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An $M^{[X]}/G(a,b)/1$ queueing system with two heterogeneous service, Server breakdown and repair, Multiple vacation, Closedown, Balking and Stand-by server.

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Abstract: *The objective of this paper is to study the behaviour of the server breakdown and repair in an $M^{[X]}/G(a,b)/1$ queueing system with two heterogeneous service, multiple vacation, closedown, balking and stand-by server. After a first stage of service to a batch of customers the main server serves the second stage of service to the same batch of customers. The main server may breakdown at any stage of service with exponential rate $'\alpha'$, then the main server go for a repair service which follows exponential with rate $'\eta'$ and the interrupted stage of service to a batch of customers is transferred to the stand-by server and he starts a interrupted stage service to that batch of customers a fresh. After the completion of first stage of service the stand-by server serves the second stage of service to the same batch of customers. Let $'(1-\pi)'$ be the probability that a batch of arriving customers balk during the busy period of the stand-by server. If the system size becomes zero before main server repair completed then the stand-by server decide to stay in the system and he starts service to the next batch only when the queue size reaches is at least $'a'$. At the instant of repair completion, if the stand-by server is busy in any stage of service the interrupted stage of service to a batch of customers is transferred to the main server and he starts a interrupted stage service to that batch of customers a fresh. Suppose at the instant of repair completion or at the instant of the second stage of service served by the main server if queue length is less than $'a'$ then the server perform a closedown work. After that, the server leaves for multiple vacation of random length. After a completion of vacation, if the queue length is still less than $'a'$ he leaves for another vacation and so on until he finds minimum $'a'$ customers in the queue. After a vacation, if the server finds at least $'a'$ customers waiting for service, say $'\varepsilon'$, then he serves a batch of size min(ε,b) customers, where $b \geq a$.*

Keywords: *Bulk service, Two heterogeneous service, Multiple vacation, Stand-by service.*

I. Introduction

General bulk service rule was first introduced by Neuts [1]. General bulk service rule states that the server will start to provide service only when atleast $'a'$ units are present in the queue and maximum service capacity is $'b'$. On completion of a batch of service, if less than $'a'$ customers are present in the queue then server has to wait upto the queue length reaches the value $'a'$. If less than or equal to $'b'$ and greater than or equal to $'a'$ customers are in the queue, then all the existing customers are taken into service. If greater than or equal to $'b'$ customers are in the queue, then $'b'$ customers are taken into service. Downton [2] obtained the waiting time distribution of bulk service queues by considering random arrivals and random service time distribution. In bulk service queueing problem Jaiswal [3] conformed the result of Downton[2]. He derived waiting time distribution using embedded Markov-chain technique. He assumed that the batches are either fixed queue length or whole queue, whatever is lower. That is, if more than $'a'$ customers are in the queue, then $'a'$ customers are taken into service. If less than $'a'$ customers are in the queue, then all the existing customers are taken into service. Arumuganathan and Jeyakumar [6] analyzed $M^{[X]}/G(a,b)/1$ queueing model with multiple vacation and closedown time. In closedown time queueing system, on completion of a service, if the queue length is less than the minimum service capacity of the server, the server performs closedown work. In $M^{[X]}/G(a,b)/1$ queueing system with multiple vacation, on completion of a service, if the queue size is at-most $'a-1'$, the server leaves for a vacation of random length. After a vacation, if the queue length is still less than $'a'$, then the server leaves for another vacation and so on, this will continue until the queue length reaches at least $'a'$. After a vacation, if the queue size is greater than or equal to $'a'$, say ξ, then the server serves min (ξ,b), where $b \geq a$. Recently, Madan et al. [4] studied steady state analysis of two $M^{X}/M(a,b)/1$ queue models with random breakdowns. They considered that the repair time is exponential for one model and deterministic for another one. The most of the studies on vacation queues with server breakdown conclude that if breakdown occurs the service is immediately allowed to be interrupted. In some cases, it is not possible to disturb the server immediately before completing a batch of service or it is possible to continue or extend the service up to some more time even if the breakdown occurs. A Stand-by server which operates only during the repair period of the Main server is considered in this paper. The Stand-by server model is introduced by Madan[5]. He considers a system which has, apart from the regular channel, the provision of a stand-by which is employed only during the repair times of the regular service channel. The stand-by service may not be as efficient as the main service channel, but still may contribute a lot to avoid the queue becoming out of bounds during the failure times of the main service channel. Such situations are not uncommon.

This paper is organised as follows. In section 2 the queuing problem is defined. The system equations have been developed in sections 3. The probability generating function (PGF) of the queue length distribution in steady state is obtained in section 4. Various performance measures of the queuing system are derived in section 5. A computational study is illustrated in section 6. Conclusions are given in section 7.

II. Model Description

This paper deals with arrival follows Compound Poisson with intensity rate λ, main server and stand-by server's service, vacation and closedown follows general distribution, breakdown and repair follows exponential distribution, $(1-\pi)$ be the probability that the arriving batch balks only when the stand-by server busy. After a first stage of service to a batch of customers the main server serves the second stage of service to the same batch of customers. The main server may breakdown at any stage of service with exponential rate $'\alpha'$, then the main server go for a repair service which follows exponential with rate $'\eta'$ and the interrupted stage of service to a batch of customers is transferred to the stand-by server and he starts a interrupted stage service to that batch of customers a fresh. After the completion of first stage of service the stand-by server serves the second stage of service to the same batch of customers. If the system size becomes zero before main server repair completed then the stand-by server decide to stay in the system and he starts service to the next batch only when the queue size reaches is at least $'a'$. At the instant of repair completion, if the stand-by server is busy in any stage of service the interrupted stage of service to a batch of customers is transferred to the main server and he starts a interrupted stage service to that batch of customers a fresh. Suppose at the instant of repair completion or at the instant of completion of second stage service served by the main server if queue length is less than $'a'$ then the server perform a closedown work. After that, the server leaves for multiple vacation of random length. After a completion of vacation, if the queue length is still less than $'a'$ he leaves for another vacation and so on until he finds minimum $'a'$ customers in the queue. After a vacation, if the server finds at least $'a'$ customers waiting for service, say $'\varepsilon'$, then he serves a batch of size $\min(\varepsilon,b)$ customers, where $b \geq a$.

2.1 Notations

The following notations are used in this paper.

λ - Arrival rate.

X- Group size random variable.

Customers balk with probability $(1-\pi)$

$Pr(X=k)=g_k$.

$X(z)$ - the Probability Generating Function (PGF) of X.

$S_e^{(1)}(.)$, $S_e^{(2)}(.)$, $S_t^{(1)}(.)$, $S_t^{(2)}(.)$, $V(.)$ and $C(.)$ represent the Cumulative Distribution Function (CDF) of first stage, second stage service time of main server, first stage, second stage service time of stand-by- server, vacation time and closedown time and their corresponding probability density functions are $s_e^{(1)}(x)$, $s_e^{(2)}(x)$, $s_t^{(1)}(x)$, $s_t^{(2)}(x)$, $v(x)$ and $c(x)$ respectively.

$S_{e_1}^0(t)$, $S_{e_2}^0(t)$, $S_{t_1}^0(t)$, $S_{t_2}^0(t)$, $V^0(t)$ and $C^0(t)$ represent the remaining time of first stage, second stage service given by main server, first stage, second stage service given by stand-by- server, vacation time and closedown time at time t respectively.

$\tilde{S}_e^{(1)}(\theta)$, $\tilde{S}_e^{(2)}(\theta)$, $\tilde{S}_t^{(1)}(\theta)$, $\tilde{S}_t^{(2)}(\theta)$, $\tilde{V}(\theta)$ and $\tilde{C}(\theta)$ represent the Laplace Stieltjes transform (LST) of $S_e^{(1)}$, $S_e^{(2)}$, $S_t^{(1)}$, $S_t^{(2)}$, V and C respectively.

For the further development of the queueing system, let us define the random variable as:
$\varepsilon(t)$ = (1),(2),(3),(4) and (5) denotes main server busy, vacation, closedown, stand-by server busy and stand-by server idle respectively.
$Z(t) = j$, if the server is on j^{th} vacation, $j \geq 1$
$N_s(t)$ =Number of customers in service station at time t
$N_q(t)$ = Number of customers in the queue at time t.

Define the probabilities as,

$$I_n(t)\Delta t = Pr\{N_s(t) = n, \varepsilon(t) = 5\}, 0 \leq n \leq a-1,$$

$$P_{m,n}^{(i)}(x,t)\Delta t = Pr\{N_s(t) = m, N_q(t) = n, x \leq S_{e_i}^0(t) \leq x + \Delta t, \varepsilon(t) = 1\},$$
$$a \leq m \leq b,\ n \geq 0,\ i = 1,2,$$

$$B_{m,n}^{(i)}(x,t)\Delta t = Pr\{N_s(t) = n, N_q(t) = n, x \leq S_{t_i}^0(t) \leq x + \Delta t, \varepsilon(t) = 4\},$$
$$a \leq m \leq b,\ n \geq 0,\ i = 1,2,$$

$$Q_{l,j}(x,t)\Delta t = Pr\{Z(t) = l, N_q(t) = j, x \leq V^0(t) \leq x + \Delta t, \varepsilon(t) = 2\},\ l \geq 1,\ j \geq 0,$$

$$C_n(x,t)\Delta t = Pr\{N_q(t) = n, x \leq C^0(t) \leq x + dx, \varepsilon(t) = 3\},\ n \geq 0.$$

