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On ray properties of Hurwitz polynomials

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Abstract: *In this paper, we investigate some geometric properties of the Hurwitz set which corresponds to the set of stable monic polynomials in a parameter space. We firstly consider the segment stability. After we study properties of rays in the Hurwitz sets, which corresponds with inclusion or non-inclusion of certain rays in the Hurwitz sets.*
Keywords: *Hurwitz polynomials, monic polynomials, ray properties, segment stability*

I. Introduction

The celebrated theorem Kharitonov [1] on the stability of prisms of polynomials gave an impetus to the research in this old and ever-important field and in the last decades many new results concerning stability of diamonds, edges, segments, polygones, polytopes etc. have been obtained (see [2-15]). A remarkable new approach has been towards understanding the geometry (and topology) of (all or part of) stable polynomials.

First of all, we identify a non-monic polynomial $p(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n$ with the point (or vector) $(a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}$. A stable (or Hurwitz) polynomial is a polynomial with roots lying in the open left half of the complex plane. (A necessary but not sufficient condition for stability is that all of $a_0, a_1, \ldots, a_n$ have the same sign. There are well-known necessary and sufficient conditions for stability such as the Routh-Hurwitz and Hermite-Biehler criterions and the separation property [16-17]) We will denote the set of such vectors by $\mathcal{H}^n \subset \mathbb{R}^{n+1}$ and the subset of $\mathcal{H}^n$ with positive leading coefficients $(a_0 > 0)$ with $\mathcal{H}_+^n$. The important special case of monic polynomials $(a_0 = 1)$, which for the consideration of stability are equivalent to the general case, are thus identified with vectors of the form $(1, a_1, \ldots, a_n)$. On the other hand, they are often identified with the vector $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ and this causes a minor nuisance of notation. To prevent ambiguity, we will denote the set of stable monic polynomials by $\mathcal{H}_1^n$ if they are taken as elements $(1, a_1, \ldots, a_n)$ of $\mathbb{R}^{n+1}$, and by $\widetilde{\mathcal{H}}_1^n$ if they are taken as elements $(a_1, a_2, \ldots, a_n)$ of $\mathbb{R}^n$. Unless explicitly stated otherwise, we will represent the nth order monic polynomials $p(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n$ with $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$.

Thus, the open sets $\mathcal{H}_+^n \subset \mathbb{R}^{n+1}$ and $\widetilde{\mathcal{H}}_1^n \subset \mathbb{R}^n$ are defined as follows:

- $(a_0, a_1, \ldots, a_n) \in \mathcal{H}_+^n \iff a_0 > 0$ and the polynomial $p(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n$ is stable,
- $(a_1, a_2, \ldots, a_n) \in \widetilde{\mathcal{H}}_1^n \iff$ the polynomial $p(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n$ is stable.

It is obvious that for $k > 0$ and $p = (a_1, a_2, \ldots, a_n) \in \widetilde{\mathcal{H}}_1^n$

- $kp \in \widetilde{\mathcal{H}}_1^n \iff$ the polynomial $p_k(s) = s^n + k a_1 s^{n-1} + k a_2 s^{n-2} \cdots + k a_n$ is stable.

The first geometric property of interest is the convexity and it is well-known that $\widetilde{\mathcal{H}}_1^n$ (and thus $\mathcal{H}_+^n$) is non-convex. The next question of interest is the following: Given two elements from $\mathcal{H}_+^n$ (or $\widetilde{\mathcal{H}}_1^n$), under which conditions it can be stated that the segment in $\mathbb{R}^{n+1}$ (or in $\mathbb{R}^n$) with these end points belong to $\mathcal{H}_+^n$ (or $\widetilde{\mathcal{H}}_1^n$)? Several authors gave results and discussions in this direction (see [4,6]), but the most important result is due to Rantzer [3] and implies the others. In Section 2, we give a simple new case (Remark 1) and some important consequence (Corollary 1 and Corollary 2) not obtainable by Rantzer's theorem.

