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# Algorithmic Approach for Computing Generalized Parikh Matrices of Binary Words and 2D Binary Arrays

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***Abstract:*** *The Generalized Parikh vectors and the Generalized Parikh matrices have been studied to give a better numerical representation of words & 2D binary arrays. Algorithms to find these Generalized Parikh matrices of binary words and 2D binary arrays are given in the paper. Also some conditions related to the commutativity of the product of these Generalized Parikh matrices have been discussed.*

***Keywords:*** *Parikh vectors, Parikh matrices, binary words, 2D binary arrays, Generalized Parikh vectors, Generalized Parikh matrices, path automaton.*

## I. Introduction

The study of the numerical properties of words has gained more interest recently. Counting the number of occurrences of the alphabets in a word was initially recorded as the components of the Parikh vector by R.J. Parikh (1966) [1]. From this root various branches of study have evolved. Mateescu et al. (2001) [2] introduced the Parikh matrices of words which gave the count of the alphabets as well as the count of the sub word occurrences in the word. As the Parikh vector and Parikh matrices were not injective studies were made on the M-ambiguous words and many conditions like ratio property, weak ratio property for ambiguity where given by Subramanian et al. (2009) [3]. Another branch of study that had grown along with this was the study of Generalized Parikh vectors by Sironmoney and Rajkumar Dare (1985) [4] which gave numerical representation of words using the positions in which they occur in words. This was further extended as the study of Generalized Parikh matrices of words by Huldah Samuel (2013) [5].

All these studies that were made for words have been made for 2D binary arrays recently. Subramanian et al. (2011) introduced the Parikh vectors and Parikh matrices of 2D binary arrays [6], [7]. The conditions for M-ambiguity have been studied. Subramanian et al. studied about the product of Parikh matrices and commutativity (2012) [8] and left an open question to study about the commutativity of the product of Generalized Parikh matrices. With this motivation the authors have defined & studied the Generalized Parikh vectors & Generalized Parikh matrices of 2D binary arrays (2015) [9]. This paper gives an algorithmic approach to find the Generalized Parikh matrices of binary words & 2D binary arrays and displays few results related to the commutativity of their products. Also an interesting observation is made in which every decimal number N finds a place in the Generalized Parikh matrix of its binary equivalent.

## II. Preliminaries

Let $\Sigma$ be the non-empty finite set of alphabets. A word $w \in \Sigma^*$ is a string of alphabets over $\Sigma$. The length of a word w is $|w|$. The number of occurrences of the alphabet a in the word w is denoted by $\#_a(w)$ or $|w|_a$. The mirror image of a word w is denoted by mi(w).

**Definition 1:**
For any word $w \in \Sigma^*$ and $\Sigma = \{a_1, a_2, \ldots, a_k\}$ the Parikh vector of the word w is

$$\psi(w) = (\#_{a_1}(w), \#_{a_2}(w), \ldots, \#_{a_k}(w))$$

**Definition 2:**
If w is any word over $\Sigma^*$ and $\Sigma = \{a_1, a_2, \ldots, a_k\}$ the Generalized Parikh vector of w is p(w) defined by

$$p(w) = (p_1, p_2, \ldots, p_k) \quad \text{where} \quad p_i = \sum_{j \in A_i} \frac{1}{2^j} \quad \text{and}$$

$A_i$ gives all the positions of the letters $a_i$ in the word w.

**Definition 3:**
If $\Sigma = \{a_1, a_2, \ldots, a_k\}$ the Parikh matrix mapping of a word is defined by $\psi : (\Sigma^*, ., \lambda) \rightarrow (M_{k+1}, ., I_{k+1})$ defined by $\psi(a_q) = (m_{ij})$, $1 \leq i, j \leq k+1$ such that (i) $m_{i,j} = 1$ (ii) $m_{j,j+1} = 1$ and (iii) all other elements are zero.

**Definition 4:**
The Parikh matrix of a word $w = a_1 a_2 \dots a_n$ is the product of the Parikh matrices of the alphabets in w.

i.e., $\psi(w) = \psi(a_1)\, \psi(a_2)\, \psi(a_3) \dots \psi(a_n)$

**Definition 5:**
The Generalized Parikh matrix mapping of a word is the same as the Parikh matrix mapping defined above except for $m_{j,j+1} = r$ where r is the $j^{th}$ coordinate of the Generalized Parikh vector of w.