III. Queue Size Distribution

Now, the following equations are obtained for the above queueing system, using supplementary variables technique:

$$P_{i,0}^{(1)}(x-\Delta t, t+\Delta t) = (1-\lambda\Delta t)(1-\alpha\Delta t)P_{i,0}^{(1)}(x,t) + \sum_{m=a}^{b} P_{m,i}^{(2)}(0,t)s_e^{(1)}(x)\Delta t$$
$$+\sum_{l=1}^{\infty} Q_{l,i}(0,t)s_e^{(1)}(x)\Delta t + \eta\int_0^{\infty} B_{i,0}^{(1)}(y,t)dy s_e^{(1)}(x)\Delta t,\ a \leq i \leq b \quad (1)$$

$$P_{i,j}^{(1)}(x-\Delta t, t+\Delta t) = (1-\lambda\Delta t)(1-\alpha\Delta t)P_{i,j}^{(1)}(x,t) + \sum_{k=1}^{j} P_{i,j-k}^{(1)}(x,t)\lambda g_k\Delta t$$
$$+\eta\int_0^{\infty} B_{i,j}^{(1)}(y,t)dy s_e^{(1)}(x)\Delta t,\ a \leq i \leq b-1,\ j \geq 1 \quad (2)$$

$$P_{b,j}^{(1)}(x-\Delta t, t+\Delta t) = (1-\lambda\Delta t)(1-\alpha\Delta t)P_{b,j}^{(1)}(x,t) + \sum_{m=a}^{b} P_{m,b+j}^{(2)}(0,t)s_e^{(1)}(x)\Delta t$$
$$+\sum_{l=1}^{\infty} Q_{l,b+j}(0,t)s_e^{(1)}(x)\Delta t + \eta\int_0^{\infty} B_{b,j}^{(1)}(y,t)dy s_e^{(1)}(x)\Delta t$$
$$+\sum_{k=1}^{j} P_{b,j-k}^{(1)}(x,t)\lambda g_k\Delta t,\ j \geq 1 \quad (3)$$

$$P_{i,0}^{(2)}(x-\Delta t, t+\Delta t) = (1-\lambda\Delta t)(1-\alpha\Delta t)P_{i,0}^{(2)}(x,t) + P_{i,0}^{(1)}(0,t)s_e^{(2)}(x)\Delta t$$
$$+\eta\int_0^{\infty} B_{i,0}^{(2)}(y,t)dy s_e^{(2)}(x)\Delta t,\ a \leq i \leq b \quad (4)$$

$$P_{i,j}^{(2)}(x-\Delta t, t+\Delta t) = (1-\lambda\Delta t)(1-\alpha\Delta t)P_{i,j}^{(2)}(x,t) + P_{i,j}^{(1)}(0,t)s_e^{(2)}(x)\Delta t$$
$$+\sum_{k=1}^{j} P_{i,j-k}^{(2)}(x,t)\lambda g_k\Delta t + \eta\int_0^{\infty} B_{i,j}^{(2)}(y,t)dy s_e^{(2)}(x)\Delta t,\quad a \leq i \leq b,\ j \geq 1 \quad (5)$$

$$I_0(t+\Delta t) = (1-\lambda\Delta t)(1-\eta\Delta t)I_0(t) + \sum_{m=a}^{b} B_{m,0}^{(2)}(0,t)\Delta t \tag{6}$$

$$I_n(t+\Delta t) = (1-\lambda\Delta t)(1-\eta\Delta t)I_n(t) + \sum_{m=a}^{b} B_{m,n}^{(2)}(0,t)\Delta t + \sum_{k=1}^{n} I_{n-k}(t)\lambda g_k \Delta t, \quad 1 \le n \le a-1 \tag{7}$$

$$B_{i,0}^{(1)}(x-\Delta t, t+\Delta t) = (1-\lambda\Delta t)(1-\eta\Delta t)B_{i,0}^{(1)}(x,t) + \sum_{m=a}^{b} B_{m,i}^{(2)}(0,t)s_t^{(1)}(x)\Delta t + \sum_{n=0}^{a-1} I_n(t)\lambda g_{i-n} s_t^{(1)}(x)\Delta t + \alpha\int_0^\infty P_{i,0}^{(1)}(y,t)dy s_t^{(1)}(x)\Delta t + \lambda(1-\pi)B_{i,0}^{(1)}(x,t)\Delta t, \; a \le i \le b \tag{8}$$

$$B_{i,j}^{(1)}(x-\Delta t, t+\Delta t) = (1-\lambda\Delta t)(1-\eta\Delta t)B_{i,j}^{(1)}(x,t) + \pi\sum_{k=1}^{j} B_{i,j-k}^{(1)}(x,t)\lambda g_k \Delta t + \alpha\int_0^\infty P_{i,j}^{(1)}(y,t)dy s_t^{(1)}(x)\Delta t + \lambda(1-\pi)B_{i,j}^{(1)}(x,t)\Delta t, \quad a \le i \le b-1, \; j \ge 1 \tag{9}$$

$$B_{b,j}^{(1)}(x-\Delta t, t+\Delta t) = (1-\lambda\Delta t)(1-\eta\Delta t)B_{b,j}^{(1)}(x,t) + \pi\sum_{k=1}^{j} B_{b,j-k}^{(1)}(x,t)\lambda g_k \Delta t + \alpha\int_0^\infty P_{b,j}^{(1)}(y,t)dy s_t^{(1)}(x)\Delta t + \sum_{m=a}^{b} B_{m,b+j}^{(2)}(0,t)s_t^{(1)}(x)\Delta t + \sum_{n=0}^{a-1} I_n(t)\lambda g_{b+j-n} s_t^{(1)}(x)\Delta t + \lambda(1-\pi)B_{b,j}^{(1)}(x,t)\Delta t, \; j \ge 1 \tag{10}$$

$$B_{i,0}^{(2)}(x-\Delta t, t+\Delta t) = (1-\lambda\Delta t)(1-\eta\Delta t)B_{i,0}^{(2)}(x,t) + \lambda(1-\pi)B_{i,0}^{(2)}(x,t)\Delta t + B_{i,0}^{(1)}(0,t)s_t^{(2)}(x)\Delta t + \alpha\int_0^\infty P_{i,0}^{(2)}(y,t)dy s_t^{(2)}(x)\Delta t, \; a \le i \le b \tag{11}$$

$$B_{i,j}^{(2)}(x-\Delta t, t+\Delta t) = (1-\lambda\Delta t)(1-\eta\Delta t)B_{i,j}^{(2)}(x,t) + \pi\sum_{k=1}^{j} B_{i,j-k}^{(2)}(x,t)\lambda g_k \Delta t + B_{i,j}^{(1)}(0,t)s_t^{(2)}(x)\Delta t + \alpha\int_0^\infty P_{i,j}^{(2)}(y,t)dy s_t^{(2)}(x)\Delta t + \lambda(1-\pi)B_{i,j}^{(2)}(x,t)\Delta t, \; a \le i \le b, \; j \ge 1 \tag{12}$$

$$C_n(x-\Delta t, t+\Delta t) = (1-\lambda\Delta t)C_n(x,t) + \sum_{m=a}^{b} P_{m,n}^{(2)}(0,t)c(x)\Delta t + \sum_{k=1}^{n} C_{n-k}(x,t)\lambda g_k \Delta t + \eta I_n(t)c(x)\Delta t, n \le a-1 \tag{13}$$

$$C_n(x-\Delta t, t+\Delta t) = (1-\lambda\Delta t)C_n(x,t) + \sum_{k=1}^{n} C_{n-k}(x,t)\lambda g_k \Delta t, n \ge a \tag{14}$$

$$Q_{1,0}(x-\Delta t, t+\Delta t) = (1-\lambda\Delta t)Q_{1,0}(x,t) + C_0(0,t)v(x)\Delta t \tag{15}$$

$$Q_{1,n}(x-\Delta t, t+\Delta t) = (1-\lambda\Delta t)Q_{1,n}(x,t) + \sum_{k=1}^{n} Q_{1,n-k}(x,t)\lambda g_k \Delta t + C_n(0,t)v(x)\Delta t, \quad n \ge 1 \tag{16}$$

$$Q_{j,0}(x-\Delta t, t+\Delta t) = (1-\lambda\Delta t)Q_{j,0}(x,t) + Q_{j-1,0}(0,t)v(x)\Delta t, \; j \ge 2 \tag{17}$$

$$Q_{j,n}(x-\Delta t, t+\Delta t) = (1-\lambda\Delta t)Q_{j,n}(x,t) + Q_{j-1,n}(0,t)v(x)\Delta t$$

$$+\sum_{k=1}^{n}Q_{j,n-k}(x,t)\lambda g_k\Delta t,\ j\geq 2, 1\leq n\leq a-1 \quad (18)$$

$$Q_{j,n}(x-\Delta t,t+\Delta t)=(1-\lambda\Delta t)Q_{j,n}(x,t)+\sum_{k=1}^{n}Q_{j,n-k}(x,t)\lambda g_k\Delta t,\ j\geq 2, n\geq a. \quad (19)$$

From the above equations, the steady state queue size equations are obtained as follows:

$$-\frac{d}{dx}P_{i,0}^{(1)}(x)=-(\lambda+\alpha)P_{i,0}^{(1)}(x)+\sum_{m=a}^{b}P_{m,i}^{(2)}(0)s_e^{(1)}(x)+\sum_{l=1}^{\infty}Q_{l,i}(0)s_e^{(1)}(x)$$
$$+\eta\int_0^{\infty}B_{i,0}^{(1)}(y)dys_e^{(1)}(x),\ a\leq i\leq b \quad (20)$$

$$-\frac{d}{dx}P_{i,j}^{(1)}(x)=-(\lambda+\alpha)P_{i,j}^{(1)}(x)+\sum_{k=1}^{j}P_{i,j-k}^{(1)}(x)\lambda g_k+\eta\int_0^{\infty}B_{i,j}^{(1)}(y)dys_e^{(1)}(x),$$
$$a\leq i\leq b-1,\ j\geq 1 \quad (21)$$

$$-\frac{d}{dx}P_{b,j}^{(1)}(x)=-(\lambda+\alpha)P_{b,j}^{(1)}(x)+\sum_{m=a}^{b}P_{m,b+j}^{(2)}(0)s_e^{(1)}(x)+\sum_{l=1}^{\infty}Q_{l,b+j}(0)s_e^{(1)}(x)$$
$$+\eta\int_0^{\infty}B_{b,j}^{(1)}(y)dys_e^{(1)}(x)+\sum_{k=1}^{j}P_{b,j-k}^{(1)}(x)\lambda g_k,\ j\geq 1 \quad (22)$$

