx - \frac{1}{2}(T + I)x\right\|^2}{2\|y - Ty\|\|Tx - x\|}\right] + \beta\left[\frac{\|y - Ty\|^2\left\|\frac{1}{2}(T + I)x - x\right\| + \left\|Tx - \frac{1}{2}(T + I)x\right\|^2\|y - Ty\|}{\|y - Ty\|\|Tx - x\| + \|y - Ty\|\left\|Tx - \frac{1}{2}(T + I)x\right\|}\right] + \gamma\left[\frac{\|y - Ty\| + \|Tx - x\|}{2}\right] + \delta\left[\frac{\left\|\frac{1}{2}(T + I)x - x\right\| + \left\|Tx - \frac{1}{2}(T + I)x\right\| + \|y - Ty\|}{2}\right] + \eta\left\|\frac{1}{2}(T + I)x - Tx\right\| + \xi\left[\frac{\|y - Ty\|\|Tx - x\|}{\left\|\frac{1}{2}(T + I)x - Tx\right\|}\right]$$

$$\|z - x\| \le \alpha\left[\frac{\|y - Ty\|^2\|Tx - x\| + \|y - Ty\|\frac{1}{4}\|Tx - x\|^2}{2\|y - Ty\|\|Tx - x\|}\right] + \beta\left[\frac{\|y - Ty\|^2\frac{1}{2}\|Tx - x\| + \frac{1}{4}\|Tx - x\|^2\|y - Ty\|}{\|y - Ty\|\|Tx - x\| + \|y - Ty\|\frac{1}{2}\|Tx - x\|}\right] + \gamma\left[\frac{\|y - Ty\| + \|Tx - x\|}{2}\right] + \delta\left[\frac{\frac{1}{2}\|Tx - x\| + \frac{1}{2}\|Tx - x\| + \|y - Ty\|}{2}\right] + \eta\frac{1}{2}\|x - Tx\| + \xi\left[\frac{\|y - Ty\|\|Tx - x\|}{\frac{1}{2}\|x - Tx\|}\right]$$

$$\|z - x\| \le \alpha\left[\frac{\|y - Ty\|\|Tx - x\|\left(\|y - Ty\| + \frac{1}{4}\|Tx - x\|\right)}{2\|y - Ty\|\|Tx - x\|}\right] + \beta\left[\frac{\|y - Ty\|\|Tx - x\|\left(\frac{1}{2}\|y - Ty\| + \frac{1}{4}\|x - Tx\|\right)}{\frac{3}{2}\|y - Ty\|\|Tx - x\|}\right] + \gamma\left[\frac{\|y - Ty\| + \|Tx - x\|}{2}\right] + 2\xi\|y - Ty\| + \delta\left[\frac{\|Tx - x\| + \|y - Ty\|}{2}\right] + \frac{\eta}{2}\|x - Tx\|$$

$$\|z - x\| \ge \frac{\alpha}{2}\|y - Ty\| + \frac{\alpha}{8}\|Tx - x\| + \frac{\beta}{3}\|y - Ty\| + \frac{\beta}{6}\|Tx - x\| + \frac{\gamma}{2}\|y - Ty\| + \frac{\gamma}{2}\|Tx - x\| + \frac{\delta}{2}\|Tx - x\| + \frac{\delta}{2}\|Ty - y\| + \frac{\eta}{2}\|Tx - x\| + 2\xi\|y - Ty\|$$

$$\|z - x\| \le \left(\frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{\delta}{2} + 2\xi\right)\|y - Ty\| + \left(\frac{\alpha}{8} + \frac{\beta}{6} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2}\right) \quad (2.5)$$

$$\|v - x\| = \|2y - z - x\| = \left\|2\frac{1}{2}(T + I)x - Ty - x\right\| = \|Tx - Ty\|$$

