IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 12, Issue 6 Ver. V (Nov. - Dec.2016), PP 25-37
www.iosrjournals.org

Cubic ideals of Γ-near rings

V. Chinnadurai[1], K. Bharathivelan[2]
[1]*Associate professor, Department of Mathematics, Annamalai University, India.*
[2]*Research scholar, Department of Mathematics, Annamalai University, India.*

Abstract: *In this paper, we introduce the notion of cubic ideals of Γ-near-rings which is a combination of an interval-valued fuzzy set and a fuzzy set. Interval-valued fuzzy set is another generalization of fuzzy sets that was introduced by Lofti Asker Zadeh. In order to obtain cubic sub Γ-near-ring, cubic ideals of Γ-near-rings, direct product of cubic ideals in Γ-near-rings is also a cubic ideal, intersection of any family of cubic ideals of Γ-near-ring is also a cubic ideal of Γ-near-ring, cubic level set on Γ-near- ring and strongest cubic relation on Γ-near-ring. Subsequently we prove that a necessary and sufficient condition for a cubic ideal and its characteristic function and how the images and inverse- images of cubic ideals of Γ-near-rings become cubic ideals of Γ-near-rings are studied.*
Keywords: *Cubic ideal, cubic homomorphism, ideal, near-ring, Γ-near-ring.*

I. Introduction

Zadeh [1] introduced the notion of fuzzy set in 1965. In 1975, Zadeh [2] introduced the notion of interval-valued fuzzy set, where the values of the membership function are closed subintervals of [0,1] instead of a single value from it. In 1971, Rosenfeld [3] defined fuzzy subgroup and gave some of its properties. In 1991, Abou-Zaid [4] introduced the notion of fuzzy subnear-rings and fuzzy ideals in near-rings. Jun, Kim [5] and Davaz [6] applied a few concepts of interval valued fuzzy subsets in near-rings. Thillaigovindan et al [7] introduced the notion of interval-valued fuzzy ideals of near-rings. Jun et al [8] introduced the concept of cubic sets. This structure encompasses interval-valued fuzzy set and fuzzy set. Chinnadurai et al [9, 10] introduced the notion of cubic bi-ideals of near-rings and cubic ideals of Γ semigroups. Γ-near-rings were defined by Bh. Satyanarayana [14] and ideal theory in Γ-near-rings was studied by Bh. Satyanarayana. The purpose of this paper to introduce the notion of cubic ideals of Γ-near-rings and homomorphism in cubic ideals of Γ-near-rings. We investigate some basic results, examples and properties.

II. Preliminaries

We recall the following definitions for subsequent use.
Definition 2.1. [4] A non-empty set R with two binary operations "+" and "·" is called a near-ring if
$i)$ $(R,+)$ is a group
$ii)$ $(R,\ \cdot)$ is a semigroup
$iii)$ $x\cdot(y+z)=x\cdot y+x\cdot z$ for all $x,y,z\in R$.
we use the word near-ring to mean left near-ring. We denote xy instead of $x\cdot y$, note that $x0=0$ and $x(-y)=-xy$ but in general $x0\neq 0$ for some $x\in R$.
Definition 2.2. [4] An ideal I of a near-ring R is a subset of R such that
$i)$ $(I,+)$ is a normal subgroup of $(R,+)$
$ii)$ $RI\subseteq I$
$iii)$ $((x+i)y-xy)\in I$ for any $i\in I$ and $x,y\in R$.
Note that I is a left ideal of R if it satisfies (i) and (ii) and I is a right ideal of R if it satisfies (i) and (iii).
Definition 2.3. [11] Let $(M,+)$ be a group and Γ be a non-empty set. Then M is said to be a Γ-near-ring, if there exist a mapping $M\times\Gamma\times M\to M$ (The image of (x,α,y) is denoted by $x\alpha y$) satisfies the following conditions
$i)$ $(x+y)\alpha z=x\alpha z+y\alpha z$
$ii)$ $(x\alpha y)\beta z=x\alpha(y\beta z)$ for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$.
Definition 2.4. [11] Let M be a Γ-near-ring. A normal subgroup $(I,+)$ of $(M,+)$ is called
$i)$ a left ideal if $x\alpha(y+i)-x\alpha y\in I$ for all $x,y\in M,\alpha\in\Gamma$, i ∈ I
$ii)$ a right ideal if $i\alpha x\in I$ for all $x\in M,\alpha\in\Gamma$, i ∈ I
$iii)$ an ideal if it is both a left ideal and a right ideal of M
A Γ-near-ring M is said to be a zero-symmetric if $a\alpha 0=0$ for all $a\in M$ and $\alpha\in\Gamma$, where 0 is the additive identity in M.
Definition 2.5. [13] A mapping $\mu: X\to[0,1]$ is called a fuzzy subset of X.
Definition 2.6. [4] Let R be a near-ring and μ be a fuzzy subset of R. We say μ is a fuzzy subnear-ring of R if

$i)\ \mu(x-y) \geq \min\{\mu(x),\mu(y)\}$
$ii)\ \mu(xy) \geq \min\{\mu(x),\mu(y)\}$ for all $x,y \in R$.
μ is called a fuzzy ideal of R. if μ is a fuzzy subnear-ring of R and
$iii)\ \mu(y+x-y) = \mu(x)$
$iv)\ \mu(xy) \geq \mu(y)$
$v)\ \mu\big((x+i)y-xy\big) \geq \mu(i)$ for any $x,y,i \in R$.
Note that μ is a fuzzy left ideal of R if it satisfies $(i),(ii),(iii)$ and (iv) and μ is a fuzzy right ideal of R if it satisfies $(i),(ii),(iii)$ and (v).

Definition 2.7. [13] Let X be a non-empty set. A mapping $\bar{\mu}: X \to D[0,1]$ is called interval-valued fuzzy set (in short i-v), where D[0,1] denote the family of all closed sub intervals of [0,1] and $\bar{\mu}(x) = [\mu^-(x),\mu^+(x)]$ for all $x \in X$, where μ^- and μ^+ are fuzzy subsets of X such that $\mu^-(x) \leq \mu^+(x)$ for all $x \in X$.

Definition 2.8. [7] An interval-valued fuzzy subset $\bar{\mu}$ of a near-ring R is called an sub near-ring of R if
$i)\ \bar{\mu}(x-y) \geq \min\{\bar{\mu}(x),\bar{\mu}(y)\}$
$ii)\ \bar{\mu}(xy) \geq \min\{\bar{\mu}(x),\bar{\mu}(y)\}$ for all $x,y \in R$.
An interval-valued fuzzy subset $\bar{\mu}$ of a near-ring R is called an interval-valued ideal of R if $\bar{\mu}$ is an interval-valued fuzzy subnear-ring of R and
$iii)\ \bar{\mu}(y+x-y) = \bar{\mu}(x)$
$iv)\ \bar{\mu}(xy) \geq \bar{\mu}(y)$
$v)\ \bar{\mu}\big((x+i)y-xy\big) \geq \bar{\mu}(i)$ for any $x,y,i \in R$.
Note that $\bar{\mu}$ is an i-v fuzzy left ideal of R if it satisfies $(i),(ii),(iii)$ and (iv) and $\bar{\mu}$ is an i-v fuzzy right ideal of R if it satisfies $(i),(ii),(iii)$ and (v).

Definition 2.9. [8] Let X be a non-empty set. A cubic set $\mathcal{A}$ in X is a structure of the form $\mathcal{A}=\{\langle x,\bar{\mu}_A(x),\lambda(x)\rangle : x \in X\}$ and denoted by $\mathcal{A}=\langle\bar{\mu}_A,\lambda\rangle$, where $\bar{\mu}_A = [\mu_A^-,\mu_A^+]$ is an interval-valued fuzzy set (briefly, IVF) in X and λ is a fuzzy set in X.

Definition 2.10. [9] Let $\mathcal{A} = <\bar{\mu},\gamma>$ be a cubic set of S. Define $U(\mathcal{A};\tilde{t},n) = \{x \in S | \bar{\mu}(x) \geq \tilde{t}\ and\ \gamma(x) \leq n\}$ where $\tilde{t} \in D[0,1]$ and $n \in [0,1]$ is called the cubic level set of $\mathcal{A}$.

Definition 2.11. [12] For any non-empty subset G of a set X, the characteristic cubic set of G is defined to be a structure $\chi_G(x) =< x,\bar{\mu}_{\chi_G}(x),\gamma_{\chi_G}(x): x \in X >$ which is briefly denoted by
$\chi_G(x) =< \bar{\mu}_{\chi_G}(x),\gamma_{\chi_G}(x) >$ where $\bar{\mu}_{\chi_G}(x) = \begin{cases}[1,1] & if\ x \in G\\ [0,0] & otherwise\end{cases}$ and $\gamma_{\chi_G}(x) = \begin{cases}0 & if\ x \in G\\ 1 & otherwise\end{cases}$

Definition 2.12. [9] Let $\mathcal{A}_i = \langle\bar{\mu}_i,\omega_i\rangle$ be cubic bi-ideals of near-rings N_i for $i = 1,2,3,\dots,n$. Then the cubic direct product of $\mathcal{A}_i, (i = 1,2,3,\dots,n)$ is a function $(\bar{\mu}_1 \times \bar{\mu}_2 \times,\dots,\times \bar{\mu}_n): R_1 \times R_2 \times \dots \times R_n \to D[0,1]$, $(\omega_1 \times \omega_2 \times,\dots,\times \omega_n): R_1 \times R_2 \times \dots \times R_n \to [0,1]$ defined by
$(\bar{\mu}_1 \times \bar{\mu}_2 \times,\dots,\times \bar{\mu}_n)(x_1,x_2,\dots,x_n) = \min\{\bar{\mu}_1(x_1),\bar{\mu}_2(x_2),\dots,\bar{\mu}_n(x_n)\}$ and
$(\omega_1 \times \omega_2 \times,\dots,\times \omega_n)(x_1,x_2,\dots,x_n) = \max\{\omega_1(x_1),\omega_2(x_2),\dots,\omega_n(x_n)\}$.

Definition 2.13. [9] Let $\mathcal{A} = \langle\bar{\mu},\omega\rangle$ be a cubic set of N. Then the strongest cubic relation on N is a cubic relation α with respect to $\mathcal{A} = \langle\bar{\mu},\omega\rangle$ given by $\alpha(x,y) = \{\big((x,y),\beta(x,y),\gamma(x,y)\big) | \ x,y \in N\}$, where β is an interval-valued fuzzy relation with respect to $\bar{\mu}$ defined by $\beta(x,y) = \min\{\bar{\mu}(x),\bar{\mu}(y)\}$ and γ is a fuzzy relation with respect to ω defined by $\gamma(x,y) = \max\{\omega(x),\omega(y)\}$.

Definition 2.14. [7] Let R and S be near-rings. A map $\theta: R \to S$ is called a (near-ring) homomorphism if $\theta(x+y) = \theta(x)+\theta(y)$ and $\theta(xy) = \theta(x)\theta(y)$ for all $x,y \in R$.

Definition 2.15. [9] Let f be a mapping from a set R to R_1. Let $\mathcal{A}_1 =< \bar{\mu}_1,\omega_1 >$ be a cubic set of R and $\mathcal{A}_2 =< \bar{\mu}_2,\omega_2 >$ be a cubic set of R_1. Then the pre-image $f^{-1}(\mathcal{A}_2) =< f^{-1}(\bar{\mu}_2),f^{-1}(\omega_2) >$ is a cubic set of R defined by $f^{-1}(\mathcal{A}_2)(x) =< f^{-1}(\bar{\mu}_2)(x),f^{-1}(\omega_2)(x) >=< \bar{\mu}_2\big(f(x)\big),\omega_2\big(f(x)\big) >$. The image $f(\mathcal{A}_1) =< f(\bar{\mu}_1),f(\omega_1) >$ is a cubic set of R_1 defined by $f(\mathcal{A}_1)(x) =< f(\bar{\mu}_1)(x),f(\omega_1)(x) >$
where $f(\bar{\mu}_1)(x) = \begin{cases}\sup_{y \in f^{-1}(x)}\bar{\mu}(y) & if\ f^{-1}(x) \neq \emptyset\\ [0,0] & otherwise\end{cases}$ and $f(\omega_1)(x) = \begin{cases}\inf_{y \in f^{-1}(x)}\lambda(y) & if\ f^{-1}(x) \neq \emptyset\\ 1 & otherwise\end{cases}$

III. Main Results

In this section, we introduce the notion of cubic ideals of Γ-near rings and discuss some of its properties. Throughout this paper, N denotes Γ-near-ring unless otherwise specified.