$$-\frac{d}{dx}P_{i,0}^{(2)}(x)=-(\lambda+\alpha)P_{i,0}^{(2)}(x)+P_{i,0}^{(1)}(0)s_e^{(2)}(x)+\eta\int_0^{\infty}B_{i,0}^{(2)}(y)dys_e^{(2)}(x),\ a\leq i\leq b \quad (23)$$

$$-\frac{d}{dx}P_{i,j}^{(2)}(x)=-(\lambda+\alpha)P_{i,j}^{(2)}(x)+P_{i,j}^{(1)}(0)s_e^{(2)}(x)+\eta\int_0^{\infty}B_{i,j}^{(2)}(y)dys_e^{(2)}(x)$$
$$+\sum_{k=1}^{j}P_{i,j-k}^{(2)}(x)\lambda g_k,\ a\leq i\leq b,\ j\geq 1 \quad (24)$$

$$(\lambda+\eta)I_0=\sum_{m=a}^{b}B_{m,0}^{(2)}(0) \quad (25)$$

$$(\lambda+\eta)I_n=\sum_{m=a}^{b}B_{m,n}^{(2)}(0)+\sum_{k=1}^{n}I_{n-k}\lambda g_k,\ 1\leq n\leq a-1 \quad (26)$$

$$-\frac{d}{dx}B_{i,0}^{(1)}(x)=-(\lambda+\eta)B_{i,0}^{(1)}(x)+\sum_{m=a}^{b}B_{m,i}^{(2)}(0)s_t^{(1)}(x)+\sum_{n=0}^{a-1}I_n\lambda g_{i-n}s_t^{(1)}(x)$$
$$+\alpha\int_0^{\infty}P_{i,0}^{(1)}(y)dys_t^{(1)}(x)+\lambda(1-\pi)B_{i,0}^{(1)}(x),\ a\leq i\leq b \quad (27)$$

$$-\frac{d}{dx}B_{i,j}^{(1)}(x)=-(\lambda+\eta)B_{i,j}^{(1)}(x)+\pi\sum_{k=1}^{j}B_{i,j-k}^{(1)}(x)\lambda g_k+\alpha\int_0^{\infty}P_{i,j}^{(1)}(y)dys_t^{(1)}(x)$$
$$+\lambda(1-\pi)B_{i,j}^{(1)}(x),\ a\leq i\leq b-1,\ j\geq 1 \quad (28)$$

$$-\frac{d}{dx}B_{b,j}^{(1)}(x)=-(\lambda+\eta)B_{b,j}^{(1)}(x)+\pi\sum_{k=1}^{j}B_{b,j-k}^{(1)}(x)\lambda g_k+\alpha\int_0^{\infty}P_{b,j}^{(1)}(y)dys_t^{(1)}(x)$$
$$+\sum_{m=a}^{b}B_{m,b+j}^{(2)}(0)s_t^{(1)}(x)+\sum_{n=0}^{a-1}I_n\lambda g_{b+j-n}s_t^{(1)}(x)+\lambda(1-\pi)B_{b,j}^{(1)}(x),\quad j\geq 1 \quad (29)$$

$$-\frac{d}{dx}B_{i,0}^{(2)}(x)=-(\lambda+\eta)B_{i,0}^{(2)}(x)+\lambda(1-\pi)B_{i,0}^{(2)}(x)+B_{i,0}^{(1)}(0)s_t^{(2)}(x)$$

$$+\alpha\int_0^\infty P_{i,0}^{(2)}(y)dys_t^{(2)}(x),\ a\le i\le b \tag{30}$$

$$-\frac{d}{dx}B_{i,j}^{(2)}(x)=-(\lambda+\eta)B_{i,j}^{(2)}(x)+\pi\sum_{k=1}^{j}B_{i,j-k}^{(2)}(x)\lambda g_k+\alpha\int_0^\infty P_{i,j}^{(2)}(y)dys_t^{(2)}(x)$$
$$+B_{i,j}^{(1)}(0)s_t^{(2)}(x)+\lambda(1-\pi)B_{i,j}^{(2)}(x),\ a\le i\le b,\ j\ge 1 \tag{31}$$

$$-\frac{d}{dx}C_n(x)=-\lambda C_n(x)+\sum_{m=a}^{b}P_{m,n}^{(2)}(0)c(x)+\sum_{k=1}^{n}C_{n-k}(x)\lambda g_k+\eta I_n c(x),\ n\le a-1 \tag{32}$$

$$-\frac{d}{dx}C_n(x)=-\lambda C_n(x)+\sum_{k=1}^{n}C_{n-k}(x)\lambda g_k,\ n\ge a \tag{33}$$

$$-\frac{d}{dx}Q_{1,0}(x)=-\lambda Q_{1,0}(x)+C_0(0)v(x) \tag{34}$$

$$-\frac{d}{dx}Q_{1,n}(x)=-\lambda Q_{1,n}(x)+\sum_{k=1}^{n}Q_{1,n-k}(x)\lambda g_k+C_n(0)v(x),\ n\ge 1 \tag{35}$$

$$-\frac{d}{dx}Q_{j,0}(x)=-\lambda Q_{j,0}(x)+Q_{j-1,0}(0)v(x),\ j\ge 2 \tag{36}$$

$$-\frac{d}{dx}Q_{j,n}(x)=-\lambda Q_{j,n}(x)+Q_{j-1,n}(0)v(x)+\sum_{k=1}^{n}Q_{j,n-k}(x)\lambda g_k,\ j\ge 2, 1\le n\le a-1 \tag{37}$$

$$-\frac{d}{dx}Q_{j,n}(x)=-\lambda Q_{j,n}(x)+\sum_{k=1}^{n}Q_{j,n-k}(x)\lambda g_k,\ j\ge 2, n\ge a. \tag{38}$$

Taking LST on both sides of equations (20) to (38) except (25) and (26), we have

$$\theta\tilde{P}_{i,0}^{(1)}(\theta)-P_{i,0}^{(1)}(0)=(\lambda+\alpha)\tilde{P}_{i,0}^{(1)}(\theta)-\sum_{m=a}^{b}P_{m,i}^{(2)}(0)\tilde{S}_e^{(1)}(\theta)-\sum_{l=1}^{\infty}Q_{l,i}(0)\tilde{S}_e^{(1)}(\theta)$$
$$-\eta\int_0^\infty B_{i,0}^{(1)}(y)dy\tilde{S}_e^{(1)}(\theta),\ a\le i\le b \tag{39}$$

$$\theta\tilde{P}_{i,j}^{(1)}(\theta)-P_{i,j}^{(1)}(0)=(\lambda+\alpha)\tilde{P}_{i,j}^{(1)}(\theta)-\sum_{k=1}^{j}\tilde{P}_{i,j-k}^{(1)}(\theta)\lambda g_k-\eta\int_0^\infty B_{i,j}^{(1)}(y)dy\tilde{S}_e^{(1)}(\theta),$$
$$a\le i\le b-1,\ j\ge 1 \tag{40}$$

$$\theta\tilde{P}_{b,j}^{(1)}(\theta)-P_{b,j}^{(1)}(0)=(\lambda+\alpha)\tilde{P}_{b,j}^{(1)}(\theta)-\sum_{m=a}^{b}P_{m,b+j}^{(2)}(0)\tilde{S}_e^{(1)}(\theta)-\sum_{l=1}^{\infty}Q_{l,b+j}(0)\tilde{S}_e^{(1)}(\theta)$$
$$-\eta\int_0^\infty B_{b,j}^{(1)}(y)dy\tilde{S}_e^{(1)}(\theta)-\sum_{k=1}^{j}\tilde{P}_{b,j-k}^{(1)}(\theta)\lambda g_k,\ j\ge 1 \tag{41}$$

$$\theta\tilde{P}_{i,0}^{(2)}(\theta)-P_{i,0}^{(2)}(0)=(\lambda+\alpha)\tilde{P}_{i,0}^{(2)}(\theta)-P_{i,0}^{(1)}(0)\tilde{S}_e^{(2)}(\theta)-\eta\int_0^\infty B_{i,0}^{(2)}(y)dy\tilde{S}_e^{(2)}(\theta),$$
$$a\le i\le b \tag{42}$$

$$\theta\tilde{P}_{i,j}^{(2)}(\theta)-P_{i,j}^{(2)}(0)=(\lambda+\alpha)\tilde{P}_{i,j}^{(2)}(\theta)-P_{i,j}^{(1)}(0)\tilde{S}_e^{(2)}(\theta)-\eta\int_0^\infty B_{i,j}^{(2)}(y)dy\tilde{S}_e^{(2)}(\theta)$$
$$-\sum_{k=1}^{j}\tilde{P}_{i,j-k}^{(2)}(\theta)\lambda g_k,\ a\le i\le b,\ j\ge 1 \tag{43}$$

$$\theta\tilde{B}_{i,0}^{(1)}(\theta)-B_{i,0}^{(1)}(0)=(\lambda+\eta)\tilde{B}_{i,0}^{(1)}(\theta)-\sum_{m=a}^{b}B_{m,i}^{(2)}(0)\tilde{S}_t^{(1)}(\theta)-\sum_{n=0}^{a-1}I_n\lambda g_{i-n}\tilde{S}_t^{(1)}(\theta)$$
$$-\alpha\int_0^\infty P_{i,0}^{(1)}(y)dy\tilde{S}_t^{(1)}(\theta)-\lambda(1-\pi)\tilde{B}_{i,0}^{(1)}(\theta),\ a\le i\le b \tag{44}$$

$$\theta\widetilde{B}_{i,j}^{(1)}(\theta)-B_{i,j}^{(1)}(0)=(\lambda+\eta)\widetilde{B}_{i,j}^{(1)}(\theta)-\pi\sum_{k=1}^{j}\widetilde{B}_{i,j-k}^{(1)}(\theta)\lambda g_k-\alpha\int_0^\infty P_{i,j}^{(1)}(y)dy\widetilde{S}_t^{(1)}(\theta)$$
$$-\lambda(1-\pi)\widetilde{B}_{i,j}^{(1)}(\theta),\ a\le i\le b-1,\ j\ge 1 \quad (45)$$