Section 3 contains the main results where we investigate some other geometric properties of rays, but before stating them we want to introduce some additional terminology. Given a vector $p \in \mathbb{R}^n$ (which corresponds to a monic polynomial of degree n), we call the set $\{kp : k > 0\} \subset \mathbb{R}^n$ the radial ray through p. Likewise, we will call the set $\{kp : k \geq 1\} \subset \mathbb{R}^n$ the radial ray starting at p and the set $\{kp : 0 < k \leq 1\} \subset \mathbb{R}^n$ the radial ray till p. Now we state the properties proven in Section 3. Given any vector $p \in \widetilde{\mathcal{H}}_1^n$ $(n \geq 3)$, there exists $k_0 \in (0,1)$ such that the part $\{kp : 0 < k \leq k_0\}$ of the radial ray till p lies outside $\widetilde{\mathcal{H}}_1^n$ and the part $\{kp : k_0 < k \leq 1\}$ lies inside $\widetilde{\mathcal{H}}_1^n$ (Theorem 1).

On the other hand, for every $n \geq 2$ there is a vector $p \in \widetilde{\mathcal{H}}_1^n$ (actually infinitely many) such that the radial ray starting at p lies completely inside $\widetilde{\mathcal{H}}_1^n$ (Theorem 2). For $n = 2, 3$ and 4 all radial rays starting at any $p \in \widetilde{\mathcal{H}}_1^n$ lie completely in $\widetilde{\mathcal{H}}_1^n$.

For $n \geq 5$ there exists a vector $p \in \widetilde{\mathcal{H}}_1^n$ (actually infinitely many) such that for a certain $k_0 > 1$ the part $\{kp : 1 < k \leq k_0\}$ of the radial ray starting at p lies in $\widetilde{\mathcal{H}}_1^n$, but the part $\{kp : k \geq k_0\}$ lies outside $\widetilde{\mathcal{H}}_1^n$ (Corollary 3).

II. Segment-Stability And Properties Concerning Rays

The following result comes from [6]: Given two stable polynomials $p(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$ $(a_0 > 0)$ and $q(s) = b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n$ $(b_0 > 0)$ then the segment $[p, q]$ is stable if $a_i = b_i$ either for even entries or odd entries (consult also [8,9,13]).

Proposition 1 Let $p(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$ $(a_0 > 0)$ and $q(s) = b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n$ $(b_0 > 0)$ be stable polynomials. If even (or odd) part of $q(s)$ is a positive scalar multiple of the even (or odd) part of $p(s)$ then the segment $[p, q]$ of their convex combinations is also stable.

It is enough to see this for the case of even parts, the case of odd parts being similar. One can re-arrange $p(s)$ and $q(s)$ as $p(s) = h(s^2) + sg_1(s^2)$, $q(s) = kh(s^2) + sg_2(s^2)$ where $k > 0$ is a fixed scalar. Denote $q_*(s) = \frac{q(s)}{k}$, then the convex combination of $p(s)$ and $q_*(s)$ is stable by [6]. Hence for every $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\lambda_1 + \lambda_2 > 0$ the polynomial $\lambda_1 p(s) + \lambda_2 q_*(s)$ is stable, since

$$\lambda_1 p(s) + \lambda_2 q_*(s) = (\lambda_1 + \lambda_2)\left[\frac{\lambda_1}{\lambda_1 + \lambda_2} p(s) + \frac{\lambda_2}{\lambda_1 + \lambda_2} q_*(s)\right].$$

Therefore, assigning $\lambda_1 = (1 - \lambda)$ and $\lambda_2 = k\lambda$ the polynomial $\lambda_1 p(s) + \lambda_2 q_*(s) = (1 - \lambda)p(s) + \lambda q(s)$ is stable for all $\lambda \in [0,1]$.

Corollary 1 Let $p(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$ and $q(s) = s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n$ be two stable polynomials. Identify $p(s)$ with $(a_1, a_2, \ldots, a_n)$ and $q(s)$ with $(b_1, b_2, \ldots, b_n)$ and assume that $(b_1, b_2, \ldots, b_n) = k(a_1, a_2, \ldots, a_n)$ for a positive scalar k. Then the segment $[p, q]$ in $\mathbb{R}^n$ is stable. In other words, segments on radial rays with stable end points are stable.