Definition 3.1. A cubic set $\mathcal{A} = \langle\bar{\mu},\omega\rangle$ of a Γ-near ring N is called cubic sub Γ-near ring of N if
$i)\ \bar{\mu}(x-y) \geq \min\{\bar{\mu}(x),\bar{\mu}(y)\}$ and $\omega(x-y) \leq \max\{\omega(x),\omega(y)\}$
$ii)\ \bar{\mu}(x\alpha y) \geq \min\{\bar{\mu}(x),\bar{\mu}(y)\}$ and $\omega(x\alpha y) \leq \max\{\omega(x),\omega(y)\}$ for all $x,y \in N$ and $\alpha \in \Gamma$.

Definition 3.2. A cubic set $\mathcal{A} = \langle\bar{\mu},\omega\rangle$ of a Γ-near ring N is called cubic ideal of N if $\mathcal{A} = \langle\bar{\mu},\omega\rangle$ is a cubic sub Γ-near ring of N and

$iii)\ \bar{\mu}(y+x-y) \geq \bar{\mu}(x)$ and $\omega(y+x-y) \leq \omega(x)$
$iv)\ \bar{\mu}(x\alpha y) \geq \bar{\mu}(y)$ and $\omega(x\alpha y) \leq \omega(y)$
$v)\ \bar{\mu}\big((x+z)\alpha y - x\alpha y\big) \geq \bar{\mu}(z)$ and $\omega\big((x+z)\alpha y - x\alpha y\big) \leq \omega(z)$ for all $x, y, z \in N$ and $\alpha \in \Gamma$.
Note that $\mathcal{A} = \langle \bar{\mu}, \omega \rangle$ is a cubic left ideal of N if it satisfies $(i), (ii), (iii)$ and (iv), and $\mathcal{A} = \langle \bar{\mu}, \omega \rangle$ is a cubic right ideal of N if it satisfies $(i), (ii), (iii)$ and (v).
Example 3.3. Let $N = \{0,1,2\}_{\oplus 3}$ and $\Gamma = \{\alpha, \beta\}$. Define a binary operation addition modulo 3 on N and a mapping $N \times \Gamma \times N \to N$ by the following tables

⊕ 3	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

α	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

β	0	1	2
0	0	0	0
1	0	1	2
2	0	1	2

Clearly $(i)(N, \oplus\ 3)$ is a group $(ii) x\alpha(y+z) = x\alpha y + x\alpha z$ $(iii)(x\alpha y)\beta z = x\alpha(y\beta z)$ for every $x, y, z \in N$ and $\alpha, \beta \in \Gamma$. Then N is a Γ-near-ring.
Define a cubic set $\mathcal{A} = \langle \bar{\mu}, \omega \rangle$ in N as follows

N	$\bar{\mu}(x)$	$\omega(x)$
0	[0.6,0.8]	0.2
1	[0.3,0.4]	0.6
2	[0.1,0.2]	0.8

Thus $\mathcal{A} = \langle \bar{\mu}, \omega \rangle$ is a cubic ideal of Γ-near-ring N.
Theorem 3.4. Let N be a Γ-near-ring and $\{\mathcal{A}_i\} = < \bar{\mu}_i, \omega_i | i \in \wedge >$ be a non-empty family of cubic subsets of N. If $\{\mathcal{A}_i\}$ is a cubic ideal of N then $\sqcap_{i\in\wedge} \mathcal{A}_i = < \cap_{i\in\wedge} \bar{\mu}_i, \cup_{i\in\wedge} \omega_i >$ is a cubic ideal of N.
Proof: Let $\mathcal{A}_i = < \bar{\mu}_i, \omega_i | i \in \wedge >$ be a family of cubic ideals of N. Let $x, y, z \in N, \alpha \in \Gamma$ and $\cap\, \bar{\mu}_i(x) = (inf\ \bar{\mu}_i)(x) = \inf \bar{\mu}_i(x), \cup\, \omega_i(x) = (sup\ \omega_i)(x) = \sup \omega_i(x)$.

$(\cap_{i\in I} \bar{\mu}_i)(x-y) = \inf\{\bar{\mu}_i\,(x-y)\}_{i\in I}$
$\geq \inf\{\min\{\bar{\mu}_i(x), \bar{\mu}_i(y)\}\}_{i\in I}$
$= \min\{\inf\{\bar{\mu}_i(x)\}_{i\in I}, \inf\{\bar{\mu}_i(y)\}_{i\in I}\}$
$= \min\{\cap_{i\in I} \bar{\mu}_i(x), \cap_{i\in I} \bar{\mu}_i(y)\}$

. $(\cup_{i\in I} \omega_i)(x-y) = \sup\{\omega_i\,(x-y)\}_{i\in I}$
$\leq \sup\{\max\{\omega_i(x), \omega_i(y)\}\}_{i\in I}$
$= \max\{\sup\{\omega_i(x)\}_{i\in I}, \sup\{\omega_i(y)\}_{i\in I}\}$
$= \max\{\cup_{i\in I} \omega_i(x), \cup_{i\in I} \omega_i(y)\}$

$(\cap_{i\in I} \bar{\mu}_i)(x\alpha y) = \inf\{\bar{\mu}_i\,(x\alpha y)\}_{i\in I}$
$\geq \inf\{\min\{\bar{\mu}_i(x), \bar{\mu}_i(y)\}\}_{i\in I}$
$= \min\{\inf\{\bar{\mu}_i(x)\}_{i\in I}, \inf\{\bar{\mu}_i(y)\}_{i\in I}\}$
$= \min\{\cap_{i\in I} \bar{\mu}_i(x), \cap_{i\in I} \bar{\mu}_i(y)\}$

. $(\cup_{i\in I} \omega_i)(x\alpha y) = \sup\{\omega_i\,(x\alpha y)\}_{i\in I}$
$\leq \sup\{\max\{\omega_i(x), \omega_i(y)\}\}_{i\in I}$
$= \max\{\sup\{\omega_i(x)\}_{i\in I}, \sup\{\omega_i(y)\}_{i\in I}\}$
$= \max\{\cup_{i\in I} \omega_i(x), \cup_{i\in I} \omega_i(y)\}$

$(\cap_{i\in I} \bar{\mu}_i)(y+x-y) = \inf\{\bar{\mu}_i\,(y+x-y)\}_{i\in I}$
$\geq \inf\{\bar{\mu}_i(x)\}_{i\in I}$
$= \cap_{i\in I} \bar{\mu}_i(x)$

$(\cup_{i\in I} \omega_i)(y+x-y) = \sup\{\omega_i\,(y+x-y)\}_{i\in I}$
$\leq \sup\{\omega_i(x)\}_{i\in I}$
$= \cap_{i\in I} \omega_i(x)$

$(\cap_{i\in I} \bar{\mu}_i)(x\alpha y) = \inf\{\bar{\mu}_i\,(x\alpha y)\}_{i\in I}$
$\geq \inf\{\bar{\mu}_i(y)\}_{i\in I}$
$= \cap_{i\in I} \bar{\mu}_i(y)$

. $(\cup_{i\in I} \omega_i)(x\alpha y) = \sup\{\omega_i\,(x\alpha y)\}_{i\in I}$
$\leq \sup\{\omega_i(y)\}_{i\in I}$

$$= \cap_{i\in I}\, \omega_i(y)$$

$$(\cap_{i\in I}\, \bar{\mu}_i)\big((x+z)\alpha y - x\alpha y\big) = \inf\, \{\bar{\mu}_i\left((x+z)\alpha y - x\alpha y\right)\}_{i\in I}$$
$$\geq \inf\{\bar{\mu}_i(z)\}_{i\in I}$$
$$= \cap_{i\in I}\, \bar{\mu}_i(z)$$

. $(\cup_{i\in I}\, \omega_i)\big((x+z)\alpha y - x\alpha y\big) = \sup\{\omega_i\left((x+z)\alpha y - x\alpha y\right)\}_{i\in I}$
$$\leq \sup\{\omega_i(z)\}_{i\in I}$$
$$= \cap_{i\in I}\, \omega_i(z)$$

Hence $\sqcap_{i\in\wedge}\mathcal{A}_i = < \cap_{i\in\wedge}\bar{\mu}_i, \cup_{i\in\wedge}\omega_i >$ is a cubic ideal of Γ-near-ring N.

Theorem 3.5. A cubic set $\mathcal{A} = \langle\bar{\mu}, \omega\rangle$ in N is a cubic ideal of N if and only if μ^-, μ^+ and ω are fuzzy ideals of N.

Proof: Let $\mathcal{A} = \langle\bar{\mu}, \omega\rangle$ be a cubic ideal of N. For any $x, y, z \in N, \alpha \in \Gamma$ then we have

$$[\mu^-(x-y), \mu^+(x-y)] = \bar{\mu}(x-y)$$
$$\geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$$
$$= \min\{[\mu^-(x), \mu^+(x)], [\mu^-(y), \mu^+(y)]\}$$
$$= [\min\{\mu^-(x), \mu^-(y)\}, \min\{\mu^+(x)\mu^+(y)\}]$$

It follows that $\mu^-(x-y) \geq \min\{\mu^-(x), \mu^-(y)\}$ and $\mu^+(x-y) \geq \min\{\mu^+(x)\mu^+(y)\}$

Clearly $\omega(x-y) \leq \max\{\omega(x), \omega(y)\}$

$$[\mu^-(x\alpha y), \mu^+(x\alpha y)] = \bar{\mu}(x\alpha y)$$
$$\geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$$
$$= \min\{[\mu^-(x), \mu^+(x)], [\mu^-(y), \mu^+(y)]\}$$
$$= [\min\{\mu^-(x), \mu^-(y)\}, \min\{\mu^+(x)\mu^+(y)\}]$$

It follows that $\mu^-(x\alpha y) \geq \min\{\mu^-(x), \mu^-(y)\}$ and $\mu^+(x\alpha y) \geq \min\{\mu^+(x)\mu^+(y)\}$

Clearly $\omega(x\alpha y) \leq \max\{\omega(x), \omega(y)\}$

$$[\mu^-(y+x-y), \mu^+(y+x-y)] = \bar{\mu}(y+x-y)$$
$$\geq \bar{\mu}(x)$$
$$= [\mu^-(x), \mu^+(x)]$$

It follows that $\mu^-(y+x-y) \geq \mu^-(x)$ and $\mu^+(y+x-y) \geq \mu^+(x)$

Clearly $\omega(y+x-y) \leq \omega(x)$

$$[\mu^-(x\alpha y), \mu^+(x\alpha y)] = \bar{\mu}(x\alpha y)$$
$$\geq \bar{\mu}(y)$$
$$= [\mu^-(y), \mu^+(y)]$$

It follows that $\mu^-(x\alpha y) \geq \mu^-(y)$ and $\mu^+(x\alpha y) \geq \mu^+(y)$

Clearly $\omega(x\alpha y) \leq \omega(y)$

$$\big[\mu^-\big((x+z)\alpha y - x\alpha y\big), \mu^+\big((x+z)\alpha y - x\alpha y\big)\big] = \bar{\mu}\big((x+z)\alpha y - x\alpha y\big)$$
$$\geq \bar{\mu}(z)$$
$$= [\mu^-(z), \mu^+(z)]$$

It follows that $\mu^-\big((x+z)\alpha y - x\alpha y\big) \geq \mu^-(z)$ and $\mu^+\big((x+z)\alpha y - x\alpha y\big) \geq \mu^+(z)$

Clearly $\omega\big((x+z)\alpha y - x\alpha y\big) \leq \omega(z)$

Hence μ^-, μ^+ and ω are fuzzy ideals of N.