$$\theta\widetilde{B}_{b,j}^{(1)}(\theta)-B_{b,j}^{(1)}(0)=(\lambda+\eta)\widetilde{B}_{b,j}^{(1)}(\theta)-\pi\sum_{k=1}^{j}\widetilde{B}_{b,j-k}^{(1)}(\theta)\lambda g_k-\alpha\int_0^\infty P_{b,j}^{(1)}(y)dy\widetilde{S}_t^{(1)}(\theta)$$
$$-\sum_{m=a}^{b}B_{m,b+j}^{(2)}(0)\widetilde{S}_t^{(1)}(\theta)-\sum_{n=0}^{a-1}I_n\lambda g_{b+j-n}\widetilde{S}_t^{(1)}(\theta)-\lambda(1-\pi)\widetilde{B}_{b,j}^{(1)}(\theta),$$
$$j\ge 1 \quad (46)$$

$$\theta\widetilde{B}_{i,0}^{(2)}(\theta)-B_{i,0}^{(2)}(0)=(\lambda+\eta)\widetilde{B}_{i,0}^{(2)}(\theta)-\lambda(1-\pi)\widetilde{B}_{i,0}^{(2)}(\theta)-B_{i,0}^{(1)}(0)\widetilde{S}_t^{(2)}(\theta)$$
$$-\alpha\int_0^\infty P_{i,0}^{(2)}(y)dy\widetilde{S}_t^{(2)}(\theta),\ a\le i\le b \quad (47)$$

$$\theta\widetilde{B}_{i,j}^{(2)}(\theta)-B_{i,j}^{(2)}(0)=(\lambda+\eta)\widetilde{B}_{i,j}^{(2)}(\theta)-\pi\sum_{k=1}^{j}\widetilde{B}_{i,j-k}^{(2)}(\theta)\lambda g_k-\alpha\int_0^\infty P_{i,j}^{(2)}(y)dy\widetilde{S}_t^{(2)}(\theta)$$
$$-B_{i,j}^{(1)}(0)\widetilde{S}_t^{(2)}(\theta)-\lambda(1-\pi)\widetilde{B}_{i,j}^{(2)}(\theta),a\le i\le b,\ j\ge 1 \quad (48)$$

$$\theta\widetilde{C}_n(\theta)-C_n(0)=\lambda\widetilde{C}_n(\theta)-\sum_{m=a}^{b}P_{m,n}^{(2)}(0)\widetilde{C}(\theta)-\sum_{k=1}^{n}\widetilde{C}_{n-k}(\theta)\lambda g_k-\eta I_n\widetilde{C}(\theta),\ n\le a-1 \quad (49)$$

$$\theta\widetilde{C}_n(\theta)-C_n(0)=\lambda C_n(\theta)-\sum_{k=1}^{n}\widetilde{C}_{n-k}(\theta)\lambda g_k,\ n\ge a \quad (50)$$

$$\theta\widetilde{Q}_{1,0}(\theta)-Q_{1,0}(0)=\lambda\widetilde{Q}_{1,0}(\theta)-C_0(0)\widetilde{V}(\theta) \quad (51)$$

$$\theta\widetilde{Q}_{1,n}(\theta)-Q_{1,n}(0)=\lambda\widetilde{Q}_{1,n}(\theta)-\sum_{k=1}^{n}\widetilde{Q}_{1,n-k}(\theta)\lambda g_k-C_n(0)\widetilde{V}(\theta),\ n\ge 1 \quad (52)$$

$$\theta\widetilde{Q}_{j,0}(\theta)-Q_{j,0}(0)=\lambda\widetilde{Q}_{j,0}(\theta)-Q_{j-1,0}(0)\widetilde{V}(\theta),\ \ j\ge 2 \quad (53)$$

$$\theta\widetilde{Q}_{j,n}(\theta)-Q_{j,n}(0)=\lambda\widetilde{Q}_{j,n}(\theta)-Q_{j-1,n}(0)\widetilde{V}(\theta)-\sum_{k=1}^{n}\widetilde{Q}_{j,n-k}(\theta)\lambda g_k,\ \ j\ge 2,1\le n\le a-1 \quad (54)$$

$$\theta\widetilde{Q}_{j,n}(\theta)-Q_{j,n}(0)=\lambda\widetilde{Q}_{j,n}(\theta)-\sum_{k=1}^{n}\widetilde{Q}_{j,n-k}(\theta)\lambda g_k,\ \ j\ge 2,n\ge a. \quad (55)$$

We define the following probability generating functions:

$$\begin{aligned}
&\widetilde{P}_i^{(n)}(z,\theta)=\sum_{j=0}^{\infty}\widetilde{P}_{i,j}^{(n)}(\theta)z^j \quad P_i^{(n)}(z,0)=\sum_{j=0}^{\infty}P_{i,j}^{(n)}(0)z^j,(a\le i\le b),\ n=1,2\\
&\widetilde{B}_i^{(n)}(z,\theta)=\sum_{j=0}^{\infty}\widetilde{B}_{i,j}^{(n)}(\theta)z^j \quad B_i^{(n)}(z,0)=\sum_{j=0}^{\infty}B_{i,j}^{(n)}(0)z^j,(a\le i\le b),\ n=1,2\\
&\widetilde{Q}_l(z,\theta)=\sum_{j=0}^{\infty}\widetilde{Q}_{l,j}(\theta)z^j \quad Q_l(z,0)=\sum_{j=0}^{\infty}Q_{l,j}(0)z^j,\ l\ge 1\\
&\widetilde{C}(z,\theta)=\sum_{n=0}^{\infty}\widetilde{C}_n(\theta)z^n \quad C(z,0)=\sum_{n=0}^{\infty}C_n(0)z^n.
\end{aligned} \quad (56)$$

By multiplying the equations (39) to (55) with suitable power of z^n and summing over $n=0$ to ∞, and using equation (56), we have

$$(\theta-u(z))\widetilde{P}_i^{(1)}(z,\theta)=P_i^{(1)}(z,0)-\widetilde{S}_e^{(1)}(\theta)[\eta\widetilde{B}_i^{(1)}(z,0)+\sum_{m=a}^{b}P_{m,i}^{(2)}(0)+\sum_{l=1}^{\infty}Q_{l,i}(0)],$$
$$a\le i\le b-1 \quad (57)$$

$$z^b(\theta-u(z))\widetilde{P}_b^{(1)}(z,\theta)=z^bP_b^{(1)}(z,0)+\widetilde{S}_e^{(1)}(\theta)\sum_{j=0}^{b-1}[\sum_{m=a}^{b}P_{m,j}^{(2)}(0)z^j+\sum_{l=1}^{\infty}Q_{l,j}(0)z^j]$$
$$-\widetilde{S}_e^{(1)}(\theta)[z^b\eta\widetilde{B}_b^{(1)}(z,0)+\sum_{m=a}^{b}P_m^{(2)}(z,0)+\sum_{l=1}^{\infty}Q_l(z,0)] \quad (58)$$

$$(\theta-u(z))\widetilde{P}_i^{(2)}(z,\theta)=P_i^{(2)}(z,0)-\widetilde{S}_e^{(2)}(\theta)[\eta\widetilde{B}_i^{(2)}(z,0)+P_i^{(1)}(z,0)],\ a\le i\le b \quad (59)$$

$$(\theta-v(z))\widetilde{B}_i^{(1)}(z,\theta)=B_i^{(1)}(z,0)-\widetilde{S}_t^{(1)}(\theta)[\alpha\widetilde{P}_i^{(1)}(z,0)+\sum_{m=a}^{b}B_{m,i}^{(2)}(0)+\sum_{n=0}^{a-1}I_n\lambda g_{i-n}],$$
$$a\le i\le b-1 \quad (60)$$

$$z^b(\theta-v(z))\widetilde{B}_b^{(1)}(z,\theta)=z^bB_b^{(1)}(z,0)-\widetilde{S}_t^{(1)}(\theta)f(z) \quad (61)$$

$$(\theta-v(z))\widetilde{B}_i^{(2)}(z,\theta)=B_i^{(2)}(z,0)-\widetilde{S}_t^{(2)}(\theta)[\alpha\widetilde{P}_i^{(2)}(z,0)+B_i^{(1)}(z,0)],\ a\le i\le b \quad (62)$$

$$(\theta-w(z))\widetilde{C}(z,\theta)=C(z,0)-\widetilde{C}(\theta)[\eta I(z)+\sum_{m=a}^{b}\sum_{n=0}^{a-1}P_{m,n}^{(2)}(0)z^n] \quad (63)$$

$$(\theta-w(z))\widetilde{Q}_1(z,\theta)=Q_1(z,0)-\widetilde{V}(\theta)C(z,0) \quad (64)$$

$$(\theta-w(z))\widetilde{Q}_j(z,\theta)=Q_j(z,0)-\widetilde{V}(\theta)\sum_{n=0}^{a-1}Q_{j-1,n}(0)z^n,\ j\ge 2, \quad (65)$$

where

$$u(z)=\lambda+\alpha-\lambda X(z), v(z)=\pi(\lambda-\lambda X(z))+\eta, w(z)=\lambda-\lambda X(z),$$

and $f(z)=z^b\alpha\widetilde{P}_b^{(1)}(z,0)+\sum_{m=a}^{b}B_m^{(2)}(z,0)-\sum_{i=a}^{b-1}\sum_{m=a}^{b}B_{m,i}^{(2)}(0)z^i$

$$-(\lambda+\eta-\lambda X(z))I(z)-\lambda\sum_{i=a}^{b-1}\sum_{n=0}^{a-1}I_ng_{i-n}z^i.$$

Substitute $\theta=u(z)$ in (57) to (59), we have

$$P_i^{(1)}(z,0)=\widetilde{S}_e^{(1)}(u(z))[\eta\widetilde{B}_i^{(1)}(z,0)+\sum_{m=a}^{b}P_{m,i}^{(2)}(0)+\sum_{l=1}^{\infty}Q_{l,i}(0)],\ a\le i\le b-1 \quad (66)$$

$$z^bP_b^{(1)}(z,0)=\widetilde{S}_e^{(1)}(u(z))[z^b\eta\widetilde{B}_b^{(1)}(z,0)+\sum_{m=a}^{b}P_m^{(2)}(z,0)+\sum_{l=1}^{\infty}Q_l(z,0)]$$
$$-\widetilde{S}_e^{(1)}(u(z))\sum_{j=0}^{b-1}[\sum_{m=a}^{b}P_{m,j}^{(2)}(0)z^j+\sum_{l=1}^{\infty}Q_{l,j}(0)z^j] \quad (67)$$

$$P_i^{(2)}(z,0)=\widetilde{S}_e^{(2)}(u(z))[\eta\widetilde{B}_i^{(2)}(z,0)+P_i^{(1)}(z,0)],\ a\le i\le b. \quad (68)$$