Proof. Either the even or odd parts of p and q are proportional according to n being odd or even. The result follows from Proposition 1. ☐

Corollary 2 If the radial ray emanating from the origin enters the $\widetilde{\mathcal{H}}_1^n$ and then leaves it, it cannot re-enter it. In other words, for $p \in \widetilde{\mathcal{H}}_1^n$ if $k_0 p \notin \widetilde{\mathcal{H}}_1^n$ for $k_0 < 1$ then $kp \notin \widetilde{\mathcal{H}}_1^n$ for any $k < k_0$ and similarly if $k_1 p \notin \widetilde{\mathcal{H}}_1^n$ for $k_1 > 1$, then $kp \notin \widetilde{\mathcal{H}}_1^n$ for any $k > k_1$.

We now prove the theorems stated in the introduction.

Theorem 1 For any vector $p \in \widetilde{\mathcal{H}}_1^n$, $(n \geq 3)$, there exists $k_0 \in (0,1)$ such that

- $kp \notin \widetilde{\mathcal{H}}_1^n$ for all k with $0 < k \leq k_0$
- $kp \in \widetilde{\mathcal{H}}_1^n$ for all k with $k_0 < k \leq 1$

Proof. By the separation property of stable polynomials, a necessary and sufficient condition for $p(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$ to be stable is that the curve $p(j\omega)$, where $0 \leq \omega < \infty$, cuts the real and imaginary axes alternatively n times precisely.

If $n = 4m$ then for

$$k_* = -\frac{\omega_*^n}{a_n - a_{n-2}\omega_*^2 + \cdots - a_2\omega_*^{n-2}}$$

we have $0 < k_* < 1$ and $p_{k_*}(j\omega_*) = 0$, where $p_k(s) = s^n + ka_1 s^{n-1} + ka_2 s^{n-2} \cdots + ka_n$ and ω_* corresponds with the point of intersection with the real axis. If $n = 4m + 1$ then for

$$k_* = -\frac{\omega_*^n}{a_{n-1}\omega_* - a_{n-3}\omega_*^3 + \cdots - a_2\omega_*^{n-2}}$$

we have $0 < k_* < 1$ and $p_{k_*}(j\omega_*) = 0$, where ω_* corresponds with the point of intersection with the imaginary axis. Similar procedure can be applied to the cases $n = 4m + 2$ and $n = 4m + 3$. Thus for any $n \geq 3$ and any $p \in \widetilde{\mathcal{H}}_1^n$ there exists $k_* \in (0,1)$ such that $k_*(a_1, a_2, \ldots, a_n) \notin \widetilde{\mathcal{H}}_1^n$. From Corollary 2 the desired result follows. ☐

Theorem 1 shows that if we move radially towards the origin starting from an arbitrary polynomial $p \in \widetilde{\mathcal{H}}_1^n$, then we certainly leave $\widetilde{\mathcal{H}}_1^n$.

The following properties are about what can happen when we move in reverse direction.

Theorem 2 For $n \geq 2$ there exists infinitely many $p \in \widetilde{\mathcal{H}}_1^n$ such that $kp \in \widetilde{\mathcal{H}}_1^n$ for all $k \geq 1$.

To prove this theorem we first prove the following proposition.

Proposition 2 Let $q(s) = a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n$, $(a_1 > 0)$ be a stable polynomial. Then there exists $\varepsilon_0 > 0$ such that for all ε with $0 < \varepsilon \leq \varepsilon_0$ the polynomial $p_\varepsilon(s) = \varepsilon s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$ is stable.

Proof. Let n be an even number. Then we can write $q(s) = q_1(s^2) + sq_2(s^2)$, where $q_1(u)$ and $q_2(u)$ are polynomials of order $m = \frac{n-2}{2}$. Let $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_m$ denote the roots of $q_1(u)$ and $q_2(u)$ respectively. Then by the Hermite-Biehler theorem

$$v_1 < u_1 < v_2 < u_2 < \cdots < v_m < u_m < 0.$$

The polynomial $p_\varepsilon(s)$ can be written as $p_\varepsilon(s) = [\varepsilon(s^2)^{m+1} + q_1(s^2)] + sq_2(s^2)$. If we look into graphs of functions $y = q_1(u)$ and $y = -\varepsilon u^{m+1}$, we see that these graphs, for small $\varepsilon > 0$, intersect each other in $m+1$ points and when $\varepsilon \to 0$, m of these intersection points approaches to $u_1, u_2, \dots, u_m$, whereas the other root to $-\infty$. Therefore for the roots $u_0^\varepsilon, u_1^\varepsilon, u_2^\varepsilon, \dots, u_m^\varepsilon$ of $\varepsilon u^{m+1} + q_1(u)$, there exists $\varepsilon_0 > 0$ satisfying

$$u_0^\varepsilon < v_1 < u_1^\varepsilon < v_2 < \cdots < v_m < u_m^\varepsilon < 0$$

for all $0 < \varepsilon \le \varepsilon_0$. Then by the Hermite-Biehler theorem the stability of $p_\varepsilon(s)$ follows. The case of odd n can be carried out similarly. ☐