Conversely suppose that μ^-, μ^+ and ω are fuzzy ideals of N. Let $x, y, z \in N, \alpha \in \Gamma$.

$$\bar{\mu}(x-y) = [\mu^-(x-y), \mu^+(x-y)]$$
$$\geq [\min\{\mu^-(x), \mu^-(y)\}, \min\{\mu^+(x), \mu^+(y)\}]$$
$$= \min\{[\mu^-(x), \mu^+(x)], [\mu^-(y), \mu^+(y)]\}$$
$$\geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$$

Clearly $\omega(x-y) \leq \max\{\omega(x), \omega(y)\}$

$$\bar{\mu}(x\alpha y) = [\mu^-(x\alpha y), \mu^+(x\alpha y)]$$
$$\geq [\min\{\mu^-(x), \mu^-(y)\}, \min\{\mu^+(x)\mu^+(y)\}]$$
$$= \min\{[\mu^-(x), \mu^+(x)], [\mu^-(y), \mu^+(y)]\}$$
$$\geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$$

Clearly $\omega(x\alpha y) \leq \max\{\omega(x), \omega(y)\}$

$$\bar{\mu}(y+x-y) = [\mu^-(y+x-y), \mu^+(y+x-y)]$$
$$\geq [\mu^-(x), \mu^+(x)]$$
$$\geq \bar{\mu}(x)$$

Clearly $\omega(y+x-y) \leq \omega(x)$

$$\bar{\mu}(x\alpha y) = [\mu^-(x\alpha y), \mu^+(x\alpha y)]$$
$$\geq [\mu^-(y), \mu^+(y)]$$
$$= \bar{\mu}(y)$$

Clearly $\omega(x\alpha y) \leq \omega(y)$
$\bar{\mu}((x+z)\alpha y - x\alpha y) = [\mu^-((x+z)\alpha y - x\alpha y), \mu^+((x+z)\alpha y - x\alpha y)]$
$\geq [\mu^-(z), \mu^+(z)]$
$= \bar{\mu}(z)$
Clearly $\omega((x+z)\alpha y - x\alpha y) \leq \omega(z)$
Hence $\mathcal{A} = \langle \bar{\mu}, \omega \rangle$ is a cubic ideal of N.

Theorem 3.6. Let H be a non-empty subset of N. Then H is an ideal of N if and only if the characteristic cubic set $\chi_H =< \bar{\mu}_{\chi_H}, \omega_{\chi_H} >$ of H in N is a cubic ideal of N.

Proof: Let H be a ideal of N. Let $x, y, z \in N$ and $\alpha \in \Gamma$. Suppose that
$\bar{\mu}_{\chi_H}(x-y) < \min\{\bar{\mu}_{\chi_H}(x), \bar{\mu}_{\chi_H}(y)\}$ and $\omega_{\chi_H}(x-y) > \max\{\omega_{\chi_H}(x), \omega_{\chi_H}(y)\}$ for some $x, y \in N$.
Then $\bar{\mu}_{\chi_H}(x-y) = \bar{0}$, $\bar{\mu}_{\chi_H}(x) = \bar{1} = \bar{\mu}_{\chi_H}(y)$ and $\omega_{\chi_H}(x-y) = 1, \omega_{\chi_H}(x) = 0 = \omega_{\chi_H}(y)$.
This implies that $x, y \in H$ but $x - y \notin H$ a contradiction.
Hence $\bar{\mu}_{\chi_H}(x-y) \geq \min\{\bar{\mu}_{\chi_H}(x), \bar{\mu}_{\chi_H}(y)\}$ and $\omega_{\chi_H}(x-y) \leq \max\{\omega_{\chi_H}(x), \omega_{\chi_H}(y)\}$.
Again assume that $\bar{\mu}_{\chi_H}(x\alpha y) < \min\{\bar{\mu}_{\chi_H}(x), \bar{\mu}_{\chi_H}(y)\}$ and $\omega_{\chi_H}(x\alpha y) > \max\{\omega_{\chi_H}(x), \omega_{\chi_H}(y)\}$ for some $x, y \in N$ *and* $\alpha \in \Gamma$. This implies that $\bar{\mu}_{\chi_H}(x\alpha y) = \bar{0}$, $\bar{\mu}_{\chi_H}(x) = \bar{1} = \bar{\mu}_{\chi_H}(y)$ and $\omega_{\chi_H}(x\alpha y) = 1, \omega_{\chi_H}(x) = 0 = \omega_{\chi_H}(y)$. This implies that $x, y \in H$ and $x\alpha y \notin H$, which is a contradiction.
Thus $\bar{\mu}_{\chi_H}(x\alpha y) \geq \min\{\bar{\mu}_{\chi_H}(x), \bar{\mu}_{\chi_H}(y)\}$ and $\omega_{\chi_H}(x\alpha y) \leq \max\{\omega_{\chi_H}(x), \omega_{\chi_H}(y)\}$.
Let us assume that $\bar{\mu}_{\chi_H}(y+x-y) < \bar{\mu}_{\chi_H}(x)$ and $\omega_{\chi_H}(y+x-y) > \omega_{\chi_H}(x)$. This implies that $\bar{\mu}_{\chi_H}(x) = \bar{1}$, $\bar{\mu}_{\chi_H}(y+x-y) = \bar{0}$ and $\omega_{\chi_H}(x) = 0, \omega_{\chi_H}(y+x-y) = 1$. So, $x \in H$ and $y + x - y \notin H$, which is a contradiction. Thus $\bar{\mu}_{\chi_H}(y+x-y) \geq \bar{\mu}_{\chi_H}(x)$ and $\omega_{\chi_H}(y+x-y) \leq \omega_{\chi_H}(x)$.
Assume that $\bar{\mu}_{\chi_H}(x\alpha y) < \bar{\mu}_{\chi_H}(y)$ and $\omega_{\chi_H}(x\alpha y) > \omega_{\chi_H}(y)$ implies $\bar{\mu}_{\chi_H}(x\alpha y) = \bar{0}$, $\bar{\mu}_{\chi_H}(y) = \bar{1}$ and $\omega_{\chi_H}(x\alpha y) = 1, \omega_{\chi_H}(y) = 0$. This implies that $y \in H$ and $x\alpha y \notin H$, which is a contradiction. Thus $\bar{\mu}_{\chi_H}(x\alpha y) \geq \bar{\mu}_{\chi_H}(y)$ and $\omega_{\chi_H}(x\alpha y) \leq \omega_{\chi_H}(y)$. Similarly we can prove $\bar{\mu}_{\chi_H}((x+z)\alpha y - x\alpha y) \geq \bar{\mu}_{\chi_H}(z)$ and $\omega_{\chi_H}((x+z)\alpha y - x\alpha y) \leq \omega_{\chi_H}(z)$.
Therefore $\chi_H =< \bar{\mu}_{\chi_H}, \omega_{\chi_H} >$ is cubic ideal of N.
Conversely, assume that $\chi_H =< \bar{\mu}_{\chi_H}, \omega_{\chi_H} >$ is cubic ideal of N. Let $x, y \in H$. Then
$\bar{\mu}_{\chi_H}(x) = \bar{1} = \bar{\mu}_{\chi_H}(y)$ and $\omega_{\chi_H}(x) = 0 = \omega_{\chi_H}(y)$.
$\bar{\mu}_{\chi_H}(x-y) \geq \min\{\bar{\mu}_{\chi_H}(x), \bar{\mu}_{\chi_H}(y)\} = \min\{\bar{1}, \bar{1}\} = \bar{1}$.
$\omega_{\chi_H}(x-y) \leq \max\{\omega_{\chi_H}(x), \omega_{\chi_H}(y)\} = \min\{0,0\} = 0$.
This implies that $\bar{\mu}_{\chi_H}(x-y) = \bar{1}$ and $\omega_{\chi_H}(x-y) = 0$. Thus $x - y \in H$.
Let $x, y \in H$ and $\alpha \in \Gamma$. Then $\bar{\mu}_{\chi_H}(x) = \bar{1} = \bar{\mu}_{\chi_H}(y)$ and $\omega_{\chi_H}(x) = 0 = \omega_{\chi_H}(y)$.
$\bar{\mu}_{\chi_H}(x\alpha y) \geq \min\{\bar{\mu}_{\chi_H}(x), \bar{\mu}_{\chi_H}(y)\} = \min\{\bar{1}, \bar{1}\} = \bar{1}$.
$\omega_{\chi_H}(x\alpha y) \leq \max\{\omega_{\chi_H}(x), \omega_{\chi_H}(y)\} = \min\{0,0\} = 0$.
This implies that $\bar{\mu}_{\chi_H}(x\alpha y) = \bar{1}$ and $\omega_{\chi_H}(x\alpha y) = 0$. Thus $x\alpha y \in H$.
Let $x \in H$ and $y \in N$. Then $\bar{\mu}_{\chi_H}(x) = \bar{1}$ and $\omega_{\chi_H}(x) = 0$. We have
$\bar{\mu}(y+x-y) \geq \bar{\mu}(x) = \bar{1}$ and $\omega(y+x-y) \leq \omega(x) = 0$. So, $y + x - y \in H$.
Again, let $y \in H, x \in N$ and $\alpha \in \Gamma$ be such that $\bar{\mu}_{\chi_H}(y) = \bar{1}$ and $\omega_{\chi_H}(y) = 0$.
$\bar{\mu}(x\alpha y) \geq \bar{\mu}(y) = \bar{1}$ and $\omega(x\alpha y) \leq \omega(y) = 0$, which implies that $x\alpha y \in H$.
Similarly we can prove that $(x+z)\alpha y - x\alpha y \in H$. Therefore H is an ideal of N.

Theorem 3.7. If $\mathcal{A} = < \bar{\mu}, \omega >$ be any cubic set of N. Then $\mathcal{A} = < \bar{\mu}, \omega >$ is a cubic ideal of N if and only if every non-empty cubic level set $U(\mathcal{A}\,; \tilde{t}, n)$ is an ideal of N.

Proof: Assume that $\mathcal{A} = < \bar{\mu}, \omega >$ be a cubic ideal of N.
Let $x, y, z \in U(\mathcal{A}\,; \tilde{t}, n)$ for all $\tilde{t} \in D[0,1]$ and $n \in [0,1]$. Then $\bar{\mu}(x) \geq \tilde{t}, \bar{\mu}(y) \geq \tilde{t}, \bar{\mu}(z) \geq \tilde{t}$ and $\omega(x) \leq n, \omega(y) \leq n, \omega(z) \leq n$.
Now suppose $x, y \in U(\mathcal{A}\,; \tilde{t}, n)$ then by definition of cubic ideal of N
$\bar{\mu}(x-y) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\} \geq \min\{\tilde{t}, \tilde{t}\} \geq \tilde{t}$ and
$\omega(x-y) \leq \max\{\omega(x), \omega(y)\} \leq \max\{n, n\} \leq n$. Hence $x - y \in U(\mathcal{A}\,; \tilde{t}, n)$.
Suppose $x, y \in U(\mathcal{A}\,; \tilde{t}, n)$ and $\alpha \in \Gamma$ then
$\bar{\mu}(x\alpha y) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\} \geq \min\{\tilde{t}, \tilde{t}\} \geq \tilde{t}$ and
$\omega(x\alpha y) \leq \max\{\omega(x), \omega(y)\} \leq \max\{n, n\} \leq n$. Hence $x\alpha y \in U(\mathcal{A}\,; \tilde{t}, n)$.
Let $x \in U(\mathcal{A}\,; \tilde{t}, n)$ and $y \in N$. We know that $\bar{\mu}(y+x-y) \geq \bar{\mu}(x) \geq \tilde{t}$ and $\omega(y+x-y) \leq \omega(x) \leq n$ this implies that $y + x - y \in U(\mathcal{A}\,; \tilde{t}, n)$.
Let $y \in U(\mathcal{A}\,; \tilde{t}, n), x \in N$ and $\alpha \in \Gamma$. Then $\bar{\mu}(x\alpha y) \geq \bar{\mu}(y) = \tilde{t}$ and $\omega(x\alpha y) \leq \omega(y) = n$.
This implies that $x\alpha y \in U(\mathcal{A}\,; \tilde{t}, n)$.