Substitute $\theta=v(z)$ in (60) to (62), we have

$$B_i^{(1)}(z,0)=\widetilde{S}_t^{(1)}(v(z))[\alpha\widetilde{P}_i^{(1)}(z,0)+\sum_{m=a}^{b}B_{m,i}^{(2)}(0)+\sum_{n=0}^{a-1}I_n\lambda g_{i-n}],\ a\le i\le b-1 \quad (69)$$

$$z^bB_b^{(1)}(z,0)=\widetilde{S}_t^{(1)}(v(z))f(z) \quad (70)$$

$$B_i^{(2)}(z,0)=\widetilde{S}_t^{(2)}(v(z))[\alpha\widetilde{P}_i^{(2)}(z,0)+B_i^{(1)}(z,0)],\ a\le i\le b. \quad (71)$$

Substitute $\theta=w(z)$ in (63) to (65), we have

$$C(z,0)=\widetilde{C}(w(z))[\eta I(z)+\sum_{m=a}^{b}\sum_{n=0}^{a-1}P_{m,n}^{(2)}(0)z^n] \quad (72)$$

$$Q_1(z,0)=\widetilde{V}(w(z))C(z,0) \quad (73)$$

$$Q_j(z,0) = \tilde{V}(w(z)) \sum_{n=0}^{a-1} Q_{j-1,n}(0) z^n, \; j \geq 2. \tag{74}$$

Substitute (66) to (74) in (57) to (65), we have

$$\tilde{P}_i^{(1)}(z,\theta) = \frac{(\tilde{S}_e^{(1)}(u(z)) - \tilde{S}_e^{(1)}(\theta))[\eta \tilde{B}_i^{(1)}(z,0) + \sum_{m=a}^{b} P_{m,i}^{(2)}(0) + \sum_{l=1}^{\infty} Q_{l,i}(0)]}{(\theta - u(z))}, \quad a \leq i \leq b-1 \tag{75}$$

$$\tilde{P}_b^{(1)}(z,\theta) = \frac{(\tilde{S}_e^{(1)}(\theta) - \tilde{S}_e^{(1)}(u(z))) \sum_{j=0}^{b-1} [\sum_{m=a}^{b} P_{m,j}^{(2)}(0) z^j + \sum_{l=1}^{\infty} Q_{l,j}(0) z^j] + (\tilde{S}_e^{(1)}(u(z)) - \tilde{S}_e^{(1)}(\theta))[z^b \eta \tilde{B}_b^{(1)}(z,0) + \sum_{m=a}^{b} P_m^{(2)}(z,0) + \sum_{l=1}^{\infty} Q_l(z,0)]}{z^b(\theta - u(z))} \tag{76}$$

$$\tilde{P}_i^{(2)}(z,\theta) = \frac{(\tilde{S}_e^{(2)}(u(z)) - \tilde{S}_e^{(2)}(\theta))[\eta \tilde{B}_i^{(2)}(z,0) + P_i^{(1)}(z,0)]}{(\theta - u(z))}, \; a \leq i \leq b \tag{77}$$

$$\tilde{B}_i^{(1)}(z,\theta) = \frac{(\tilde{S}_t^{(1)}(v(z)) - \tilde{S}_t^{(1)}(\theta))[\alpha \tilde{P}_i^{(1)}(z,0) + \sum_{m=a}^{b} B_{m,i}^{(2)}(0) + \sum_{n=0}^{a-1} I_n \lambda g_{i-n}]}{(\theta - v(z))} \tag{78}$$

$$\tilde{B}_b^{(1)}(z,\theta) = \frac{(\tilde{S}_t^{(1)}(v(z)) - \tilde{S}_t^{(1)}(\theta)) f(z)}{z^b(\theta - v(z))} \tag{79}$$

$$\tilde{B}_i^{(2)}(z,\theta) = \frac{(\tilde{S}_t^{(2)}(v(z)) - \tilde{S}_t^{(2)}(\theta))[\alpha \tilde{P}_i^{(2)}(z,0) + B_i^{(1)}(z,0)]}{(\theta - v(z))}, \; a \leq i \leq b \tag{80}$$

$$\tilde{C}(z,\theta) = \frac{(\tilde{C}(w(z)) - \tilde{C}(\theta))[\eta I(z) + \sum_{m=a}^{b} \sum_{n=0}^{a-1} P_{m,n}^{(2)}(0) z^n]}{(\theta - w(z))} \tag{81}$$

$$\tilde{Q}_1(z,\theta) = \frac{(\tilde{V}(w(z)) - \tilde{V}(\theta))C(z,0)}{(\theta - w(z))} \tag{82}$$

$$\tilde{Q}_j(z,\theta) = \frac{(\tilde{V}(w(z)) - \tilde{V}(\theta))\sum_{n=0}^{a-1} Q_{j-1,n}(0)z^n}{(\theta - w(z))}, \; j \geq 2. \tag{83}$$

IV. Probability Generating Function Of Queue Size

4.1 PGF of queue size at an arbitrary time epoch

The PGF of the queue size at an arbitrary time epoch is obtained as

$$P(z) = \sum_{i=a}^{b} \tilde{P}_i^{(1)}(z,0) + \sum_{i=a}^{b} \tilde{P}_i^{(2)}(z,0) + I(z) + \sum_{i=a}^{b} \tilde{B}_i^{(1)}(z,0) + \sum_{i=a}^{b} \tilde{B}_i^{(2)}(z,0) + \tilde{C}(z,0) + \sum_{l=1}^{\infty} \tilde{Q}_l(z,0). \tag{84}$$

By substitute $\theta = 0$ on the equations (75) to (83) then the equation (84) becomes

$$P(z) = \frac{\begin{aligned} & K_1(z)\sum_{i=a}^{b-1}(z^b - z^i)c_i + K_2(z)\sum_{i=a}^{b-1}(z^b - z^i)d_i \\ & + [(\tilde{V}(w(z)) - 1)(K_1(z) - K_3(z))]\sum_{n=0}^{a-1} q_n z^n \\ & + [w(z)K_3(z) - (\lambda + \eta - \lambda X(z))K_2(z)]I(z) - K_1(z)\sum_{n=0}^{a-1} p_n z^n \\ & + \tilde{C}(w(z))(\tilde{V}(w(z))K_1(z) + (1 - \tilde{V}(w(z)))K_3(z))(\eta I(z) + \sum_{n=0}^{a-1} p_n z^n) \end{aligned}}{w(z)K_3(z)}. \tag{85}$$

This represents the PGF of number of customers in queue at an arbitrary time epoch,
where

$$\sum_{i=a}^{b-1}\left(\sum_{i=a}^{b} P_{m,i}^{(2)}(0) + \sum_{l=1}^{\infty} Q_{l,i}(0)\right) = \sum_{i=a}^{b-1} c_i,$$

$$\sum_{i=a}^{b-1}\left(\sum_{i=a}^{b} B_{m,i}^{(2)}(0) + \sum_{n=0}^{a-1} I_n \lambda g_{i-n}\right) = \sum_{i=a}^{b-1} d_i,$$

$$\sum_{i=a}^{b} P_{m,i}^{(2)}(0) = p_i, \; \sum_{l=1}^{\infty} Q_{l,i}(0) = q_i,$$

$$
\begin{aligned}
K_1(z) = {} & u(z)v(z)w(z)\{z^b\alpha[1-\tilde{S}_t^{(1)}(v(z))\tilde{S}_t^{(2)}(v(z))] \\
& + v(z)(z^b - \tilde{S}_t^{(1)}(v(z))\tilde{S}_t^{(2)}(v(z)))(1-\tilde{S}_e^{(1)}(u(z))\tilde{S}_e^{(2)}(u(z))) \\
& + \alpha\tilde{S}_e^{(1)}(u(z))\tilde{S}_t^{(2)}(v(z))(1-\tilde{S}_t^{(1)}(v(z)))(1-\tilde{S}_e^{(2)}(u(z))) \\
& - z^b\alpha\tilde{S}_e^{(1)}(u(z))[\tilde{S}_e^{(2)}(u(z))(1-\tilde{S}_t^{(2)}(v(z))) + (1-\tilde{S}_t^{(1)}(v(z)))\tilde{S}_t^{(2)}(v(z))]\} \\
& - z^b\alpha\eta(\alpha+v(z))w(z)(1-\tilde{S}_e^{(1)}(u(z)))(1-\tilde{S}_e^{(2)}(u(z)))\times \\
& (1-\tilde{S}_t^{(1)}(v(z)))(1-\tilde{S}_t^{(2)}(v(z))),
\end{aligned}
$$

$$
\begin{aligned}
K_2(z) = {} & u(z)v(z)w(z)\{z^b\eta[1-\tilde{S}_e^{(1)}(u(z))\tilde{S}_e^{(2)}(u(z))] \\
& + u(z)(z^b - \tilde{S}_e^{(1)}(u(z))\tilde{S}_e^{(2)}(u(z)))(1-\tilde{S}_t^{(1)}(v(z))\tilde{S}_t^{(2)}(v(z))) \\
& + \eta\tilde{S}_t^{(1)}(v(z))\tilde{S}_e^{(2)}(u(z))(1-\tilde{S}_e^{(1)}(u(z)))(1-\tilde{S}_t^{(2)}(v(z))) \\
& - z^b\eta\tilde{S}_t^{(1)}(v(z))[\tilde{S}_t^{(2)}(v(z))(1-\tilde{S}_e^{(2)}(u(z))) + (1-\tilde{S}_e^{(1)}(u(z)))\tilde{S}_e^{(2)}(u(z))]\} \\
& - z^b\alpha\eta(\eta+u(z))w(z)(1-\tilde{S}_e^{(1)}(u(z)))(1-\tilde{S}_e^{(2)}(u(z)))\times \\
& (1-\tilde{S}_t^{(1)}(v(z)))(1-\tilde{S}_t^{(2)}(v(z))),
\end{aligned}
$$