A binary array X over $\Sigma^{**}$ is a rectangular array of symbols of $\Sigma = \{a, b\}$ with m rows and n columns & has size $m \times n$.

**Definition 6:**
The Parikh matrix of a binary array X is $\begin{pmatrix} 1 & p & r \\ p & 1 & q \\ s & q & 1 \end{pmatrix}$ where p and q are the sums of counts of a's and b's in every row and column. r is the sum of the count of subword ab occurrences in every row and s is the sum of the count of the subword ab occurrences in every column.

The basic operations like column concatenation and row concatenation are used with usual meanings. For example if $A = \begin{pmatrix} a & a \\ a & b \end{pmatrix}$ and $B = \begin{pmatrix} b & a \\ b & b \end{pmatrix}$ then the row concatenation is $\begin{matrix} A \\ . \\ B \end{matrix} = \begin{pmatrix} a & a \\ a & b \\ b & a \\ b & b \end{pmatrix}$ and the column concatenation is $A.B = \begin{pmatrix} a & a & b & a \\ a & b & b & b \end{pmatrix}$. The mirror image of A about base is $mi_B(A) = \begin{pmatrix} a & b \\ a & a \end{pmatrix}$ and the mirror image of A about rightmost vertical is $mi_{RV}(A) = \begin{pmatrix} a & a \\ b & a \end{pmatrix}$

## III. Generalized parikh vectors and generalized parikh matrices of 2d binary arrays

All the definitions given below are introduced by the authors Julie and Baskar Babujee [9], [10]

**Definition 7:**
If a 2D array $X \in \Sigma^{**}$ and $\Sigma = \{a_1, a_2, \dots, a_n\}$ the Generalized Parikh vector of X is

$$\mathcal{P}(X) = (p_1, p_2, \dots, p_n) \quad \text{where} \quad p_i = \sum_{i,j \in A_{ij}} \frac{1}{2^{i+j}}$$

where $A_{ij}$ gives all the positions of the letter $a_i$ in the 2D array X.

By definition if $\Sigma = \{a, b\}$, the Generalized Parikh vector of the binary array X will be

$$\mathcal{P}(X) = (p_1, p_2) \quad \text{where} \quad p_i = \sum_{i,j \in A_{ij}} \frac{1}{2^{i+j}}$$

**Example 1:**
Let $X = \begin{pmatrix} b & a & b \\ b & a & b \\ a & b & a \end{pmatrix}$ the Generalized Parikh vector of X is

$$\mathcal{P}(X) = \left( \frac{1}{2^{1+2}} + \frac{1}{2^{2+2}} + \frac{1}{2^{3+1}} + \frac{1}{2^{3+3}}, \frac{1}{2^{1+1}} + \frac{1}{2^{1+3}} + \frac{1}{2^{2+1}} + \frac{1}{2^{2+3}} + \frac{1}{2^{3+2}} \right) = \left( \frac{17}{64}, \frac{1}{2} \right)$$

**Definition 8:**
The Generalized Parikh matrix mapping of 2D array is $G : (\Sigma^*, ., \lambda) \to (M_{k+1}, ., I_{k+1})$.
Let $X = (a_{ij})_{m \times n}$ be a 2D array where $a_{ij} \in \Sigma = \{a_1, a_2, \dots, a_k\}$. The Generalized Parikh matrix of the array X is the product of the Generalized Parikh matrices of every $a_{ij}$ in X.

$$G(X) = G^{(a_{11})} G^{(a_{12})} \dots G^{(a_{1n})} G^{(a_{21})} \dots G^{(a_{2n})} \dots G^{(a_{m1})} \dots G^{(a_{mn})}$$

where each $G^{(a_{ij})} = (m_{ij})$ with (i) $m_{i,i} = 1$ for $1 \le i \le k+1$, (ii) $m_{j,j+1} = r$ where r is $r^{th}$ coordinate of the Generalized Parikh vector of X where $1 \le i, j \le k+1$ and (iii) all other $m_{ij}$'s are zero.