Let $z \in U(\mathcal{A}\,;\tilde{t},n), x,y \in N$ and $\alpha \in \Gamma$. By definition $\bar{\mu}\big((x+z)\alpha y - x\alpha y\big) \geq \bar{\mu}(z) = \tilde{t}$ and $\omega\big((x+z)\alpha y - x\alpha y\big) \leq \omega(z) = n$. Which implies that $\big((x+z)\alpha y - x\alpha y\big) \in U(\mathcal{A}\,;\tilde{t},n)$.
Therefore $U(\mathcal{A}\,;\tilde{t},n)$ is an ideal of N.
Conversely, let $\tilde{t} \in D[0,1]$ and $n \in [0,1]$ be such that $U(\mathcal{A}\,;\tilde{t},n) \neq \emptyset$ and $U(\mathcal{A}\,;\tilde{t},n)$ is a bi-ideal of N.
Suppose we assume that $\bar{\mu}(x-y) \ngeq \min\{\bar{\mu}(x),\bar{\mu}(y)\}$ (or) $\omega(x-y) \nleq \max\{\omega(x),\omega(y)\}$
If $\bar{\mu}(x-y) \ngeq \min\{\bar{\mu}(x),\bar{\mu}(y)\}$ then there exist $\tilde{t}_1 \in D[0,1]$ such that $\bar{\mu}(x-y) < \tilde{t}_1 < \min\{\bar{\mu}(x),\bar{\mu}(y)\}$
Hence $x,y \in U(\mathcal{A}\,;\tilde{t}_1,\max\{\omega(x),\omega(y)\})$ but $x-y \notin U(\mathcal{A}\,;\tilde{t}_1,\max\{\omega(x),\omega(y)\})$. This is a contradiction.
If $\omega(x-y) \nleq \max\{\omega(x),\omega(y)\}$, then there exist $n_1 \in [0,1]$ such that $\omega(x-y) > n_1 > \max\{\omega(x),\omega(y)\}$ implies $x,y \in U(\mathcal{A}\,;\min\{\bar{\mu}(x),\bar{\mu}(y)\},n_1)$ and $x-y \notin U(\mathcal{A}\,;\min\{\bar{\mu}(x),\bar{\mu}(y)\},n_1)$. This gives a contradiction.
Hence $\bar{\mu}(x-y) \geq \min\{\bar{\mu}(x),\bar{\mu}(y)\}$ and $\omega(x-y) \leq \max\{\omega(x),\omega(y)\}$.
Let $x,y \in N$ and $\alpha \in N$. Suppose $\bar{\mu}(x\alpha y) \ngeq \min\{\bar{\mu}(x),\bar{\mu}(y)\}$ (or) $\omega(x\alpha y) \nleq \max\{\omega(x),\omega(y)\}$
If $\bar{\mu}(x\alpha y) \ngeq \min\{\bar{\mu}(x),\bar{\mu}(y)\}$ then there exist $\tilde{t}_1 \in D[0,1]$ such that $\bar{\mu}(x\alpha y) < \tilde{t}_1 < \min\{\bar{\mu}(x),\bar{\mu}(y)\}$
Hence $x,y \in U(\mathcal{A}\,;\tilde{t}_1,\max\{\omega(x),\omega(y)\})$ but $x\alpha y \notin U(\mathcal{A}\,;\tilde{t}_1,\max\{\omega(x),\omega(y)\})$. This is a contradiction.
If $\omega(x\alpha y) \nleq \max\{\omega(x),\omega(y)\}$, then there exist $n_1 \in [0,1]$ such that $\omega(x\alpha y) > n_1 > \max\{\omega(x),\omega(y)\}$.
Hence $x,y \in U(\mathcal{A}\,;\min\{\bar{\mu}(x),\bar{\mu}(y)\},n_1)$ and $x\alpha y \notin U(\mathcal{A}\,;\min\{\bar{\mu}(x),\bar{\mu}(y)\},n_1)$ which is a contradiction.
Hence $\bar{\mu}(x\alpha y) \geq \min\{\bar{\mu}(x),\bar{\mu}(y)\}$ and $\omega(x\alpha y) \leq \max\{\omega(x),\omega(y)\}$.
Let $x,y \in N$. Suppose $\bar{\mu}(y+x-y) \ngeq \bar{\mu}(x)$ (or) $\omega(y+x-y) \nleq \omega(x)$. If $\bar{\mu}(y+x-y) \ngeq \bar{\mu}(x)$ then there exist $\tilde{t}_1 \in D[0,1]$ such that $\bar{\mu}(y+x-y) < \tilde{t}_1 < \bar{\mu}(x)$. So, $x \in U\big(\mathcal{A}\,;\tilde{t}_1,\omega(x)\big)$ but $y+x-y \notin U\big(\mathcal{A}\,;\tilde{t}_1,\omega(x)\big)$. This is a contradiction. If $\omega(y+x-y) \nleq \omega(x)$ then there exist $n_1 \in [0,1]$ such that $\omega(y+x-y) > n_1 > \omega(x)$. So, $x \in U(\mathcal{A}\,;\bar{\mu}(x),n_1)$ but $y+x-y \notin U(\mathcal{A}\,;\bar{\mu}(x),n_1)$. This is a contradiction. Thus $\bar{\mu}(y+x-y) \geq \bar{\mu}(x)$ and $\omega(y+x-y) \geq \omega(x)$.
Let $x,y \in N$ and $\alpha \in N$. Suppose $\bar{\mu}(x\alpha y) \ngeq \bar{\mu}(y)$ (or) $\omega(x\alpha y) \nleq \omega(y)$. If $\bar{\mu}(x\alpha y) \ngeq \bar{\mu}(y)$ then there exist $\tilde{t}_1 \in D[0,1]$ such that $\bar{\mu}(x\alpha y) < \tilde{t}_1 < \bar{\mu}(y)$ implies that $y \in U\big(\mathcal{A}\,;\tilde{t}_1,\omega(y)\big)$ but $x\alpha y \notin U\big(\mathcal{A}\,;\tilde{t}_1,\omega(y)\big)$. This is a contradiction. If $\omega(x\alpha y) \nleq \omega(y)$, then there exist $n_1 \in [0,1]$ such that $\omega(x\alpha y) > n_1 > \omega(y)$ implies $y \in U(\mathcal{A}\,;\bar{\mu}(y),n_1)$ and $x\alpha y \notin U(\mathcal{A}\,;\bar{\mu}(y),n_1)$ which is a contradiction.
Hence $\bar{\mu}(x\alpha y) \geq \bar{\mu}(y)$ and $\omega(x\alpha y) \leq \omega(y)$.
Assume that $\bar{\mu}((x+z)\alpha y - x\alpha y) \ngeq \bar{\mu}(z)$ (or) $\omega\big((x+z)\alpha y - x\alpha y\big) \nleq \omega(z)$. If $\bar{\mu}((x+z)\alpha y - x\alpha y) \ngeq \bar{\mu}(z)$ then there exist $\tilde{t}_1 \in D[0,1]$ such that $\bar{\mu}\big((x+z)\alpha y - x\alpha y\big) < \tilde{t}_1 < \bar{\mu}(z)$. So, $z \in U\big(\mathcal{A}\,;\tilde{t}_1,\omega(z)\big)$ but $(x+z)\alpha y - x\alpha y \notin U\big(\mathcal{A}\,;\tilde{t}_1,\omega(z)\big)$. This is a contradiction. If $\omega((x+z)\alpha y - x\alpha y) \nleq \omega(z)$ then there exist $n_1 \in [0,1]$ such that $\omega\big((x+z)\alpha y - x\alpha y\big) > n_1 > \omega(z)$. So, $z \in U(\mathcal{A}\,;\bar{\mu}(z),n_1)$ but $y+x-y \notin U(\mathcal{A}\,;\bar{\mu}(z),n_1)$. This is a contradiction.
Thus $\bar{\mu}((x+z)\alpha y - x\alpha y) \geq \bar{\mu}(z)$ and $\omega\big((x+z)\alpha y - x\alpha y\big) \geq \omega(z)$.
Therefore $\mathcal{A} = <\bar{\mu},\omega>$ is a cubic ideal of N.

Theorem 3.8. Let $\mathcal{A} = <\bar{\mu},\omega>$ be a cubic set of N and $\vartheta(x,y) = \big\{\big((x,y),\bar{\tau}(x,y),\delta(x,y)\big) \mid x,y \in N\big\}$ be a strongest cubic relation with respect to ϑ. Then $\mathcal{A} = <\bar{\mu},\omega>$ is a cubic ideal of N if and only if ϑ is a cubic ideal of $N \times N$.

Proof: Assume that $\mathcal{A} = <\bar{\mu},\omega>$ is a cubic ideal of N. Let $x_1,x_2,y_1,y_2,z_1,z_2 \in N$. Then $x = (x_1,x_2), y = (y_1,y_2), z = (z_1,z_2) \in N \times N$ and $\alpha = (\alpha_1,\alpha_2) \in \Gamma$ we have

$$\begin{aligned}
\bar{\tau}(x-y) &= \bar{\tau}((x_1,x_2)-(y_1,y_2)) \\
&= \bar{\tau}(x_1-y_1,x_2-y_2) \\
&= \min\{\bar{\mu}(x_1-y_1),\bar{\mu}(x_2-y_2)\} \\
&\geq \min\{\min\{\bar{\mu}(x_1),\bar{\mu}(y_1)\},\min\{\bar{\mu}(x_2),\bar{\mu}(y_2)\}\} \\
&= \min\{\min\{\bar{\mu}(x_1),\bar{\mu}(x_2)\},\min\{\bar{\mu}(y_1),\bar{\mu}(y_2)\}\} \\
&= \min\{\bar{\tau}(x_1,x_2),\bar{\tau}(y_1,y_2)\} \\
&= \min\{\bar{\tau}(x),\bar{\tau}(y)\}
\end{aligned}$$

$$\begin{aligned}
\delta(x-y) &= \delta((x_1,x_2)-(y_1,y_2)) \\
&= \delta(x_1-y_1,x_2-y_2) \\
&= \max\{\omega(x_1-y_1),\omega(x_2-y_2)\} \\
&\leq \max\{\max\{\omega(x_1),\omega(y_1)\},\max\{\omega(x_2),\omega(y_2)\}\} \\
&= \max\{\max\{\omega(x_1),\omega(x_2)\},\max\{\omega(y_1),\omega(y_2)\}\} \\
&= \max\{\delta(x_1,x_2),\delta(y_1,y_2)\} \\
&= \max\{\delta(x),\delta(y)\}
\end{aligned}$$

$$\begin{aligned}
\bar{\tau}(x\alpha y) &= \bar{\tau}((x_1,x_2)(\alpha_1,\alpha_2)(y_1,y_2)) \\
&= \bar{\tau}(x_1\alpha_1 y_1,x_2\alpha_2 y_2) \\
&= \min\{\bar{\mu}(x_1\alpha_1 y_1),\bar{\mu}(x_2\alpha_2 y_2)\} \\
&\geq \min\{\min\{\bar{\mu}(x_1),\bar{\mu}(y_1)\},\min\{\bar{\mu}(x_2),\bar{\mu}(y_2)\}\} \\
&= \min\{\min\{\bar{\mu}(x_1),\bar{\mu}(x_2)\},\min\{\bar{\mu}(y_1),\bar{\mu}(y_2)\}\}
\end{aligned}$$