$$
\begin{aligned}
K_3(z) = {} & z^b z^b[u(z)v(z) - \alpha\eta(1-\tilde{S}_e^{(1)}(u(z)))(1-\tilde{S}_t^{(1)}(v(z)))] \\
& \times[u(z)v(z) - \alpha\eta(1-\tilde{S}_e^{(2)}(u(z)))(1-\tilde{S}_t^{(2)}(v(z)))] \\
& + u^2v^2\tilde{S}_e^{(1)}(u(z))\tilde{S}_e^{(2)}(u(z))\tilde{S}_t^{(1)}(v(z))\tilde{S}_t^{(2)}(v(z)) \\
& - z^b u(z)v(z)\tilde{S}_e^{(1)}(u(z))[u(z)v(z)\tilde{S}_e^{(2)}(u(z)) \\
& + \alpha\eta(1-\tilde{S}_t^{(1)}(v(z)))(1-\tilde{S}_e^{(2)}(u(z)))\tilde{S}_t^{(2)}(v(z))] \\
& - z^b u(z)v(z)\tilde{S}_t^{(1)}(v(z))[u(z)v(z)\tilde{S}_t^{(2)}(v(z)) \\
& + \alpha\eta(1-\tilde{S}_e^{(1)}(u(z)))(1-\tilde{S}_t^{(2)}(v(z)))\tilde{S}_e^{(2)}(u(z))].
\end{aligned}
$$

REMARK 1.

Equation (85) has $a+2b$ unknowns $c_a, c_{a+1}, ..., c_{b-1}, d_a, d_{a+1}, ..., d_{b-1}, p_0, p_1, ..., p_{a-1}$, $q_0, q_1, ..., q_{a-1}$ and $I_0, I_1, ..., I_{a-1}$.We develop the following theorems to express q_i in terms of p_i and I_i in such a way that the numerator has only $2b$ constants. Now equation (85) gives the PGF of the number of customers involving only 2b unknowns. By Rouche's theorem of complex variables it can be proved that $K_3(z)$ in the denominator of equation (85) has $2b-1$ zeros inside and one on the unit circle $|z|=1$. Since $P(z)$ is analytic within and on the unit circle, the numerator must vanish at these points, which gives $2b$ equations in $2b$ unknowns. We can solve these equations by any suitable numerical technique. Thus, equation (85) gives the PGF of queue size at arbitrary time.

REMARK 2.

The probability generating function has to satisfy $P(1)=1$. Applying L' Hopital's rule in (85), we get $N''(1) = 2(-\lambda X1)K_3^{'}(1)$.Since p_i, q_i and I_i are probabilities, it follows that $N''(1)$ must be positive. Thus $P(1)=1$ if and only if the $2(-\lambda X1)K_3^{'}(1) >$ 0. If

$$\rho = \frac{\lambda X1(\pi\alpha+\eta) \quad ((1-\tilde{S}_e^{(1)}(\alpha))(1-\tilde{S}_e^{(2)}(\alpha))(1-\tilde{S}_t^{(1)}(\eta))(1-\tilde{S}_e^{(2)}(\eta)) - (1-\tilde{S}_e^{(1)}(\alpha)\tilde{S}_e^{(2)}(\alpha))(1-\tilde{S}_t^{(1)}(\eta)\tilde{S}_t^{(2)}(\eta)))}{b\alpha\eta(\tilde{S}_e^{(1)}(\alpha)[\tilde{S}_e^{(2)}(\alpha)(\tilde{S}_t^{(1)}(\eta)+\tilde{S}_t^{(2)}(\eta)-1)+\tilde{S}_t^{(2)}(\eta)] + \tilde{S}_t^{(1)}(\eta)[\tilde{S}_t^{(2)}(\eta)(\tilde{S}_e^{(1)}(\alpha)+\tilde{S}_e^{(2)}(\alpha)-1)+\tilde{S}_2^{(2)}(\alpha)])}$$

then $\rho < 1$ is the condition to be satisfied for the existence of steady state for the model under consideration.

4.2 RESULT

Theorem:

$q_n = \sum_{i=0}^{n} K_i(\eta I_i + p_i), n = 0,1,2,...,a-1$, where

$K_n = \frac{h_n + \sum_{i=1}^{n}\gamma_i K_{n-i}}{1-\gamma_0}$, $n = 1,2,...,a-1$ with $K_0 = \frac{\gamma_0\beta_0}{1-\gamma_0}, h_n = \sum_{i=0}^{n}(\eta I_i + p_i)\beta_{n-i}$,

where γ_i's and β_i's are the probabilities of the $'i'$ customers arrive during vacation time and closedown time respectively.

Proof:

Using equations (73) and (74), $\sum_{l=1}^{\infty} Q_j(z,0)$ simplifies to

$$\sum_{n=0}^{\infty} q_n z^n = \tilde{V}(\lambda - \lambda X(z))[\tilde{C}(\lambda - \lambda X(z))(\eta\sum_{n=0}^{a-1} I_n z^n + \sum_{n=0}^{a-1} p_n z^n) + \sum_{n=0}^{a-1} q_n z^n]$$

$$= \sum_{n=0}^{\infty}\gamma_n z^n[\sum_{j=0}^{\infty}\beta_j z^j(\eta\sum_{n=0}^{a-1} I_n z^n + \sum_{n=0}^{a-1} p_n z^n) + \sum_{n=0}^{a-1} q_n z^n]$$

$$= \sum_{n=0}^{a-1}(\sum_{i=0}^{n}\gamma_{n-i}h_i)z^n + \sum_{n=a}^{\infty}[\sum_{k=0}^{n-(a-1)}\gamma_k\sum_{n=0}^{a-1}(\eta I_i + p_i)\beta_{n-i-k}]z^n$$

$$+ \sum_{n=a}^{\infty}\sum_{k=n-(a-2)}^{n}\gamma_k\sum_{i=0}^{a-k}\beta_i(I_{n-i-k} + p_{n-i-k})z^n + \sum_{n=0}^{a-1}(\sum_{i=0}^{n} q_i\gamma_{n-i})z^n + \sum_{n=a}^{\infty}(\sum_{i=0}^{a-1} q_i\gamma_{n-i})z^n$$

equating the coefficients of z^n on both sides of the above equation for $n = 0,1,2,...,a-1$, we have

$$q_n = \sum_{j=0}^{n}\sum_{i=0}^{n-j}\gamma_i\beta_{n-i-j}(\eta I_j + p_j) + \sum_{i=0}^{n}\gamma_{n-i}q_i \, .$$

On solving for q_n, we have

$$q_n = \frac{\sum_{j=0}^{n} h_{n-j}(\eta I_j + p_j) + \sum_{i=0}^{n-1}\gamma_{n-i}q_i}{(1-\gamma_0)}.$$

The co – efficient of $(\eta I_n + p_n)$ in q_n is $\frac{\gamma_0\beta_0}{1-\gamma_0} = K_0$ and

the co – efficient of $(\eta I_{n-1} + p_{n-1})$ in $q_n = \frac{[h_1 + \gamma_1.\text{co} - \text{efficiento f}(\eta I_{n-1} + p_{n-1})\text{in } q_{n-1}]}{(1-\gamma_0)}$.

$$= \frac{h_1 + \gamma_1.K_0}{(1-\gamma_0)} = K_1.$$

Similarly, we get

$$K_n = \frac{h_n + \sum_{i=1}^{n} \gamma_i K_{n-i}}{1-\gamma_0}, n = 1,2,...,a-1.$$

4.3 Particular Case

Case 1:

When there are no balking, breakdown, repair , Stand-by service, second stage service and closedown then equation (85) reduces to

$$P(z) = \frac{(\tilde{S}_e^{(1)}(u(z))-1)\sum_{i=a}^{b-1}(z^b - z^i)c_i + (z^b - 1)(\tilde{V}(w(z))-1)\sum_{n=0}^{a-1} c_n z^n}{(-w(z))(z^b - \tilde{S}_e^{(1)}(u(z)))}$$

which coincide with Arumuganathan and Jeyakumar [7], if the setup time is zero, $N = a$ and no closedown.

Case 2:

When there are no balking, breakdown, stand-by service and second stage service then equation (85) reduces to

$$P(z) = \frac{(\tilde{S}_e^{(1)}(u(z))-1)\sum_{i=a}^{b-1}(z^b - z^i)c_i + (z^b - 1)(\tilde{V}(w(z))-1)\sum_{n=0}^{a-1} p_n z^n + (\tilde{C}(w(z))\tilde{V}(w(z))-1)(z^b-1)\sum_{n=0}^{a-1} p_n z^n}{(-w(z))(z^b - \tilde{S}_e^{(1)}(u(z)))}$$

which coincide with Senthilnathan. B et.al [8], if no server breakdown.

Case 3:

When there are no balking, breakdown, stand-by service, closedown and a=b=N, then the equation (85) reduces to

$$P(z) = \frac{(z-1)(1-\tilde{V}(\lambda - \lambda X(z)))\sum_{n=0}^{N-1} c_n z^n}{(\lambda - \lambda X(z))[z - \tilde{S}_e^{(1)}(\lambda - \lambda X(z))\tilde{S}_e^{(2)}(\lambda - \lambda X(z))]}$$

which coincide with Senthil Kumar. M et.al [9], if no retrial.