**Example 2:**

Let $X = \begin{pmatrix} b & a \\ b & a \end{pmatrix}$. The Generalized Parikh matrix of X is G(X)

$$G(X)=G^{(b)}G^{(a)}G^{(b)}G^{(a)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2^2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1/2^3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2^3 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1/2^4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3/16 & 1/64 \\ 0 & 1 & 3/8 \\ 0 & 0 & 1 \end{pmatrix}$$

**Definition 9:**

The set $\phi_{com}(X)$ is the set of all Generalized Parikh matrix commutators of X given by

$$\phi_{com}(X) = \{Y : G(X)\, G(Y) = G(Y)\, G(X)\}$$

i.e., It is the set of all Y such that the product of the Generalized Parikh matrices of X & Y commute.

**Definition 10:**

The path automata [11], [12] $MP_n = (Q, \Sigma, \delta_1, u_1, \{u_n\})$ where Q is the first set of states $\{u_1, u_2, \ldots, u_n\}$, the alphabet set $\Sigma$ is the singleton {a} and $\delta$ is defined by $\delta(u_i, a) = u_{i+1}$ for $1 \le i \le n-1$.

The Cartesian product of $MP_m$ and $MP_n$ is the grid automata $MP_m \times MP_n$ and when m = n it is the square grid automata.

## IV. Generalized Parikh Matrices of 1D Binary Words

Throughout this section a word means a binary word over $\Sigma = \{a, b\}$. When the length of the word is n and large, multiplication of n matrices manually takes much time and we give an algorithm to calculate the Generalized Parikh matrix of a word w.

### 3.1 Algorithm to find Generalized Parikh matrix of a word

Step 1. Read the string $w = x_1 x_2 \ldots x_n$

Step 2. Initialize n =length of the string w, count=1.

Also initialize (A) )= $(a_{ij})$, (B) )= $(b_{ij})$, (GPM) = $(g_{ij})$ as 3× 3 matrices.

Step 3. for i = 1 to 3 do

for j = 1 to 3 do

(GPM) = $(g_{ij})$

if i = j, $g_{ij} = 1$

if $i \neq j$, $g_{ij} = 0$

Step 4. for count = 1 to n do

If the $k^{th}$ alphabet is 'a' the following statements are executed (if $x_k = a$)

for i = 1 to 3 do

for j = 1 to 3 do

(A)= $(a_{ij})$

if i = j, $a_{ij} = 1$

if i < j, $a_{ij} = 0$

$a_{12} = \frac{1}{2^k}$

$a_{13} = 0$

$a_{23} = 0$

(GPM) = (GPM)(A)

else

The $k^{th}$ alphabet is 'b' and the following statements are executed ($x_k = b$)

for i = 1 to 3 do

for j = 1 to 3 do

(B)= $(b_{ij})$

if i = j, $b_{ij} = 1$

if i < j, $b_{ij} = 0$

$b_{23} = \frac{1}{2^k}$

$b_{12} = 0$

$b_{13} = 0$

(GPM)= (GPM)(B)

Step 5. Print the matrix (GPM)

Using the above algorithm programmed in JAVA the Generalized Parikh matrix of a few words have been calculated and have been enumerated here.

(i) $w_1 = a^6 b^7$.

The generalized Parikh matrix of $w_1 = \pi(w_1) = \begin{pmatrix} 1 & 63/64 & 8001/524288 \\ 0 & 1 & 127/8192 \\ 0 & 0 & 1 \end{pmatrix}$

(ii) $w_2$ = abbabaabbaaabbbb.

The generalized Parikh matrix of $w_2 = \pi(w_2) = \begin{pmatrix} 1 & 2407/4096 & 56007945/268435456 \\ 0 & 1 & 27023/65536 \\ 0 & 0 & 1 \end{pmatrix}$

(iii) $w_3 = b^{20}a$.

The generalized Parikh matrix of $w_3 = \pi(w_3) = \begin{pmatrix} 1 & 1/2097152 & 0 \\ 0 & 1 & 1048575/1048576 \\ 0 & 0 & 1 \end{pmatrix}$

Since there is no sub word ab in the word $w_3$, the first row third column element is zero.