$= \min\{\bar{\tau}(x_1, x_2), \bar{\tau}(y_1, y_2)\}$
$= \min\{\bar{\tau}(x), \bar{\tau}(y)\}$

$\delta(x\alpha y) = \delta((x_1, x_2)(\alpha_1, \alpha_2)(y_1, y_2))$
$= \delta(x_1\alpha_1 y_1, x_2\alpha_2 y_2)$
$= \max\{\omega(x_1\alpha_1 y_1), \omega(x_2\alpha_2 y_2)\}$
$\leq \max\{\max\{\omega(x_1), \omega(y_1)\}, \max\{\omega(x_2), \omega(y_2)\}\}$
$= \max\{\max\{\omega(x_1), \omega(x_2)\}, \max\{\omega(y_1), \omega(y_2)\}\}$
$= \max\{\delta(x_1, x_2), \delta(y_1, y_2)\}$
$= \max\{\delta(x), \delta(y)\}$

$\bar{\tau}(y + x - y) = \bar{\tau}((y_1, y_2) + (x_1, x_2) - (y_1, y_2))$
$= \bar{\tau}(y_1 + x_1 - y_1, y_2 + x_2 - y_2)$
$= \min\{\bar{\mu}(y_1 + x_1 - y_1), \bar{\mu}(y_2 + x_2 - y_2)\}$
$\geq \min\{\bar{\mu}(x_1), \bar{\mu}(x_2)\}$
$= \bar{\tau}(x_1, x_2)$
$= \bar{\tau}(x)$

$\delta(y + x - y) = \delta((y_1, y_2) + (x_1, x_2) - (y_1, y_2))$
$= \delta(y_1 + x_1 - y_1, y_2 + x_2 - y_2)$
$= \max\{\omega(y_1 + x_1 - y_1), \omega(y_2 + x_2 - y_2)\}$
$\leq \max\{\omega(x_1), \omega(x_2)\}$
$= \delta(x_1, x_2)$
$= \delta(x)$

$\bar{\tau}(x\alpha y) = \bar{\tau}((x_1, x_2)(\alpha_1, \alpha_2)(y_1, y_2))$
$= \bar{\tau}(x_1\alpha_1 y_1, x_2\alpha_2 y_2)$
$= \min\{\bar{\mu}(x_1\alpha_1 y_1), \bar{\mu}(x_2\alpha_2 y_2)\}$
$\geq \min\{\bar{\mu}(y_1), \bar{\mu}(y_2)\}$
$= \bar{\tau}(y_1, y_2)$
$= \bar{\tau}(y)$

$\delta(x\alpha y) = \delta((x_1, x_2)(\alpha_1, \alpha_2)(y_1, y_2))$
$= \delta(x_1\alpha_1 y_1, x_2\alpha_2 y_2)$
$= \max\{\omega(x_1\alpha_1 y_1), \omega(x_2\alpha_2 y_2)\}$
| $\leq \max\{\omega(y_1), \omega(y_2)\}$
$= \bar{\tau}(y_1, y_2)$
$= \bar{\tau}(y)$

$\bar{\tau}\big((x + z)\alpha y - x\alpha y\big) = \bar{\tau}(((x_1, x_2) + (z_1, z_2))(\alpha_1, \alpha_2)(y_1, y_2) - (x_1, x_2)(\alpha_1, \alpha_2)(y_1, y_2))$
$= \bar{\tau}((x_1 + z_1)\alpha_1 y_1 - x_1\alpha_1 y_1, (x_2 + z_2)\alpha_2 y_2 - x_2\alpha_2 y_2)$
$= \min\{\bar{\mu}\big((x_1 + z_1)\alpha_1 y_1 - x_1\alpha_1 y_1\big), \bar{\mu}\big((x_2 + z_2)\alpha_2 y_2 - x_2\alpha_2 y_2\big)\}$
$\geq \min\{\bar{\mu}(z_1), \bar{\mu}(z_2)\}$
$= \bar{\tau}(z_1, z_2)$
$= \bar{\tau}(z)$

$\delta\big((x + z)\alpha y - x\alpha y\big) = \delta(((x_1, x_2) + (z_1, z_2))(\alpha_1, \alpha_2)(y_1, y_2) - (x_1, x_2)(\alpha_1, \alpha_2)(y_1, y_2))$
$= \delta((x_1 + z_1)\alpha_1 y_1 - x_1\alpha_1 y_1, (x_2 + z_2)\alpha_2 y_2 - x_2\alpha_2 y_2)$
$= \max\{\omega\big((x_1 + z_1)\alpha_1 y_1 - x_1\alpha_1 y_1\big), \omega\big((x_2 + z_2)\alpha_2 y_2 - x_2\alpha_2 y_2\big)\}$
$\leq \max\{\omega(z_1), \omega(z_2)\}$
$= \delta(z_1, z_2)$
$= \delta(z)$

Therefore ϑ is a cubic ideal of $N \times N$.
Conversely, assume that ϑ is a cubic ideal of $N \times N$.
Let $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in N \times N$ and $\alpha = (\alpha_1, \alpha_2) \in \Gamma$.

$\min\{\bar{\mu}(x_1 - y_1), \bar{\mu}(x_2 - y_2)\} = \bar{\tau}(x_1 - y_1, x_2 - y_2)$
$= \bar{\tau}((x_1, x_2) - (y_1, y_2))$
$= \bar{\tau}(x - y)$
$\geq \min\{\bar{\tau}(x), \bar{\tau}(y)\}$
$= \min\{\bar{\tau}(x_1, x_2), \bar{\tau}(y_1, y_2)\}$
$= \min\{\min\{\bar{\mu}(x_1), \bar{\mu}(x_2)\}, \min\{\bar{\mu}(y_1), \bar{\mu}(y_2)\}\}$
$= \min\{\min\{\bar{\mu}(x_1), \bar{\mu}(y_1)\}, \min\{\bar{\mu}(x_2), \bar{\mu}(y_2)\}\}$

It follows that $\bar{\mu}(x_1 - y_1) \geq \min\{\bar{\mu}(x_1), \bar{\mu}(y_1)\}$ and $\bar{\mu}(x_2 - y_2) \geq \min\{\bar{\mu}(x_2), \bar{\mu}(y_2)\}$

$\max\{\omega(x_1 - y_1), \omega(x_2 - y_2)\} = \delta(x_1 - y_1, x_2 - y_2)$
$= \delta((x_1, x_2) - (y_1, y_2))$

$$\begin{aligned}
&= \delta(x-y) \\
&\leq \max\{\delta(x), \delta(y)\} \\
&= \max\{\delta(x_1, x_2), \delta(y_1, y_2)\} \\
&= \max\{\max\{\omega(x_1), \omega(x_2)\}, \max\{\omega(y_1), \omega(y_2)\}\} \\
&= \max\{\max\{\omega(x_1), \omega(y_1)\}, \max\{\omega(x_2), \omega(y_2)\}\}
\end{aligned}$$

It follows that $\omega(x_1 - y_1) \leq \max\{\omega(x_1), \omega(y_1)\}$ and $\omega(x_2 - y_2) \leq \max\{\omega(x_2), \omega(y_2)\}$

$$\begin{aligned}
\min\{\bar{\mu}(x_1\alpha_1 y_1), \bar{\mu}(x_2\alpha_2 y_2)\} &= \bar{\tau}(x_1\alpha_1 y_1, x_2\alpha_2 y_2) \\
&= \bar{\tau}((x_1, x_2)(\alpha_1, \alpha_2)(y_1, y_2)) \\
&= \bar{\tau}(x\alpha y) \\
&\geq \min\{\bar{\tau}(x), \bar{\tau}(y)\} \\
&= \min\{\bar{\tau}(x_1, x_2), \bar{\tau}(y_1, y_2)\} \\
&= \min\{\min\{\bar{\mu}(x_1), \bar{\mu}(x_2)\}, \min\{\bar{\mu}(y_1), \bar{\mu}(y_2)\}\} \\
&= \min\{\min\{\bar{\mu}(x_1), \bar{\mu}(y_1)\}, \min\{\bar{\mu}(x_2), \bar{\mu}(y_2)\}\}
\end{aligned}$$

This implies that $\bar{\mu}(x_1\alpha_1 y_1) \geq \min\{\bar{\mu}(x_1), \bar{\mu}(y_1)\}$ and $\bar{\mu}(x_2\alpha_2 y_2) \geq \min\{\bar{\mu}(x_2), \bar{\mu}(y_2)\}$

$$\begin{aligned}
\max\{\omega(x_1\alpha_1 y_1), \omega(x_2\alpha_2 y_2)\} &= \delta(x_1\alpha_1 y_1, x_2\alpha_2 y_2) \\
&= \delta((x_1, x_2)(\alpha_1, \alpha_2)(y_1, y_2)) \\
&= \delta(x\alpha y) \\
&\leq \max\{\delta(x), \delta(y)\} \\
&= \max\{\delta(x_1, x_2), \delta(y_1, y_2)\} \\
&= \max\{\max\{\omega(x_1), \omega(x_2)\}, \max\{\omega(y_1), \omega(y_2)\}\} \\
&= \max\{\max\{\omega(x_1), \omega(y_1)\}, \max\{\omega(x_2), \omega(y_2)\}\}
\end{aligned}$$

It follows that $\omega(x_1\alpha_1 y_1) \leq \max\{\omega(x_1), \omega(y_1)\}$ and $\omega(x_2\alpha_2 y_2) \leq \max\{\omega(x_2), \omega(y_2)\}$

$$\begin{aligned}
\min\{\bar{\mu}(y_1 + x_1 - y_1), \bar{\mu}(y_2 + x_2 - y_2)\} &= \bar{\tau}(y_1 + x_1 - y_1, y_2 + x_2 - y_2) \\
&= \bar{\tau}((y_1, y_2) + (x_1, x_2) - (y_1, y_2)) \\
&= \bar{\tau}(y + x - y) \\
&\geq \bar{\tau}(x) \\
&= \bar{\tau}(x_1, x_2) \\
&= \min\{\bar{\mu}(x_1), \bar{\mu}(x_2)\}
\end{aligned}$$

Which implies that $\bar{\mu}(y_1 + x_1 - y_1) \geq \bar{\mu}(x_1)$ and $\bar{\mu}(y_2 + x_2 - y_2) \geq \bar{\mu}(x_2)$

$$\begin{aligned}
\max\{\omega(y_1 + x_1 - y_1), \omega(y_2 + x_2 - y_2)\} &= \delta(y_1 + x_1 - y_1, y_2 + x_2 - y_2) \\
&= \delta((y_1, y_2) + (x_1, x_2) - (y_1, y_2)) \\
&= \delta(y + x - y) \\
&\leq \delta(x) \\
&= \delta(x_1, x_2) \\
&= \max\{\omega(x_1), \omega(x_2)\}
\end{aligned}$$

It follows that $\omega(y_1 + x_1 - y_1) \leq \omega(x_1)$ and $\omega(y_2 + x_2 - y_2) \leq \omega(x_2)$

$$\begin{aligned}
\min\{\bar{\mu}(x_1\alpha_1 y_1), \bar{\mu}(x_2\alpha_2 y_2)\} &= \bar{\tau}(x_1\alpha_1 y_1, x_2\alpha_2 y_2) \\
&= \bar{\tau}((x_1, x_2)(\alpha_1, \alpha_2)(y_1, y_2)) \\
&= \bar{\tau}(x\alpha y) \\
&\geq \bar{\tau}(y) \\
&= \bar{\tau}(y_1, y_2) \\
&= \min\{\bar{\mu}(y_1), \bar{\mu}(y_2)\}
\end{aligned}$$