V. Some Important Performance Measures

5.1 Expected Queue Length

The mean queue length $E(Q)$ at an arbitrary time epoch is obtained by differentiating $P(z)$ at $z = 1$ and is given by

$$E(Q) = \frac{N'''D'' - N''D'''}{3(D'')^2} \quad (86)$$

where

$$D'' = 2(-\lambda X1)K_3'(1),$$
$$D''' = 3[(-\lambda X2)K_3'(1) + (-\lambda X1)K_3''(1)]$$

$$N'' = 2K_1'(1)\sum_{i=a}^{b-1}(b-i)c_i + 2K_2'(1)\sum_{i=a}^{b-1}(b-i)d_i + B''(1)I(1) + 2B'(1)I'(1)$$
$$- K_1''(1)\sum_{n=0}^{a-1}p_n - 2K_1'(1)\sum_{n=0}^{a-1}np_n + (2.C1.A'(1) + A''(1))(\eta I(1) + \sum_{n=0}^{a-1}p_n)$$
$$+ 2.A'(1)(\eta I'(1) + \sum_{n=0}^{a-1}np_n) + 2.V1.(K_1'(1) - K_3'(1))\sum_{n=0}^{a-1}q_n$$

$$N''' = 3\sum_{i=a}^{b-1}[(b-i)(K_1''(1)c_i + K_2''(1)d_i) + (b(b-1) - i(i-1))[K_1'(1)c_i + K_2'(1)d_i]]$$
$$+ 3(K_1'(1) - K_3'(1))[2.V1.\sum_{n=0}^{a-1}nq_n + V2.\sum_{n=0}^{a-1}q_n] + 3.V1.(K_1''(1) - K_3''(1))\sum_{n=0}^{a-1}q_n$$
$$+ 3[(2.C1.A'(1) + A''(1))[\eta I'(1) + \sum_{n=0}^{a-1}np_n] + A'(1)[\eta I''(1) + \sum_{n=0}^{a-1}n(n-1)p_n]]$$
$$+ 3[B''(1)I'(1) + B'(1)I''(1)] - 3[K_1''(1)\sum_{n=0}^{a-1}np_n + K_1'(1)\sum_{n=0}^{a-1}n(n-1)p_n]$$
$$+ [A'''(1) + 3.(C2.A'(1) + C1.A''(1))](\eta I(1) + \sum_{n=0}^{a-1}p_n) + B'''(1)I(1) - K_1'''(1)\sum_{n=0}^{a-1}p_n$$

$$K_1'(1) = \alpha\eta\lambda X1(\alpha+\eta)((1-\tilde{S}_e^{(1)}(\alpha))(1-\tilde{S}_e^{(2)}(\alpha))(1-\tilde{S}_t^{(1)}(\eta))(1-\tilde{S}_t^{(2)}(\eta))$$
$$-[1-\tilde{S}_e^{(1)}(\alpha)\tilde{S}_e^{(2)}(\alpha)][1-\tilde{S}_t^{(1)}(\eta)\tilde{S}_t^{(2)}(\eta)])$$

$$K_1''(1) = 2b[K_1'(1) + \alpha\eta\lambda X1(\alpha\tilde{S}_e^{(1)}(\alpha)\tilde{S}_t^{(2)}(\eta)(1-\tilde{S}_t^{(1)}(\eta))(1-\tilde{S}_e^{(2)}(\alpha))$$
$$-(\alpha+\eta\tilde{S}_t^{(1)}(\eta)\tilde{S}_t^{(2)}(\eta)[1-\tilde{S}_e^{(1)}(\alpha)\tilde{S}_e^{(2)}(\alpha)]))]$$
$$+ 2\alpha\eta\lambda X1(\alpha+\eta)[(1-\tilde{S}_e^{(1)}(\alpha))(1-\tilde{S}_e^{(2)}(\alpha))((1-\tilde{S}_t^{(1)}(\eta))S_{t21} + (1-\tilde{S}_t^{(2)}(\eta))S_{t11})$$
$$-[1-\tilde{S}_e^{(1)}(\alpha)\tilde{S}_e^{(2)}(\alpha)][\tilde{S}_t^{(1)}(\eta) + \tilde{S}_t^{(2)}(\eta)] - [1-\tilde{S}_t^{(1)}(\eta)\tilde{S}_t^{(2)}(\eta)][\tilde{S}_e^{(1)}(\alpha) + \tilde{S}_e^{(2)}(\alpha)]$$
$$+ (1-\tilde{S}_t^{(1)}(\eta))(1-\tilde{S}_t^{(2)}(\eta))((1-\tilde{S}_e^{(1)}(\alpha))S_{e21} + (1-\tilde{S}_e^{(2)}(\alpha))S_{e11})]$$
$$+ 2(\pi\alpha+\eta)(\lambda X1)^2(\alpha+\eta)[1-\tilde{S}_e^{(1)}(\alpha)\tilde{S}_e^{(2)}(\alpha)][1-\tilde{S}_t^{(1)}(\eta)\tilde{S}_t^{(2)}(\eta)]$$
$$+ \frac{K_1'(1)}{\lambda X1(\alpha+\eta)}((\alpha+\eta)\lambda X2 - 2\pi(\lambda X1)^2)$$

$$K_1'''(1) = 3((\alpha+\eta)[2(\pi\alpha+\eta)(\lambda X1)^2 - \alpha\eta\lambda X2] + 2\pi\alpha\eta(\lambda X1)^2)\times$$
$$([S_{e11}\tilde{S}_e^{(2)}(\alpha) + S_{e21}\tilde{S}_e^{(1)}(\alpha)][1-\tilde{S}_t^{(1)}(\eta)\tilde{S}_t^{(2)}(\eta))$$
$$+[S_{t11}\tilde{S}_t^{(2)}(\eta) + S_{t21}\tilde{S}_t^{(1)}(\eta)][1-\tilde{S}_e^{(1)}(\alpha)\tilde{S}_e^{(2)}(\alpha)))$$
$$+3\alpha\eta[(\alpha+\eta)\lambda X2 - 2\pi(\lambda X1)^2]\times$$
$$((1-\tilde{S}_e^{(2)}(\alpha))(1-\tilde{S}_t^{(2)}(\eta))[S_{e11}(1-\tilde{S}_t^{(1)}(\eta)) + S_{t11}(1-\tilde{S}_e^{(1)}(\alpha))]$$
$$+(1-\tilde{S}_e^{(1)}(\alpha))(1-\tilde{S}_t^{(1)}(\eta))[S_{e21}(1-\tilde{S}_t^{(2)}(\eta)) + S_{t21}(1-\tilde{S}_e^{(2)}(\alpha))])$$
$$+3\alpha\eta(\alpha+\eta)\lambda X1(2S_{e21}S_{t21}[(1-\tilde{S}_e^{(1)}(\alpha))(1-\tilde{S}_t^{(1)}(\eta)) - \tilde{S}_e^{(1)}(\alpha)\tilde{S}_t^{(1)}(\eta)]$$
$$+2S_{t11}[(1-\tilde{S}_e^{(1)}(\alpha))(S_{e21}(1-\tilde{S}_t^{(2)}(\eta)) + S_{t21}(1-\tilde{S}_e^{(2)}(\alpha)))$$
$$-S_{e21}\tilde{S}_e^{(1)}(\alpha)\tilde{S}_t^{(2)}(\eta) - S_{t21}(1-\tilde{S}_e^{(1)}(\alpha)\tilde{S}_e^{(2)}(\alpha))]$$

$$
\begin{aligned}
&+2S_{e11}[(1-\tilde{S}_e^{(2)}(\alpha))(S_{t11}(1-\tilde{S}_t^{(2)}(\eta))+S_{t21}(1-\tilde{S}_t^{(1)}(\eta)))\\
&+S_{e21}[(1-\tilde{S}_t^{(1)}(\eta))(1-\tilde{S}_t^{(2)}(\eta))-(1-\tilde{S}_t^{(1)}(\eta)\tilde{S}_t^{(2)}(\eta))]\\
&-\tilde{S}_e^{(2)}(\alpha)(S_{t11}\tilde{S}_t^{(2)}(\eta)+S_{t21}\tilde{S}_t^{(1)}(\eta))]\\
&+(1-\tilde{S}_e^{(2)}(\alpha))(1-\tilde{S}_t^{(2)}(\eta))(S_{e12}(1-\tilde{S}_t^{(1)}(\eta))+S_{t12}(1-\tilde{S}_e^{(1)}(\alpha)))\\
&+(1-\tilde{S}_e^{(1)}(\alpha))(1-\tilde{S}_t^{(1)}(\eta))(S_{e22}(1-\tilde{S}_t^{(2)}(\eta))+S_{t22}(1-\tilde{S}_e^{(2)}(\alpha)))\\
&-(1-\tilde{S}_t^{(1)}(\eta)\tilde{S}_t^{(2)}(\eta))(S_{e12}\tilde{S}_e^{(2)}(\alpha)+S_{e22}\tilde{S}_e^{(1)}(\alpha))\\
&-(1-\tilde{S}_e^{(1)}(\alpha)\tilde{S}_e^{(2)}(\alpha))(S_{t12}\tilde{S}_t^{(2)}(\eta)+S_{t22}\tilde{S}_t^{(1)}(\eta))\\
&+2b(1-\tilde{S}_e^{(1)}(\alpha))(1-\tilde{S}_t^{(1)}(\eta))[S_{e21}(1-\tilde{S}_t^{(2)}(\eta))+S_{t21}(1-\tilde{S}_e^{(2)}(\alpha))]\\
&+2b(1-\tilde{S}_e^{(2)}(\alpha))(1-\tilde{S}_t^{(2)}(\eta))[S_{e11}(1-\tilde{S}_t^{(1)}(\eta))+S_{t11}(1-\tilde{S}_e^{(1)}(\alpha))])\\
&-6b\alpha\eta\lambda X1\times(\eta[S_{e11}\tilde{S}_e^{(2)}(\alpha)+S_{e21}\tilde{S}_e^{(1)}(\alpha)]\\
&+\alpha[(1-\tilde{S}_t^{(2)}(\eta))(S_{e11}\tilde{S}_e^{(2)}(\alpha)+S_{e21}\tilde{S}_e^{(1)}(\alpha))+S_{t11}\tilde{S}_t^{(2)}(\eta)(1-\tilde{S}_e^{(1)}(\alpha))\\
&+(1-\tilde{S}_t^{(1)}(\eta))(S_{e11}\tilde{S}_t^{(2)}(\eta)+S_{t21}\tilde{S}_e^{(1)}(\alpha))+S_{t21}(\tilde{S}_t^{(1)}(\eta)-\tilde{S}_e^{(1)}(\alpha)\tilde{S}_e^{(2)}(\alpha))])\\
&+\alpha\eta(1-\tilde{S}_e^{(1)}(\alpha))(1-\tilde{S}_e^{(2)}(\alpha))(1-\tilde{S}_t^{(1)}(\eta))(1-\tilde{S}_t^{(2)}(\eta))\times\\
&(3b(b-1)\lambda X1(\alpha+\eta)-[6\pi\lambda^2X1.X2-(\alpha+\eta)\lambda X3]\\
&-3b[2\pi(\lambda X1)^2-(\alpha+\eta)\lambda X2])+6b\alpha\eta\pi(\lambda X1)^2(1-\tilde{S}_e^{(1)}(\alpha)\tilde{S}_e^{(2)}(\alpha))\\
&+3b[2(\pi\alpha+\eta)(\lambda X1)^2-\alpha\eta\lambda X2-(b-1)\alpha\eta\lambda X1]\times\\
&[(\alpha+\eta)(1-\tilde{S}_e^{(1)}(\alpha)\tilde{S}_e^{(2)}(\alpha))+\alpha\tilde{S}_t^{(2)}(\eta)[\tilde{S}_t^{(1)}(\eta)(\tilde{S}_e^{(1)}(\alpha)-1)+\tilde{S}_e^{(1)}(\alpha)(\tilde{S}_e^{(2)}(\alpha)-1)]]\\
&+(1-\tilde{S}_e^{(1)}(\alpha)\tilde{S}_e^{(2)}(\alpha))(1-\tilde{S}_t^{(1)}(\eta)\tilde{S}_t^{(2)}(\eta))(6\pi\lambda X1[\alpha\eta\lambda X2-(\pi\alpha+\eta)(\lambda X1)^2]\\
&+(\alpha+\eta)[6(\pi\alpha+\eta)\lambda^2X1X2-\pi(\lambda X1)^3-\alpha\eta\lambda X3])
\end{aligned}
$$