(iv) $w_4 = a^{25}$.

The Generalized Parikh matrix of $w_4 = \pi(x_4) = \begin{pmatrix} 1 & 33554431/33554432 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & \frac{2^{25}-1}{2^{25}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For a word with length n, if (x, y) is the Generalized Parikh vector, its components x and y will appear in the super diagonal of the Generalized Parikh matrix of the word and its sum $x + y = \frac{2^n - 1}{2^n}$. Here the length is 25 and the sum of its super diagonal elements which are the components of the Generalized Parikh vector of $w_4$ is $x + y = \frac{2^{25} - 1}{2^{25}}$ which agrees with the available result.

(v) $w_5 = b^{30}$.

The Generalized Parikh matrix of $w_5 = \pi(w_5) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1073741823}{1073741824} \\ 0 & 0 & 1 \end{pmatrix}$

and the second row third column element is $\frac{2^{30} - 1}{2^{30}}$

In general the matrix multiplication is not commutative. We discuss about some instances of the product of the Generalized Parikh matrices of words being commutative.

**Theorem 1:**

If the Generalized Parikh matrices of two words $w_1$ and $w_2$ are $\pi(w_1) = \begin{pmatrix} 1 & p_1 & r_1 \\ 0 & 1 & q_1 \\ 0 & 0 & 1 \end{pmatrix}$ and $\pi(w_2) = \begin{pmatrix} 1 & p_2 & r_2 \\ & 1 & q_2 \\ & & 1 \end{pmatrix}$ respectively then the product of their Generalized Parikh matrices $w_1$ and $w_2$ will be commutative iff $p_1q_2 = p_2q_1$.

**Proof:**

The products $\pi(w_1)\ \pi(w_2)$ and $\pi(w_2)\ \pi(w_1)$ are the same in all positions except the first row third column which has $r_1 + r_2 + p_1q_2$ in first case and $r_1 + r_2 + p_2q_1$ in the second. Hence their product will be equal only when $p_1q_2 = p_2q_1$ in which case $\pi(w_1)\ \pi(w_2) = \pi(w_2)\ \pi(w_1)$ & hence the result.

**Theorem 2:**

If $w_1$ and $w_2$ are two words with length $n_1$ & $n_2$ respectively and their Generalized Parikh vectors are $(p_1, q_1)$ and $(p_2, q_2)$ respectively, then the product of their Generalized Parikh matrices will be commutative iff $\frac{p_1}{p_2} = \frac{q_1}{q_2} = \left(\frac{2^{n_1}-1}{2^{n_2}-1}\right)2^{n_2-n_1}$.

**Proof:**

By property of Generalized Parikh vectors, if a binary word w has length n and has Generalized Parikh vector (p, q) then $p+q = \frac{2^n-1}{2^n}$. Hence here $p_1+q_1 = \frac{2^{n_1}-1}{2^{n_1}}$ and $p_2+q_2 = \frac{2^{n_2}-1}{2^{n_2}}$. Substituting for $q_1$ and $q_2$ from this in the condition $p_1q_2 = p_2q_1$ we have $\frac{p_1}{p_2} = \left(\frac{2^{n_1}-1}{2^{n_2}-1}\right)2^{n_2-n_1}$. Similarly substituting for $p_1$ and $p_2$ gives the other result.

**Corollary 1:**

When both the words are of equal length $n_1 = n_2 = n$ in the above $\frac{p_1}{p_2} = \frac{q_1}{q_2} = 1$. i.e., $p_1 = p_2$ and $q_1 = q_2$. Hence when the Generalized Parikh vectors of the words are the same their Generalized Parikh matrices product will commute.

**Observations 1:**

1) If $\Sigma = \{a, b\}$ and $w \in \Sigma^+$ such that $|w|_{ab} = |mi(w)|_{ab}$ and u = wmir(w), v = mir(w).w then $\pi(u)\ \pi(v) = \pi(v)\ \pi(u)$ i.e., the product of the Generalized Parikh matrices of u and v is commutative.
2) In general, for $w \in \Sigma^+$ with $|w|_{ab} = |mi(w)|_{ab}$ where $\Sigma = \{a, b\}$ and $u = (wmi(w))^r$ and $v = (mi(w).w)^r$ then, $\pi(u)\ \pi(v) = \pi(v)\ \pi(u)$.
3) If $\pi(w)$ is the Generalized Parikh matrix of a word w then $\pi(w)\ [\pi(w)]^n = [\pi(w)]^n\ \pi(w)$ for all $n \geq 1$.