This implies that $\bar{\mu}(x_1\alpha_1 y_1) \geq \bar{\mu}(y_1)$ and $\bar{\mu}(x_2\alpha_2 y_2) \geq \bar{\mu}(y_2)$

$$\begin{aligned}
\max\{\omega(x_1\alpha_1 y_1), \omega(x_2\alpha_2 y_2)\} &= \delta(x_1\alpha_1 y_1, x_2\alpha_2 y_2) \\
&= \delta((x_1, x_2)(\alpha_1, \alpha_2)(y_1, y_2)) \\
&= \delta(x\alpha y) \\
&\leq \delta(y) \\
&= \delta(y_1, y_2) \\
&= \max\{\omega(y_1), \omega(y_2)\}
\end{aligned}$$

This implies that $\omega(x_1\alpha_1 y_1) \leq \omega(y_1)$ and $\omega(x_2\alpha_2 y_2) \leq \omega(y_2)$

$$\begin{aligned}
&\min\{\bar{\mu}\big((x_1 + z_1)\alpha_1 y_1 - x_1\alpha_1 y_1\big), \bar{\mu}\big((x_2 + z_2)\alpha_2 y_2 - x_2\alpha_2 y_2\big)\} \\
&= \bar{\tau}((x_1 + z_1)x_1 y_1 - x_1\alpha_1 y_1, (x_2 + z_2)\alpha_2 y_2 - x_2\alpha_2 y_2) \\
&= \bar{\tau}\big(\big((x_1, x_2) + (z_1, z_2)\big)(\alpha_1, \alpha_2)(y_1, y_2) - (x_1, x_2)(\alpha_1, \alpha_2)(y_1, y_2)\big) \\
&= \bar{\tau}\big((x + z)\alpha y - x\alpha y\big) \\
&\geq \bar{\tau}(z) \\
&= \bar{\tau}(z_1, z_2) \\
&= \min\{\bar{\mu}(z_1), \bar{\mu}(z_2)\}
\end{aligned}$$

It follows that $\bar{\mu}\big((x_1 + z_1)\alpha_1 y_1 - x_1\alpha_1 y_1\big) \geq \bar{\mu}(z_1)$ and $\bar{\mu}\big((x_2 + z_2)\alpha_2 y_2 - x_2\alpha_2 y_2\big) \geq \bar{\mu}(z_2)$

$\max\{\omega((x_1+z_1)\alpha_1y_1-x_1\alpha_1y_1), \omega((x_2+z_2)\alpha_2y_2-x_2\alpha_2y_2)\}$
$= \delta((x_1+z_1)\alpha_1y_1-x_1\alpha_1y_1, (x_2+z_2)\alpha_2y_2-x_2\alpha_2y_2)$
$= \delta(((x_1,x_2)+(z_1,z_2))(\alpha_1,\alpha_2)(y_1,y_2)-(x_1,x_2)(\alpha_1,\alpha_2)(y_1,y_2))$
$= \delta((x+z)\alpha y-x\alpha y)$
$\leq \delta(z)$
$= \delta(z_1,z_2)$
$= \max\{\omega(z_1),\omega(z_2)\}$

It follows that $\omega((x_1+z_1)\alpha_1y_1-x_1\alpha_1y_1) \leq \omega(z_1)$ and $\omega((x_2+z_2)\alpha_2y_2-x_2\alpha_2y_2) \leq \omega(z_2)$

Hence $\mathcal{A} = <\bar{\mu},\omega>$ is a cubic ideal of N.

Theorem 3.9. If $\mathcal{A} = \langle\bar{\mu},\omega\rangle$ is a cubic ideal of N, then the set $N_{\mathcal{A}} = \{x \in N \mid \mathcal{A}(\mathrm{x}) = \mathcal{A}(0)\}$ is an ideal of N.

Proof: Let $\mathcal{A} = \langle\bar{\mu},\omega\rangle$ be a cubic ideal of N and $x, y \in N$, then $\mathcal{A}(\mathrm{x}) = \mathcal{A}(0)$ and $\mathcal{A}(\mathrm{y}) = \mathcal{A}(0)$.

Suppose $x, y, z \in N_{\mathcal{A}}$. Then $\bar{\mu}(x) = \bar{\mu}(y) = \bar{\mu}(z) = \bar{\mu}(0)$ and $\omega(x) = \omega(y) = \omega(z) = \omega(0)$

Since $\bar{\mu}$ is an interval-valued fuzzy ideal of N,

$\bar{\mu}(x-y) \geq \min\{\bar{\mu}(x),\bar{\mu}(y)\} = \min\{\bar{\mu}(0),\bar{\mu}(0)\} = \bar{\mu}(0)$

ω is a fuzzy ideal of N

$\omega(x-y) \leq \max\{\omega(x),\omega(y)\} = \max\{\omega(0),\omega(0)\} = \omega(0)$

Thus $x-y \in N_{\mathcal{A}}$.

Let $x, y \in N_{\mathcal{A}}$ and $\alpha \in \Gamma$.

$\bar{\mu}(x\alpha y) \geq \min\{\bar{\mu}(x),\bar{\mu}(y)\} = \min\{\bar{\mu}(0),\bar{\mu}(0)\} = \bar{\mu}(0)$ and

$\omega(x\alpha y) \leq \max\{\omega(x),\omega(y)\} = \max\{\omega(0),\omega(0)\} = \omega(0)$

This implies that $x\alpha y \in N_{\mathcal{A}.}$

For every $y \in N$ and $x \in N_{\mathcal{A}}$ we have

$\bar{\mu}(y+x-y) \geq \bar{\mu}(x) = \bar{\mu}(0)$ and $\omega(y+x-y) \leq \omega(x) = \omega(0)$

Thus $y+x-y \in N_{\mathcal{A}}$.

Let $x \in N$, $y \in N_{\mathcal{A}}$ and $\alpha \in \Gamma$ we have

$\bar{\mu}(x\alpha y) \geq \bar{\mu}(y) = \bar{\mu}(0)$ and $\omega(x\alpha y) \leq \omega(x) = \omega(0)$

Thus $x\alpha y \in N_{\mathcal{A}.}$ Similarly, we have to prove $(x+z)\alpha y - x\alpha y \in N_{\mathcal{A}}$.

Therefore, $N_{\mathcal{A}}$ is an ideal of N.

Theorem 3.10. The direct product of cubic ideals of Γ-near-ring is a cubic ideal of Γ-near-ring.

Proof: Let $\mathcal{A}_i = \langle\bar{\mu}_i,\omega_i\rangle$ be cubic ideals of Γ-near-rings N_i for $i = 1,2,3,\dots,n$.

Let $N = N_1 \times N_2 \times \dots \times N_n, \Gamma = \Gamma_1 \times \Gamma_2 \times \dots \times \Gamma_n$, and

$x = (x_1,x_2,\dots,x_n),\ y = (y_1,y_2,\dots,y_n), z = (z_1,z_2,\dots,z_n) \in N, \alpha = (\alpha_1,\alpha_2,\dots,\alpha_n) \in \Gamma$.

$\bar{\mu}_i(x-y) = \bar{\mu}_i((x_1,x_2,\dots,x_n)-(y_1,y_2,\dots,y_n))$
$= \bar{\mu}_i(x_1-y_1, x_2-y_2,\dots,x_n-y_n)$
$= \min\{\bar{\mu}_1(x_1-y_1),\bar{\mu}_2(x_2-y_2),\dots,\bar{\mu}_n(x_n-y_n)\}$
$\geq \min\{\min\{\bar{\mu}_1(x_1),\bar{\mu}_1(y_1)\},\min\{\bar{\mu}_2(x_2),\bar{\mu}_2(y_2)\},\dots,\min\{\bar{\mu}_n(x_n),\bar{\mu}_n(y_n)\}\}$
$= \min\{\min\{\bar{\mu}_1(x_1),\bar{\mu}_2(x_2),\dots,\bar{\mu}_n(x_n)\},\min\{\bar{\mu}_1(y_1),\bar{\mu}_2(y_2),\dots,\bar{\mu}_n(y_n)\}\}$
$= \min\{(\bar{\mu}_1\times\bar{\mu}_2\times,\dots,\times\bar{\mu}_n)(x_1,x_2,\dots,x_n),(\bar{\mu}_1\times\bar{\mu}_2\times,\dots,\times\bar{\mu}_n)(y_1,y_2,\dots,y_n)\}$

$\bar{\mu}_i(x-y) = \min\{\bar{\mu}_i(x),\bar{\mu}_i(y)\}$

$\omega_i(x-y) = \omega_i((x_1,x_2,\dots,x_n)-(y_1,y_2,\dots,y_n))$
$= \omega_i(x_1-y_1,x_2-y_2,\dots,x_n-y_n)$
$= \max\{\omega_1(x_1-y_1),\omega_2(x_2-y_2),\dots,\omega_n(x_n-y_n)\}$
$\leq \max\{\max\{\omega_1(x_1),\omega_1(y_1)\},\max\{\omega_2(x_2),\omega_2(y_2)\},\dots,\max\{\omega_n(x_n),\omega_n(y_n)\}\}$
$= \max\{\max\{\omega_1(x_1),\omega_2(x_2),\dots,\omega_n(x_n)\},\max\{\omega_1(y_1),\omega_2(y_2),\dots,\omega_n(y_n)\}\}$
$= \max\{(\omega_1\times\omega_2\times,\dots,\times\omega_n)(x_1,x_2,\dots,x_n),(\omega_1\times\omega_2\times,\dots,\times\omega_n)(y_1,y_2,\dots,y_n)\}$

$\omega_i(x-y) = \max\{\omega_i(x),\omega_i(y)\}$

$\bar{\mu}_i(x\alpha y) = \bar{\mu}_i((x_1,x_2,\dots,x_n)(\alpha_1,\alpha_2,\dots,\alpha_n)(y_1,y_2,\dots,y_n))$
$= \bar{\mu}_i(x_1\alpha_1y_1,x_2\alpha_2y_2,\dots,x_n\alpha_ny_n)$
$= \min\{\bar{\mu}_1(x_1\alpha_1y_1),\bar{\mu}_2(x_2\alpha_2y_2),\dots,\bar{\mu}_n(x_n\alpha_ny_n)\}$
$\geq \min\{\min\{\bar{\mu}_1(x_1),\bar{\mu}_1(y_1)\},\min\{\bar{\mu}_2(x_2),\bar{\mu}_2(y_2)\},\dots,\min\{\bar{\mu}_n(x_n),\bar{\mu}_n(y_n)\}\}$
$= \min\{\min\{\bar{\mu}_1(x_1),\bar{\mu}_2(x_2),\dots,\bar{\mu}_n(x_n)\},\min\{\bar{\mu}_1(y_1),\bar{\mu}_2(y_2),\dots,\bar{\mu}_n(y_n)\}\}$
$= \min\{(\bar{\mu}_1\times\bar{\mu}_2\times,\dots,\times\bar{\mu}_n)(x_1,x_2,\dots,x_n),(\bar{\mu}_1\times\bar{\mu}_2\times,\dots,\times\bar{\mu}_n)(y_1,y_2,\dots,y_n)\}$

$\bar{\mu}_i(x\alpha y) = \min\{\bar{\mu}_i(x),\bar{\mu}_i(y)\}$

$\omega_i(x\alpha y) = \omega_i((x_1,x_2,\dots,x_n)(\alpha_1,\alpha_2,\dots,\alpha_n)(y_1,y_2,\dots,y_n))$
$= \omega_i(x_1\alpha_1y_1,x_2\alpha_2y_2,\dots,x_n\alpha_ny_n)$
$= \max\{\omega_1(x_1\alpha_1y_1),\omega_2(x_2\alpha_2y_2),\dots,\omega_n(x_n\alpha_ny_n)\}$
$\leq \max\{\max\{\omega_1(x_1),\omega_1(y_1)\},\max\{\omega_2(x_2),\omega_2(y_2)\},\dots,\max\{\omega_n(x_n),{}_{-n}(y_n)\}\}$