$$\|v-x\| \le \alpha\left[\frac{\|y-Ty\|^2\|x-Tx\| + \|y-Ty\|\|x-y\|^2}{2\|y-Ty\|\|x-Tx\|}\right] + \beta\left[\frac{\|y-Ty\|^2\|y-Tx\| + \|x-y\|^2\|y-Ty\|}{\|y-Ty\|\|x-Tx\| + \|y-Ty\|\|x-y\|}\right]$$
$$+\gamma\left[\frac{\|y-Ty\| + \|x-Tx\|}{2}\right] + \delta\left[\frac{\|y-Tx\| + \|x-Ty\|}{2}\right] + \eta\|y-x\| + \xi\left[\frac{\|y-Ty\|\|x-Tx\|}{\|y-x\|}\right]$$

$$\|v-x\| \le \alpha\left[\frac{\|y-Ty\|^2\|x-Tx\| + \|y-Ty\|\left\|x-\frac{1}{2}(T+I)x\right\|^2}{2\|y-Ty\|\|x-Tx\|}\right]$$
$$+\beta\left[\frac{\|y-Ty\|^2\left\|\frac{1}{2}(T+I)x-Tx\right\| + \left\|x-\frac{1}{2}(T+I)x\right\|^2\|y-Ty\|}{\|y-Ty\|\|x-Tx\| + \|y-Ty\|\left\|x-\frac{1}{2}(T+I)x\right\|}\right]$$
$$+\gamma\left[\frac{\|y-Ty\| + \|x-Tx\|}{2}\right] + \delta\left[\frac{\left\|\frac{1}{2}(T+I)x-Tx\right\| + \left\|x-\frac{1}{2}(T+I)x\right\| + \|y-Ty\|}{2}\right]$$
$$+\eta\left\|\frac{1}{2}(T+I)x-x\right\| + \xi\left[\frac{\|y-Ty\|\|x-Tx\|}{\left\|\frac{1}{2}(T+I)x-x\right\|}\right]$$

$$\|v-x\| \le \alpha\left[\frac{\|y-Ty\|^2\|x-Tx\| + \|y-Ty\|\frac{1}{4}\|x-Tx\|^2}{2\|y-Ty\|\|x-Tx\|}\right]$$
$$+\beta\left[\frac{\|y-Ty\|^2\frac{1}{2}\|x-Tx\| + \frac{1}{4}\|x-Tx\|^2\|y-Ty\|}{\|y-Ty\|\|x-Tx\| + \|y-Ty\|\frac{1}{2}\|x-Tx\|}\right] + \gamma\left[\frac{\|y-Ty\| + \|x-Tx\|}{2}\right]$$
$$+\delta\left[\frac{\frac{1}{2}\|x-Tx\| + \frac{1}{2}\|x-Tx\| + \|y-Ty\|}{2}\right] + \eta\frac{1}{2}\|Tx-x\| + \xi\left[\frac{\|y-Ty\|\|x-Tx\|}{\frac{1}{2}\|Tx-x\|}\right]$$

$$\|v-x\| \le \alpha\left[\frac{\|y-Ty\|\|x-Tx\|\left(\|y-Ty\| + \frac{1}{4}\|x-Tx\|\right)}{2\|y-Ty\|\|x-Tx\|}\right]$$
$$+\beta\left[\frac{\|y-Ty\|\|x-Tx\|\left(\frac{1}{2}\|y-Ty\| + \frac{1}{4}\|x-Tx\|\right)}{\frac{3}{2}\|y-Ty\|\|x-Tx\|}\right] + \gamma\left[\frac{\|y-Ty\| + \|x-Tx\|}{2}\right]$$
$$+\delta\left[\frac{\|x-Tx\| + \|y-Ty\|}{2}\right] + \frac{\eta}{2}\|Tx-x\| + 2\xi\|y-Ty\|$$

$$\|v-x\| \le \frac{\alpha}{2}\|y-Ty\| + \frac{\alpha}{8}\|x-Tx\| + \frac{\beta}{3}\|y-Ty\| + \frac{\beta}{6}\|x-Tx\| + \frac{\gamma}{2}\|y-Ty\| + \frac{\gamma}{2}\|x-Tx\| + \frac{\delta}{2}\|x-Tx\|$$
$$+\frac{\delta}{2}\|Ty-y\| + \frac{\eta}{2}\|x-Tx\| + 2\xi\|y-Ty\|$$

$$\|v-x\| \le \left(\frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{\delta}{2} + 2\xi\right)\|y-Ty\| + \left(\frac{\alpha}{8} + \frac{\beta}{6} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2}\right)\|x-Tx\| \qquad (2.6)$$