Proof of Theorem 2. Let $q(s) = a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n$ be a stable polynomial. From Proposition 2 it follows that there exists $t_0 > 0$ such that for all $t \ge t_0$ the polynomial

$$p_t(s) = \frac{1}{t}s^n + a_1 s^{n-1} + \cdots + a_n = \frac{1}{t}(s^n + ta_1 s^{n-1} + \cdots + ta_n)$$

is stable. If we choose $p = (t_0 a_1, t_0 a_2, \dots, t_0 a_n)$, then $p \in \widetilde{\mathcal{H}}_1^n$ and for all $k \ge 1$ we have $kp \in \widetilde{\mathcal{H}}_1^n$. ☐

Proposition 3 For $n = 2, 3$ and 4 the property stated in Theorem 2 is true for all $p \in \widetilde{\mathcal{H}}_1^n$.

The proof is ommited.

Remark 1 It might seem that the Proposition 2 could plausibly be expected to be "naturally" true but the situation is more intricate than it seems, because there comes a surprise when we add two small terms: Let $s^n + 2s^{n-1} + \cdots$ be stable polynomial, then for no $\varepsilon > 0$ the polynomial $\varepsilon s^{n+2} + \varepsilon s^{n+1} + s^n + 2s^{n-1} + \cdots$ is stable.

Theorem 3 Let $n \ge 5$. Then for all $k > 0$, $k \ne 1$, there exists $p = (a_1, a_2, \dots, a_n) \in \widetilde{\mathcal{H}}_1^n$ such that $kp = (ka_1, ka_2, \dots, ka_n) \notin \widetilde{\mathcal{H}}_1^n$. That is to say the polynomial $p(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$ is stable but $p_k(s) = s^n + ka_1 s^{n-1} + \cdots + ka_{n-1}s + ka_n$ is not stable.

Proof. The proof is based on the Hermite-Biehler theorem. Suppose that n is an odd integer and $m = \frac{n-1}{2}$. Choose arbitrary numbers $v_1, v_2, \dots, v_m$ satisfying $v_1 < v_2 < \cdots < v_m < 0$ and define the polynomial $g(u) = (u - v_1)(u - v_2)\cdots(u - v_m) = u^m + b_1 u^{m-1} + \cdots + b_m$.

Let $k > 0$, $k \ne 1$ is given. Consider the polynomials $g_k(u) = u^m + kb_1 u^{m-1} + \cdots + kb_m$. Firstly suppose that the roots of $g_k(u)$ satisfies the condition $v'_1 < v'_2 < \cdots < v'_m < 0$. It is not difficult to see that $g(u)$ and $g_k(u)$ have no common root. Then we can find $u_1, u_2, \dots, u_m$ satisfying $v_1 < u_1 < v_2 < u_2 < \cdots < v_m < u_m < 0$ and not satisfying at least one of the following inequalities $v'_1 < u_1 < v'_2 < u_2 < \cdots < v'_m < u_m < 0$ (here we use $m \ge 2$). The Hermite-Biehler theorem ensures that $p(s) = h(s^2) + sg(s^2)$ is stable, where $h(u) = (u - u_1)(u - u_2)\cdots(u - u_m)$. If we write down $p(s)$ as $p(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$ then $p_k(s) = s^n + ka_1 s^{n-1} + \cdots + ka_{n-1}s + ka_n = kh(s^2) + sg_k(s^2)$ and the Hermite-Biehler theorem also guarantees the unstability of $p_k(s)$.

If the roots of $g_k(u)$ does not satisfy $v'_1 < v'_2 < \cdots < v'_m < 0$ then $p_k(s)$ is also unstable. By a similiar scheme one may prove the theorem for even n. ☐

Remark 2 As it is seen from the proof of Theorem 3, the point p depends on $v_1, v_2, \dots, v_m$. By changing these numbers we can obtain infinitely many p satisfying Theorem 3.