**Definition 11:**

We define the set of all Generalized Parikh matrix commutators of a word w denoted as $\pi_{com}(w)$ by

$$\pi_{com}(w) = \{u;\ \pi(w)\ \pi(u) = \pi(u)\ \pi(w)\}$$

**Observation 2:**

When $x \in \Sigma^+$, the following are equivalent

(i) $x \in \pi_{com}(x)$, $\pi_{com}(x)$ is non-empty

(ii) The concatenations $x, x^2, x^3, \ldots \in \pi_{com}(x)$.

Any decimal number can be converted into its binary equivalent in terms of zeros & ones. Some interesting results about the Generalized Parikh matrices of such binary numbers are discussed below.

**Theorem 3:**

Any number N in decimal form when converted into binary form will have a Generalized Parikh matrix $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ where $y = \frac{N}{2^n}$ and $x = \frac{2^n - N - 1}{2^n}$ where n is the length of the binary equivalent of N.

**Proof:** The number N when converted into binary form has $\frac{N}{2^n}$ as the second component of the Generalized Parikh vector and since the sum of the components of the Generalized Parikh vector of a word with length n is $\frac{2^n - 1}{2^n}$ the value of x follows.

**Observation 3:**

Every word in the language accepted by the grid automata $MP_m \times MP_n$ has m−1 zeros & n−1 ones. The Generalized Parikh vector of all its words are (m−1, n−1). Since the language $L(MP_m \times MP_n)$ has words that start with zero as well as one, the Generalized Parikh matrix of such words can be associated with the Generalized Parikh matrix of the binary equivalent of a decimal number N as below.

**Case (i):** If $w \in L(MP_m \times MP_n)$ and starts with 1, then its Generalized Parikh matrix will be the Generalized Parikh matrix of a number N in binary form and N itself occurs in the Generalized Parikh matrix of the word. In particular the Generalized Parikh matrix of such a word w will be $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ where $y = \frac{N}{2^{m+n-2}}$ and $x = \frac{2^{m+n-2} - N - 1}{2^{m+n-2}}$ as the length of every word in $L(MP_m \times MP_n)$ is m+n–2.

**Case (ii):** If the word $w \in L(MP_m \times MP_n)$ has i zeroes initially then its Generalized Parikh matrix $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ has $y = \frac{N}{2^{m+n-2}}$ where N is the decimal number whose binary form will have the Generalized Parikh matrix $\begin{pmatrix} 1 & u & w \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}$ where $v = \frac{N}{2^{(m+n-2)-i}}$.

## V. Generalized Parikh Matrices of 2D Binary Arrays

If a 2D binary array X is of size m × n then we multiply mn matrices to find the Generalized Parikh matrix of that array. When the size of the array is large, the multiplication of mn matrices manually is difficult. To ease the task we give the following algorithm to find the Generalized Parikh matrices of 2D binary arrays. The alphabet set Σ is taken to be {a, b}.

### 5.1 Algorithm to find the Generalized Parikh Matrix of 2D Binary Arrays

Step 1. Start
Step 2. Get the number of rows and number of columns.
Step 3. Get the elements of the matrix
Step 4. Initialize m = number of rows , n = number of columns, count=1
    Also initialize (A), (B), (GPM) as 3× 3 matrices.
    Initialize (X )= $(x_{ij})$ as matrix of size m× n
Step 5. for i = 1 to 3 do
    for j = 1 to 3 do
        (GPM) = $(g_{ij})$
        if i = j, $g_{ij} = 1$
        if i ≠ j, $g_{ij} = 0$
Step 6. for count = 1 to mn do
    for i = 1 to m do
        for j = 1 to n do
            if the $x_{ij}$ th alphabet is 'a' the following statements are executed (if $x_{ij}$ = a)
    for p = 1 to 3 do
        for q = 1 to 3 do
            (A) = $(a_{pq})$
            if p = q, $a_{pq} = 1$