Now, by (2.5) and (2.6), we get

$$\|z-v\| \le \|z-x\| + \|x-v\|$$

$$\|z-v\| \le 2\left(\frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{\delta}{2} + 2\xi\right)\|y-Ty\| + 2\left(\frac{\alpha}{8} + \frac{\beta}{6} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2}\right)\|x-Tx\| \qquad (2.7)$$

Also,

$$\|z-v\| = \|Ty-(2y-z)\| = \|Ty-2y+Ty\| = 2\|Ty-y\| \qquad (2.8)$$

Therefore by (2.7) and (2.8),we have

$$2\|Ty-y\| \le 2\left(\frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{\delta}{2} + 2\xi\right)\|y-Ty\| + 2\left(\frac{\alpha}{8} + \frac{\beta}{6} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2}\right)\|x-Tx\|$$

$$2\|Ty-y\| - 2\left(\frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{\delta}{2} + 2\xi\right)\|y-Ty\| \le 2\left(\frac{\alpha}{8} + \frac{\beta}{6} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2}\right)\|x-Tx\|$$

$$\|x-Tx\| \ge \frac{1-\left(\frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{\delta}{2} + 2\xi\right)}{\left(\frac{\alpha}{8} + \frac{\beta}{6} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2}\right)}\|y-Ty\|$$

$\|x-Tx\| \geq m\|y-Ty\|$ As $20\alpha+4\beta+8\gamma+8\delta+4\eta+16\xi<8$.

where $m=\dfrac{1-\left(\frac{\alpha}{2}+\frac{\beta}{3}+\frac{\gamma}{2}+\frac{\delta}{2}+2\xi\right)}{\left(\frac{\alpha}{8}+\frac{\beta}{6}+\frac{\gamma}{2}+\frac{\delta}{2}+\frac{\eta}{2}\right)}>1$

Let $R=\frac{1}{2}(T+I)$ then for every $x \in X$,

$\|R^2x-Rx\|=\|R(Rx)-Rx\|=\|Ry-y\|=\dfrac{1}{2}\|y-Ty\|=\dfrac{1}{2}\|T(Ty)-Ty\|=\dfrac{1}{2}\|Ty-y\|$

$\Rightarrow \frac{1}{2}\|y-Ty\| \Rightarrow \frac{1}{2}\|TGy-Ty\| \leq \frac{1}{2}\|Gy-y\|$

Because T is non expansive. so $\|R^2(x)-R(x)\| \leq \frac{m}{2}\|x-Tx\|$

$\|R^2x-Rx\|<\frac{m}{2}\|x-Tx\|$ By the definition of m. we claim that $R^n(x)$ is a cauchy sequence in X. Also by completeness $R^n(x)$ converges to R(x), i.e., $\lim_{n\to\infty} R^n(x)=x_0 \Rightarrow T(x_0)=x_0$. therefore x_0 is fixed point of T.

Again, $\|R^2(x)-R(x)\| \leq \dfrac{m}{2}\|x-Tx\|=\dfrac{m}{2}\|G^2x-G^2Tx\|=\dfrac{m}{2}\|G(Gx)-GT(Gx)\|$

$\leq \frac{m}{2}\|Gx-x\|$ Because G is non-expansive.

We can conclude that $G(x_0)=x_0$ that is x_0 is fixed point of G. Therefore $T(x_0)=G(x_0)=x_0$. So x_0 is a common fixed point of T and G.

The uniqueness part is obvious.

III. Conclusion

In this paper we proved fixed point theorem and common fixed point theorem in 2-Banach space. The results of this paper extend the previously known results Banach space and 2-Banach space.

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