$= \max\{\max\{\omega_1(x_1), \omega_2(x_2), \dots, \omega_n(x_n)\}, \max\{\omega_1(y_1), \omega_2(y_2), \dots, \omega_n(y_n)\}\}$
$= \max\{(\omega_1 \times \omega_2 \times, \dots, \times \omega_n)(x_1, x_2, \dots, x_n), (\omega_1 \times \omega_2 \times, \dots, \times \omega_n)(y_1, y_2, \dots, y_n)\}$
$\omega_i(x\alpha y) = \max\{\omega_i(x), \omega_i(y)\}$
$\bar{\mu}_i(y + x - y) = \bar{\mu}_i\big((y_1, y_2, \dots, y_n) + (x_1, x_2, \dots, x_n) - (y_1, y_2, \dots, y_n)\big)$
$= \bar{\mu}_i(y_1 + x_1 - y_1, y_2 + x_2 - y_2, \dots, y_n + x_n - y_n)$
$= \min\{\bar{\mu}_1(y_1 + x_1 - y_1), \bar{\mu}_2(y_2 + x_2 - y_2), \dots, \bar{\mu}_n(y_n + x_n - y_n)\}$
$\geq \min\{\bar{\mu}_1(x_1), \bar{\mu}_2(x_2), \dots, \bar{\mu}_n(x_n)\}$
$= (\bar{\mu}_1 \times \bar{\mu}_2 \times, \dots, \times \bar{\mu}_n)(x_1, x_2, \dots, x_n)$
$\bar{\mu}_i(y + x - y) = \bar{\mu}_i(x)$
$\omega_i(y + x - y) = \omega_i\big((y_1, y_2, \dots, y_n) + (x_1, x_2, \dots, x_n) - (y_1, y_2, \dots, y_n)\big)$
$= \omega_i(y_1 + x_1 - y_1, y_2 + x_2 - y_2, \dots, y_n + x_n - y_n)$
$= \max\{\omega_1(y_1 + x_1 - y_1), \omega_2(y_2 + x_2 - y_2), \dots, \omega_n(y_n + x_n - y_n)\}$
$\leq \max\{\omega_1(x_1), \omega_2(x_2), \dots, \omega_n(x_n)\}$
$= (\omega_1 \times \omega_2 \times, \dots, \times \omega_n)(x_1, x_2, \dots, x_n)$
$\omega_i(y + x - y) = \omega_i(x)$
$\bar{\mu}_i(x\alpha y) = \bar{\mu}_i\big((x_1, x_2, \dots, x_n)(\alpha_1, \alpha_2, \dots, \alpha_n)(y_1, y_2, \dots, y_n)\big)$
$= \bar{\mu}_i(x_1\alpha_1 y_1, x_2\alpha_2 y_2, \dots, x_n\alpha_n y_n)$
$= \min\{\bar{\mu}_1(x_1\alpha_1 y_1), \bar{\mu}_2(x_2\alpha_2 y_2), \dots, \bar{\mu}_n(x_n\alpha_n y_n)\}$
$\geq \min\{\bar{\mu}_1(y_1), \bar{\mu}_2(y_2), \dots, \bar{\mu}_n(y_n)\}$
$= (\bar{\mu}_1 \times \bar{\mu}_2 \times, \dots, \times \bar{\mu}_n)(y_1, y_2, \dots, y_n)$
$\bar{\mu}_i(x\alpha y) = \bar{\mu}_i(y)$
$\omega_i(x\alpha y) = \omega_i\big((x_1, x_2, \dots, x_n)(\alpha_1, \alpha_2, \dots, \alpha_n)(y_1, y_2, \dots, y_n)\big)$
$= \omega_i(x_1\alpha_1 y_1, x_2\alpha_2 y_2, \dots, x_n\alpha_n y_n)$
$= \max\{\omega_1(x_1\alpha_1 y_1), \omega_2(x_2\alpha_2 y_2), \dots, \omega_n(x_n\alpha_n y_n)\}$
$\leq \max\{\omega_1(y_1), \omega_2(y_2), \dots, \omega_n(y_n)\}$
$= (\omega_1 \times \omega_2 \times, \dots, \times \omega_n)(y_1, y_2, \dots, y_n)$
$\omega_i(x\alpha y) = \omega_i(y)$
$\bar{\mu}_i\big((x + z)\alpha y - x\alpha y\big)$
$= \bar{\mu}_i\Big(\big((x_1, x_2, \dots, x_n) + (z_1, z_2, \dots, z_n)\big)(\alpha_1, \alpha_2, \dots, \alpha_n)(y_1, y_2, \dots, y_n)$
$- (x_1, x_2, \dots, x_n)(\alpha_1, \alpha_2, \dots, \alpha_n)(y_1, y_2, \dots, y_n)\Big)$
$= \bar{\mu}_i(((x_1 + z_1)\alpha_1 y_1 - x_1\alpha_1 y_1), ((x_2 + z_2)\alpha_2 y_2 - x_2\alpha_2 y_2), \dots, ((x_n + z_n)\alpha_n y_n - x_n\alpha_n y_n))$
$= \min\{\bar{\mu}_1(y_1 + x_1 - y_1), \bar{\mu}_2(y_2 + x_2 - y_2), \dots, \bar{\mu}_n(y_n + x_n - y_n)\}$
$\geq \min\{\bar{\mu}_1(z_1), \bar{\mu}_2(z_2), \dots, \bar{\mu}_n(z_n)\}$
$= (\bar{\mu}_1 \times \bar{\mu}_2 \times, \dots, \times \bar{\mu}_n)(z_1, z_2, \dots, z_n)$
$\bar{\mu}_i\big((x + z)\alpha y - x\alpha y\big) = \bar{\mu}_i(z)$
$\omega_i\big((x + z)\alpha y - x\alpha y\big)$
$= \omega_i\Big(\big((x_1, x_2, \dots, x_n) + (z_1, z_2, \dots, z_n)\big)(\alpha_1, \alpha_2, \dots, \alpha_n)(y_1, y_2, \dots, y_n)$
$- (x_1, x_2, \dots, x_n)(\alpha_1, \alpha_2, \dots, \alpha_n)(y_1, y_2, \dots, y_n)\Big)$
$= \omega_i(((x_1 + z_1)\alpha_1 y_1 - x_1\alpha_1 y_1), ((x_2 + z_2)\alpha_2 y_2 - x_2\alpha_2 y_2), \dots, ((x_n + z_n)\alpha_n y_n - x_n\alpha_n y_n))$
$= \max\{\omega_1(y_1 + x_1 - y_1), \omega_2(y_2 + x_2 - y_2), \dots, \omega_n(y_n + x_n - y_n)\}$
$\leq \max\{\omega_1(z_1), \omega_2(z_2), \dots, \omega_n(z_n)\}$
$= (\omega_1 \times \omega_2 \times, \dots, \times \omega_n)(z_1, z_2, \dots, z_n)$
$\omega_i\big((x + z)\alpha y - x\alpha y\big) = \omega_i(z)$
Hence $\mathcal{A}_i = \langle \bar{\mu}_i, \omega_i \rangle$ is a cubic ideals of Γ-near-rings N_i.
Theorem 3.11. Let N and N_1 be two Γ-near-rings and $f: N \to N_1$ be an onto near-ring homomorphism. If $\mathcal{A} = < \bar{\mu}, \omega >$ is a cubic ideal of N then $f(\mathcal{A}) =< f(\bar{\mu}), f(\omega) >$ is a cubic ideal of N_1.
Proof: Let $\mathcal{A} = < \bar{\mu}, \omega >$ be a cubic ideal of N and $x, y \in N$.
Since $f(\bar{\mu})(x') = \sup_{f(x)=x'} \bar{\mu}(x)$ and $f(\omega)(x') = \inf_{f(x)=x'} \omega(x)$ for $x' \in N_1$
So $f(\mathcal{A}) =< f(\bar{\mu}), f(\omega) >$ is non-empty. Let $x', y' \in N_1$ and $\alpha \in \Gamma$. Then we have
$\{x |\, x \in f^{-1}(x' - y')\} \supseteq \{x - y \mid x \in f^{-1}(x')\ and\ y \in f^{-1}(y')\}$ and
$\{x |\, x \in f^{-1}(x'\alpha y')\} \supseteq \{x\alpha y \mid x \in f^{-1}(x')\ and\ y \in f^{-1}(y')\}$
$f(\bar{\mu})(x' - y') = \sup_{f(p)=x'-y'} \bar{\mu}(p)$
$\geq \sup_{f(x)=x', f(y)=y'} \bar{\mu}(x - y)$
$\geq \sup_{f(x)=x', f(y)=y'} \min\{\bar{\mu}(x), \bar{\mu}(y)\}$

$= \min\left\{\sup_{f(x)=x'} \bar{\mu}(x), \sup_{f(y)=y'} \bar{\mu}(y)\right\}$
$= \min\{f(\bar{\mu})(x'), f(\bar{\mu})(y')\}$
$f(\omega)(x'-y') = \inf_{f(p)=x'-y'} \omega(p)$
$\leq \inf_{f(x)=x', f(y)=y'} \omega(x-y)$
$\leq \inf_{f(x)=x', f(y)=y'} \max\{\omega(x), \omega(y)\}$
$= \max\left\{\inf_{f(x)=x'} \omega(x), \inf_{f(y)=y'} \omega(y)\right\}$
$= \max\{f(\omega)(x'), f(\omega)(y')\}$
$f(\bar{\mu})(x'\alpha y') = \sup_{f(p)=x'\alpha y'} \bar{\mu}(p)$
$\geq \sup_{f(x)=x', f(y)=y'} \bar{\mu}(x\alpha y)$
$\geq \text{s}Ip_{f(x)=x', f(y)=y'} \min\{\bar{\mu}(x), \bar{\mu}(y)\}$
$= \min\left\{\sup_{f(x)=x'} \bar{\mu}(x), \sup_{f(y)=y'} \bar{\mu}(y)\right\}$
$= \min\{f(\bar{\mu})(x'), f(\bar{\mu})(y')\}$
$f(\omega)(x'\alpha y') = \inf_{f(p)=x'\alpha y'} \omega(p)$
$\leq \inf_{f(x)=x', f(y)=y'} \omega(x\alpha y)$
$\leq \inf_{f(x)=x', f(y)=y'} \max\{\omega(x), \omega(y)\}$
$= \max\left\{\inf_{f(x)=x'} \omega(x), \inf_{f(y)=y'} \omega(y)\right\}$
$= \max\{f(\omega)(x'), f(\omega)(y')\}$

$f(\bar{\mu})(y'+x'-y') = \sup_{f(p)=y'+x'-y'} \bar{\mu}(p) \geq \sup_{f(x)=x', f(y)=y'} \bar{\mu}(y+x-y) \geq \sup_{f(x)=x'} \bar{\mu}(x) = f(\bar{\mu})(x')$
$f(\omega)(y'+x'-y') = \inf_{f(p)=y'+x'-y'} \omega(p) \leq \inf_{f(x)=x', f(y)=y'} \omega(y+x-y) \leq \inf_{f(x)=x'} \omega(x) = f(\omega)(x')$
$f(\bar{\mu})(x'\alpha y') = \sup_{f(p)=x'\alpha y'} \bar{\mu}(p) \geq \sup_{f(x)=x', f(y)=y'} \bar{\mu}(x\alpha y) \geq \sup_{f(y)=y'} \bar{\mu}(y) = f(\bar{\mu})(y')$
$f(\omega)(x'\alpha y') = \inf_{f(p)=x'\alpha y'} \omega(p) \leq \inf_{f(x)=x', f(y)=y'} \omega(x\alpha y) \leq \inf_{f(y)=y'} \omega(y) = f(\omega)(y')$
$f(\bar{\mu})((x'+z')\alpha y' - x'\alpha y') = \sup_{f(p)=((x'+z')\alpha y'-x'\alpha y')} \bar{\mu}(p)$
$\geq \sup_{f(x)=x', f(y)=y', f(z)=z'} \bar{\mu}((x'+z')\alpha y' - x'\alpha y')$
$\geq \sup_{f(z)=z'} \bar{\mu}(z)$
$= f(\bar{\mu})(z')$
$f(\omega)((x'+z')\alpha y' - x'\alpha y') = \inf_{f(p)=((x'+z')\alpha y'-x'\alpha y')} \omega(p)$
$\leq \inf_{f(x)=x', f(y)=y', f(z)=z'} \omega((x'+z')\alpha y' - x'\alpha y')$
$\leq \inf_{f(z)=z'} \omega(z)$
$= f(\omega)(z')$

Therefore $f(\mathcal{A}) = < f(\bar{\mu}), f(\omega) >$ is a cubic ideal of N_1.