if p< q, $a_{pq} = 0$

$a_{12} = \frac{1}{2^{i+j}}$

$a_{13} = 0$

$a_{23} = 0$

GPM = (GPM)(A)

else the $x_{ij}$ th alphabet is 'b' and the following statements are executed (if $x_{ij}$ = b)

for p = 1 to 3 do

for q = 1 to 3 do

(B )= $(b_{pq})$

if p = q, $b_{pq} = 1$

if p < q, $b_{pq} = 0$

$b_{23} = \frac{1}{2^{i+j}}$

$b_{12} = 0$

$b_{13} = 0$

GPM = (GPM)(B)

Step 7. Print the matrix (GPM).

Step 8. Stop.

Using this algorithm and designing a program in JAVA the Generalized Parikh matrices of a few samples of 2D binary arrays have been calculated and the output has been registered below.

(i) $X_1 = \begin{pmatrix} a & b \\ a & a \end{pmatrix}$

Generalized Parikh matrix of $X_1 = G(X_1) = \begin{pmatrix} 1 & 7/16 & 1/32 \\ 0 & 1 & 1/8 \\ 0 & 0 & 1 \end{pmatrix}$

(ii) $X_2 = \begin{pmatrix} a & a & a & b \\ b & a & a & a \\ b & a & b & b \\ a & a & a & b \end{pmatrix}$

Generalized Parikh matrix of $X_2 = G(X_2) = \begin{pmatrix} 1 & 81/128 & 3885/32768 \\ 0 & 1 & 63/256 \\ 0 & 0 & 1 \end{pmatrix}$

(iii) $X_3 = \begin{pmatrix} a & a & a & a & a & a & a & a & a \\ b & b & b & b & b & b & b & b & b \\ a & a & b & a & a & b & a & a & b \\ b & b & b & b & b & b & b & b & b \\ b & b & b & b & a & a & a & a & a \\ a & a & a & a & a & a & a & a & a \\ a & a & a & a & a & a & a & a & a \\ b & b & b & b & b & b & b & b & b \end{pmatrix}$

Generalized Parikh matrix of $X_3 = G(X_3) = \begin{pmatrix} 1 & 41369/65536 & \frac{1658843751}{8589934592} \\ 0 & 1 & 47567/131072 \\ 0 & 1 & 1 \end{pmatrix}$

(iv) $X_4 = \begin{pmatrix} a & a & a & a & a & a & a & a & a & a & a & a & a & a & a \\ a & a & a & a & a & a & a & a & a & a & a & a & a & a & a \\ a & a & a & a & a & a & a & a & a & a & a & a & a & a & a \\ a & a & a & a & a & a & a & a & a & a & a & a & a & a & a \\ a & a & a & a & a & a & a & a & a & a & a & a & a & a & a \\ a & a & a & a & a & a & a & a & a & a & a & a & a & a & a \\ a & a & a & a & a & a & a & a & a & a & a & a & a & a & a \\ a & a & a & a & a & a & a & a & a & a & a & a & a & a & a \\ a & a & a & a & a & a & a & a & a & a & a & a & a & a & a \\ a & a & a & a & a & a & a & a & a & a & a & a & a & a & a \\ a & a & a & a & a & a & a & a & a & a & a & a & a & a & a \\ a & a & a & a & a & a & a & a & a & a & a & a & a & a & a \\ a & a & a & a & a & a & a & a & a & a & a & a & a & a & a \\ a & a & a & a & a & a & a & a & a & a & a & a & a & a & a \\ a & a & a & a & a & a & a & a & a & a & a & a & a & a & a \end{pmatrix}$

Generalized Parikh matrix of $X_4 = G(X_4) = \begin{pmatrix} 1 & \frac{1073676289}{1073741824} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1073676289}{2^{30}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The values x and y in the super diagonal are the components of the Generalized Parikh vector of X [10]. Here $X_4$ has the Generalized Parikh vector $\left(\frac{1073676289}{2^{30}}, 0\right)$ and satisfies the property $x + y = \sum_{j=1}^{15}\sum_{i=1}^{15}\frac{1}{2^{i+j}}$ which is

$x + y = \sum_{j=1}^{n}\sum_{i=1}^{m}\frac{1}{2^{i+j}}$ for a general 2D binary array of size m × n.