Corollary 3 There exists a point $p \in \widetilde{\mathcal{H}}_1^n$, $(n \ge 5)$ with the following property: There exists a number $k_0 > 1$ such that

- $kp \in \widetilde{\mathcal{H}}_1^n$ for all $1 \le k < k_0$,
- $kp \notin \widetilde{\mathcal{H}}_1^n$ for all $k \ge k_0$.

Proof. Choose $k = 2$. Then by Theorem 3 there exists $p \in \widetilde{\mathcal{H}}_1^n$ such that $2p \notin \widetilde{\mathcal{H}}_1^n$. Then the claim follows from Corollary 2. ☐

Remark 3 There exists a radial ray in the positive quadrant of $\mathbb{R}^n$ which lies completely outside $\widetilde{\mathcal{H}}_1^n$ $(n \ge 4)$. The polynomial $p_k(s) = s^n + ks^{n-1} + ks^{n-2} + \cdots + ks + k$ is unstable for all $k > 0$. But for $n = 3$ there is no such ray.

III. Conclusion

In this paper it is established that in a parameter space of polynomials segments on radial rays with stable end points are stable. We show that there is a stable svector such that the radial ray starting at this point lies completely inside the stability region. We also show that for any positive scalar differing one, there exists a stable vector such that the multiplication of this vector by this scalar is not stable.

References

[1] V.L. Kharitonov, Asymptotic stability of an equilibrium position of a family of systems of linear differential equations, *Differential'nye Uravneniya*, 14, 1978, 2086-2088.

[2] A.C. Bartlett, C.V. Hollot, H. Lin, Root location of an entire polytope of polynomials: It suffices to check the edges, *Mathematics of Control, Signals and Systems* 1(1), 1988, 61-71.

[3] A. Rantzer, Stability conditions for polytopes of polynomials, *IEEE Transactions on Automatic Control*, 37, 1992, 79-89.

[4] I.R. Petersen, A new extension to Kharitonov's theorem, *IEEE Transactions on Automatic Control*, 35, 1990, 825-828.
[5] B.R. Barmish, *New tools for robustness of linear systems* (New York: MacMillan, 1994).
[6] S. Bialas, J. Garloff, Convex combinations of stable polynomials, *Journal of the Franklin Institute*, 319(3), 1985, 373-377.
[7] Y.C. Soh, Y.K. Foo, Generalized edge theorem, *Systems & Control Letters*, 12(3), 1989, 219-224.
[8] E.R. Panier, M.K.H. Fan, A.L. Tits, On the robust stability of polynomials with no cross-coupling between the perturbations in the coefficients of even and odd powers, *Systems & Control Letters*, 12(4), 1989, 291-299.
[9] L.J. Pujara, On the stability of uncertain polynomials with dependent coefficients, *IEEE Transactions on Automatic Control*, 35, 1990, 756-759.
[10] M.Y. Fu, A class of weak Kharitonov regions for robust stability of linear uncertain systems, *IEEE Transactions on Automatic Control*, 36, 1991, 975-978.
[11] B.R. Barmish, R. Tempo, C.V. Hollot, H.I. Kang, An extreme point result for robust stability of a diamond of polynomials, *IEEE Transactions on Automatic Control*, 37, 1992, 1460-1462.
[12] L.R. Pujara, N. Shanbhag, Some stability theorems for polygones of polynomials, *IEEE Transactions on Automatic Control*, 37, 1992, 1845-1849.
[13] E.R. Panier, M.K.H. Fan, A.L. Tits, Comments on the stability of uncertain polynomials with dependent coefficients, *IEEE Transactions on Automatic Control*, 37, 1992, 1201.
[14] V.L. Kharitonov, R. Tempo, On the stability of weighted diamond of real polynomials, *Systems & Control Letters*, 22(1), 1994, 5-7.
[15] G.R. Duan, M.Z. Wang, Properties of the entire set of Hurwitz polynomials and stability analysis of polynomial families, *IEEE Transactions on Automatic Control*, 39, 1994, 2490-2494.
[16] F.R. Gantmacher, *Applications of the theory of matrices* (New York: Interscience, 1959).
[17] E.A. Guillemin, *The mathematics of circuit analysis* (Cambridge MA: MIT Press, 1965).