Theorem 3.12. Let $f: N \to N_1$ be an onto homomorphism between Γ-near-rings N and N_1. If $\mathcal{A} = < \bar{\mu}, \omega >$ is a cubic subset of N_1 such that $f^{-1}(\mathcal{A}) = < f^{-1}(\bar{\mu}), f^{-1}(\omega) >$ is a cubic ideal of N then $\mathcal{A} = < \bar{\mu}, \omega >$ is a cubic ideal of N_1.

Proof: Let $x, y, z \in N_1$ and $\alpha \in \Gamma$. Then
$f(a) = x, f(b) = y, f(c) = z$ for some $a, b, c \in N$.
$\bar{\mu}(x-y) = \bar{\mu}(f(a) - f(b))$
$= \bar{\mu}(f(a-b))$
$= f^{-1}(\bar{\mu})(a-b)$
$\geq \min\{f^{-1}(\bar{\mu})(a), f^{-1}(\bar{\mu})(b)\}$
$= \min\{\bar{\mu}(f(a)), \bar{\mu}(f(b))\}$
$= \min\{\bar{\mu}(x), \bar{\mu}(y)\}$
$\omega(x-y) = \omega(f(a) - f(b))$
$= \omega(f(a-b))$
$= f^{-1}(\omega)(a-b)$
$\leq \max\{f^{-1}(\omega)(a), f^{-1}(\omega)(b)\}$
$= \max\{\omega(f(a)), \omega(f(b))\}$
$= \max\{\omega(x), \omega(y)\}$

$\bar{\mu}(x\alpha y) = \bar{\mu}(f(a)\alpha f(b))$
$= \bar{\mu}(f(a\alpha b))$
$= f^{-1}(\bar{\mu})(a\alpha b)$
$\geq \min\{f^{-1}(\bar{\mu})(a), f^{-1}(\bar{\mu})(b)\}$
$= \min\{\bar{\mu}(f(a)), \bar{\mu}(f(b))\}$
$= \min\{\bar{\mu}(x), \bar{\mu}(y)\}$
$\omega(x\alpha y) = \omega(f(a)\alpha f(b))$
$= \omega(f(a\alpha b))$
$= f^{-1}(\omega)(a\alpha b)$
$\leq \max\{f^{-1}(\omega)(a), f^{-1}(\omega)(b)\}$
$= \max\{\omega(f(a)), \omega(f(b))\}$
$= \max\{\omega(x), \omega(y)\}$
$\bar{\mu}(y + x - y) = \bar{\mu}(f(b) + f(a) - f(b))$
$= \bar{\mu}(f(b + a - b))$
$= f^{-1}(\bar{\mu})(b + a - b)$
$\geq f^{-1}(\bar{\mu})(a)$
$= \bar{\mu}(f(a))$
$= \bar{\mu}(x)$
$\omega(y + x - y) = \omega(f(b) + f(a) - f(b))$
$= \omega(f(b + a - b))$
$= f^{-1}(\omega)(b + a - b)$
$\leq f^{-1}(\omega)(a)$
$= \omega(f(a))$
$= \omega(x)$
$\bar{\mu}(x\alpha y) = \bar{\mu}(f(a)\alpha f(b)) = \bar{\mu}(f(a\alpha b)) = f^{-1}(\bar{\mu})(a\alpha b) \geq f^{-1}(\bar{\mu})(b) = \bar{\mu}(f(b)) = \bar{\mu}(y)$
$\omega(x\alpha y) = \omega(f(a)\alpha f(b)) = \omega(f(a\alpha b)) = f^{-1}(\omega)(a\alpha b) \leq f^{-1}(\omega)(b) = \omega(f(b)) = \omega(y)$
$\bar{\mu}((x + z)\alpha y - x\alpha y) = \bar{\mu}\left((f(a) + f(c))\alpha f(b) - f(a)\alpha f(b)\right)$
$= \bar{\mu}(f((a + c)\alpha b - a\alpha b))$
$= f^{-1}(\bar{\mu})((a + c)\alpha b - a\alpha b)$
$\geq f^{-1}(\bar{\mu})(c)$
$= \bar{\mu}(f(c))$
$= \bar{\mu}(z)$
$\omega((x + z)\alpha y - x\alpha y) = \omega\left((f(a) + f(c))\alpha f(b) - f(a)\alpha f(b)\right)$
$= \omega(f((a + c)\alpha b - a\alpha b))$
$= f^{-1}(\omega)((a + c)\alpha b - a\alpha b)$
$\leq f^{-1}(\omega)(c)$
$= \omega(f(c))$
$= \omega(z)$

Hence $\mathcal{A} = < \bar{\mu}, \omega >$ is a cubic ideal of Γ-near-ring N_1.

Theorem 3.13. Let $f: N \to N_1$ be a homomorphism between Γ- near-rings N and N_1. If $\mathcal{A} = < \bar{\mu}, \omega >$ is a cubic ideal of Γ- near-ring N_1 then $f^{-1}(\mathcal{A}) =< f^{-1}(\bar{\mu}), f^{-1}(\omega) >$ is a cubic ideal of Γ- near-ring N.

Proof: Let $\mathcal{A} = < \bar{\mu}, \omega >$ is a cubic ideal of N_1. Let $x, y, z \in N$ and $\alpha \in \Gamma$. Then
$f^{-1}(\bar{\mu})(x - y) = \bar{\mu}(f(x - y))$
$= \bar{\mu}(f(x) - f(y))$
$\geq \min\{\bar{\mu}(f(x)), \bar{\mu}(f(y))\}$
$= \min\{f^{-1}(\bar{\mu}(x)), f^{-1}(\bar{\mu}(y))\}$
$f^{-1}(\omega)(x - y) = \omega(f(x - y))$
$= \omega(f(x) - f(y))$
$\leq \max\{\omega(f(x)), \omega(f(y))\}$
$= \max\{f^{-1}(\omega(x)), f^{-1}(\omega(y))\}$
$f^{-1}(\bar{\mu})(x\alpha y) = \bar{\mu}(f(x\alpha y))$
$= \bar{\mu}(f(x)\alpha f(y))$
$\geq \min\{\bar{\mu}(f(x)), \bar{\mu}(f(y))\}$
$= \min\{f^{-1}(\bar{\mu}(x)), f^{-1}(\bar{\mu}(y))\}$
$f^{-1}(\omega)(x\alpha y) = \omega(f(x\alpha y))$
$= \omega(f(x)\alpha f(y))$

$\leq \max\{\omega(f(x)), \omega(f(y))\}$
$= \max\{f^{-1}(\omega(x)), f^{-1}(\omega(y))\}$

$f^{-1}(\bar{\mu})(y + x - y) = \bar{\mu}(f(y + x - y))$
$= \bar{\mu}(f(y) + f(x) - f(y))$
$\geq \bar{\mu}(f(x))$
$= f^{-1}(\bar{\mu}(x))$

$f^{-1}(\omega)(y + x - y) = \omega(f(y + x - y))$
$= \omega(f(y) + f(x) - f(y))$
$\leq \omega(f(x))$
$= f^{-1}(\omega(x))$

$f^{-1}(\bar{\mu})(x\alpha y) = \bar{\mu}(f(x\alpha y)) = \bar{\mu}\big(f(x)\alpha f(y)\big) \geq \bar{\mu}\big(f(y)\big) = f^{-1}(\bar{\mu}(y))$

$f^{-1}(\omega)(x\alpha y) = \omega(f(x\alpha y)) = \omega\big(f(x)\alpha f(y)\big) \leq \omega\big(f(y)\big) = f^{-1}(\omega(y))$

$f^{-1}(\bar{\mu})((x + z)\alpha y - x\alpha y) = \bar{\mu}(f((x + z)\alpha y - x\alpha y))$
$= \bar{\mu}(\big(f(x) + f(z)\big)\alpha f(y) - f(x)\alpha f(y))$
$\geq \bar{\mu}(f(z))$
$= f^{-1}(\bar{\mu}(z))$

$f^{-1}(\omega)((x + z)\alpha y - x\alpha y) = \omega(f((x + z)\alpha y - x\alpha y))$
$= \omega(\big(f(x) + f(z)\big)\alpha f(y) - f(x)\alpha f(y))$
$\leq \omega(f(z))$
$= f^{-1}(\omega(z))$

Hence, $f^{-1}(\mathcal{A}) = < f^{-1}(\bar{\mu}), f^{-1}(\omega) >$ is a cubic ideal of Γ- near-ring N.

IV. Conclusion

In the structural theory of fuzzy algebraic systems, fuzzy ideals with special properties always play in important role. In this paper we have presented cubic ideals of Γ- near-rings. We applied the interval-valued fuzzy set theory and fuzzy set theory in Γ- near-ring by their cubic ideals and obtained many results. The obtained results can be applied in various fields such as computer networks, robotics and neural networks.
In our future work on this topic we try to extend this concept to cubic bi-ideals in Γ- near-rings and cubic interior ideals in Γ- near-rings.

Acknowledgements

The authors would like to thank the referees for their valuable comments and suggestions for improving the paper.

References

Journal Papers:

[1]. L. A. Zadeh, Fuzzy sets, *Information and Computation, 8*, 1965, 338-353.
[2]. L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning I, *Information Sciences, 8,* 1975, 1-24.
[3]. A. Rosenfeld, Fuzzy groups, *Journal of Mathematical Analysis and Applications, 35*, 1971, 512-517.
[4]. S. Abou-Zaid, On fuzzy subnear-rings and ideals, *Fuzzy Sets and Systems, 44,* 1991, 139-146.
[5]. Y. B. Jun and K. H. kim, Interval-valued fuzzy R-subgroup of near-rings, *Indian Journal of Pure and Applied Mathematics, 33(1),* 2002, 71-80.
[6]. B. Davvaz, Fuzzy ideals and near-rings with interval-valued membership functions, *Journal of Sciences, Islamic republic of Iran, 12(2),* 2001, 171-175.
[7]. N. Thillaigovindan, V. Chinnadurai and S. Kadalarasi, Interval valued fuzzy ideals of near-rings, *Journal of Fuzzy Mathematics, 23(2),* 2015, 471-483.
[8]. Y.B. Jun, C.S. Kim and K.O. Yang, Cubic sets, *Annals of Fuzzy Mathematics and Informatics, 4(1),* 2012, 83-98.
[9]. V. Chinnadurai and K. Bharathivelan, Cubic bi-ideals in near-rings, *International Journal of Computer and Mathematical Sciences, 5(2),* 2016, 44-52.
[10]. V. Chinnadurai and K. Bharathivelan, Cubic ideals of Γ-semigroups, *International Journal of Current Research and Modern Education, 1(2),* 2016, 138-150.
[11]. T. Srinivas, T. Nagaiag and P. Narasimha Swamy, Anti fuzzy ideals of Γ-near-ring, Annals of Fuzzy Mathematics and Informatics, 3(2), 2012, 255-256.
[12]. Y. B. Jun and Asghar Khan, Cubic ideals of semigroups, *Honam Mathematical Journal, 35(4),* 2013, 607-623.

Books:

[13]. V. Chinnadurai, *Fuzzy ideals in algebraic structures*, Lap lambert academic publishing, Germany, (2013), ISBN 10:3659415316/ ISBN13:9783659415319.

Theses:

[14]. Bh.Satyanarayana, *Contributions to near-ring theory*, Doctoral thesis, Nagarjuna University, Andhrapradesh, 1984.