**Theorem 4:**
The Generalized Parikh matrix mapping is not one to one.
**Proof:**
$X = \begin{pmatrix} a & a & a \\ a & a & a \\ b & a & a \end{pmatrix}$ and $Y = \begin{pmatrix} a & a & a \\ a & a & b \\ a & b & a \end{pmatrix}$ have the same Generalized Parikh matrix $\begin{pmatrix} 1 & 45/64 & 21/512 \\ 0 & 1 & 1/16 \\ 0 & 0 & 1 \end{pmatrix}$

$U = \begin{pmatrix} b & a & b \\ b & a & a \\ b & a & a \end{pmatrix}$ and $V = \begin{pmatrix} b & a & b \\ b & a & b \\ a & b & a \end{pmatrix}$ have the same Generalized Parikh matrix $\begin{pmatrix} 1 & 17/64 & 19/512 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}$

Examples like X, Y and U, V show that the mapping is not one-one. There are more number of counter examples and the existential quantification proves the result.

It has been proved that if X and Y are 2D binary arrays whose Generalized Parikh vectors are the same, then the product of their Generalized Parikh matrices is commutative [5].

**Theorem 5:**
If U and V are 2D binary arrays that have same Generalized Parikh vectors then

(i) the product of the Generalized Parikh matrices of the column concatenations $(U\ .\ V)^k$, $U^k\ .\ V^k$ and $V^k\ .\ U^k$ are commutative.

(ii) the product of the Generalized Parikh matrices of the row concatenations $\begin{pmatrix} U \\ . \\ V \end{pmatrix}^k$, $\begin{matrix} U_k \\ . \\ V_k \end{matrix}$ and $\begin{matrix} V_k \\ . \\ U_k \end{matrix}$ are commutative.

**Proof:**

When U and V have same Generalized Parikh vectors, the Generalized Parikh vectors of the column concatenations $(U\ .\ V)^k$, $U^k\ .\ V^k$ and $V^k\ .\ U^k$ are the same. Commutativity follows from the fact that the product of the Generalized Parikh matrices of 2D arrays will be commutative if they have the same Generalized Parikh vector [5].

Similarly, regarding row concatenations, when U & V have the same Generalized Parikh vector, the row concatenations $\begin{pmatrix} U \\ . \\ V \end{pmatrix}^k$, $\begin{matrix} U_k \\ . \\ V_k \end{matrix}$ and $\begin{matrix} V_k \\ . \\ U_k \end{matrix}$ have the same Generalized Parikh vectors. And hence commutativity of the product of their Generalized Parikh matrices follows.

**Observation 4:**
1) In words, if u and v are such that the product of their Parikh matrices is commutative then the product of the Parikh matrices of their mirror images is also commutative. But in 2D-binary arrays if U and V are such that the product of their Generalized Parikh matrices is commutative (i.e.) if $v \in \phi_{com}(U)$
i) the product of the Generalized Parikh matrices of their respective mirror images about their base is not commutative (i.e.) $mi_B(V) \notin \phi_{com}(mi_B(U))$
ii) the product of the Generalized Parikh matrices of their respective mirror images about their rightmost vertical is not commutative (i.e.) $mi_{RV}(V) \notin \phi_{com}(mi_{RV}(U))$

2) When the 2D binary A & B have the same Generalized Parikh vectors and G(A) and G(B) are their Generalized Parikh matrices respectively then
$[G(A)]^{-1} [G(B)]^{-1} = [G(B)]^{-1} [G(A)]^{-1}$

## VI. Conclusion

Algorithms to find the Generalized Parikh matrices of 1D binary words and 2D binary arrays have been given in the paper. Any image of size m × n is treated as a 2D binary array over {0, 1} by the computers and is studied and analyzed in image processing. This paper gives a numerical representation for 2D binary arrays and it can be used for the analysis of images using numerical values. This will have a significant importance when dealt with medical images.

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