$$K_2'(1)=\alpha\eta\lambda X1(\alpha+\eta)((1-\tilde{S}_e^{(1)}(\alpha))(1-\tilde{S}_e^{(2)}(\alpha))(1-\tilde{S}_t^{(1)}(\eta))(1-\tilde{S}_t^{(2)}(\eta))$$

$$
\begin{aligned}
K_2''(1)=&[2b+\frac{(\alpha+\eta)\lambda X2-2(\lambda X1)^2}{(\alpha+\eta)\lambda X1}]K_2'(1)\\
&+2(\pi\alpha+\eta)(\lambda X1)^2(\alpha+\eta)[1-\tilde{S}_e^{(1)}(\alpha)\tilde{S}_e^{(2)}(\alpha)][1-\tilde{S}_t^{(1)}(\eta)\tilde{S}_t^{(2)}(\eta)]\\
&+2b\alpha\eta\lambda X1(\eta\tilde{S}_t^{(1)}(\eta)\tilde{S}_e^{(2)}(\alpha)(1-\tilde{S}_e^{(1)}(\alpha))(1-\tilde{S}_t^{(2)}(\eta))\\
&-\alpha\tilde{S}_e^{(1)}(\alpha)\tilde{S}_e^{(2)}(\alpha)[1-\tilde{S}_t^{(1)}(\eta)\tilde{S}_t^{(2)}(\eta)]\\
&-(\alpha+\eta)[(1-\tilde{S}_e^{(1)}(\alpha)\tilde{S}_e^{(2)}(\alpha))(S_{t11}\tilde{S}_t^{(2)}(\eta)+S_{t21}\tilde{S}_t^{(1)}(\eta))\\
&+(1-\tilde{S}_t^{(1)}(\eta)\tilde{S}_t^{(2)}(\eta))(S_{e11}\tilde{S}_e^{(2)}(\alpha)+S_{e21}\tilde{S}_e^{(1)}(\alpha))]\\
&+[(\alpha+\eta)-\lambda X1]((1-\tilde{S}_e^{(1)}(\alpha))(1-\tilde{S}_t^{(1)}(\eta))\\
&[S_{e21}(1-\tilde{S}_t^{(2)}(\eta))+S_{t21}(1-\tilde{S}_e^{(2)}(\alpha))]\\
&+(1-\tilde{S}_e^{(2)}(\alpha))(1-\tilde{S}_t^{(2)}(\eta))\\
&[S_{e11}(1-\tilde{S}_t^{(1)}(\eta))+S_{t11}(1-\tilde{S}_e^{(1)}(\alpha))]))
\end{aligned}
$$

$$
\begin{aligned}
K_3'(1)=&\frac{(\pi\alpha+\eta)K_1'(1)}{(\alpha+\eta)}+b\alpha^2\eta^2(\tilde{S}_e^{(1)}(\alpha)[\tilde{S}_e^{(2)}(\alpha)(1-\tilde{S}_t^{(1)}(\eta)-\tilde{S}_t^{(2)}(\eta))+\tilde{S}_t^{(2)}(\eta)]\\
&+\tilde{S}_t^{(1)}(\eta)[\tilde{S}_t^{(2)}(\eta)(1-\tilde{S}_e^{(1)}(\alpha)-\tilde{S}_e^{(2)}(\alpha))+\tilde{S}_e^{(2)}(\alpha)])
\end{aligned}
$$

$$
\begin{aligned}
K_2'''(1) &= 3((\alpha+\eta)[2(\pi\alpha+\eta)(\lambda X1)^2-\alpha\eta\lambda X2]+2\alpha\eta(\lambda X1)^2)\times \\
&([S_{e11}\tilde{S}_e^{(2)}(\alpha)+S_{e21}\tilde{S}_e^{(1)}(\alpha)](1-\tilde{S}_t^{(1)}(\eta)\tilde{S}_t^{(2)}(\eta)) \\
&+[S_{t11}\tilde{S}_t^{(2)}(\eta)+S_{t21}\tilde{S}_t^{(1)}(\eta)](1-\tilde{S}_e^{(1)}(\alpha)\tilde{S}_e^{(2)}(\alpha))) \\
&+3\alpha\eta[(\alpha+\eta)\lambda X2-2(\lambda X1)^2]\times \\
&((1-\tilde{S}_e^{(2)}(\alpha))(1-\tilde{S}_t^{(2)}(\eta))[S_{e11}(1-\tilde{S}_t^{(1)}(\eta))+S_{t11}(1-\tilde{S}_e^{(1)}(\alpha))] \\
&+(1-\tilde{S}_e^{(1)}(\alpha))(1-\tilde{S}_t^{(1)}(\eta))[S_{e21}(1-\tilde{S}_t^{(2)}(\eta))+S_{t21}(1-\tilde{S}_e^{(2)}(\alpha))]) \\
&+3\alpha\eta(\alpha+\eta)\lambda X1[2S_{e21}S_{t21}[(1-\tilde{S}_e^{(1)}(\alpha))(1-\tilde{S}_t^{(1)}(\eta))-\tilde{S}_e^{(1)}(\alpha)\tilde{S}_t^{(1)}(\eta)] \\
&+2S_{t11}[(1-\tilde{S}_e^{(1)}(\alpha))(S_{e21}(1-\tilde{S}_t^{(2)}(\eta))+S_{t21}(1-\tilde{S}_e^{(2)}(\alpha))) \\
&-S_{e21}\tilde{S}_e^{(1)}(\alpha)\tilde{S}_t^{(2)}(\eta)-S_{t21}(1-\tilde{S}_e^{(1)}(\alpha)\tilde{S}_e^{(2)}(\alpha))] \\
&+2S_{e11}[(1-\tilde{S}_e^{(2)}(\alpha))(S_{t11}(1-\tilde{S}_t^{(2)}(\eta))+S_{t21}(1-\tilde{S}_t^{(1)}(\eta))) \\
&+S_{e21}[(1-\tilde{S}_t^{(1)}(\eta))(1-\tilde{S}_t^{(2)}(\eta))-(1-\tilde{S}_t^{(1)}(\eta)\tilde{S}_t^{(2)}(\eta))] \\
&-\tilde{S}_e^{(2)}(\alpha)(S_{t11}\tilde{S}_t^{(2)}(\eta)+S_{t21}\tilde{S}_t^{(1)}(\eta))] \\
&+(1-\tilde{S}_e^{(2)}(\alpha))(1-\tilde{S}_t^{(2)}(\eta))(S_{e12}(1-\tilde{S}_t^{(1)}(\eta))+S_{t12}(1-\tilde{S}_e^{(1)}(\alpha))) \\
&+(1-\tilde{S}_e^{(1)}(\alpha))(1-\tilde{S}_t^{(1)}(\eta))(S_{e22}(1-\tilde{S}_t^{(2)}(\eta))+S_{t22}(1-\tilde{S}_e^{(2)}(\alpha))) \\
&-(1-\tilde{S}_t^{(1)}(\eta)\tilde{S}_t^{(2)}(\eta))(S_{e12}\tilde{S}_e^{(2)}(\alpha)+S_{e22}\tilde{S}_e^{(1)}(\alpha)) \\
&-(1-\tilde{S}_e^{(1)}(\alpha)\tilde{S}_e^{(2)}(\alpha))(S_{t12}\tilde{S}_t^{(2)}(\eta)+S_{t22}\tilde{S}_t^{(1)}(\eta)) \\
&+(1-\tilde{S}_e^{(2)}(\alpha))(1-\tilde{S}_t^{(2)}(\eta))((1-\tilde{S}_t^{(1)}(\eta))(2bS_{e11}+S_{e12})+(1-\tilde{S}_e^{(1)}(\alpha))(2bS_{t11}+S_{t12})) \\
&+(1-\tilde{S}_e^{(1)}(\alpha))(1-\tilde{S}_t^{(1)}(\eta))((1-\tilde{S}_t^{(2)}(\eta))(2bS_{e21}+S_{e22})+(1-\tilde{S}_e^{(2)}(\alpha))(2bS_{t21}+S_{t22})) \\
&-(1-\tilde{S}_e^{(1)}(\alpha)\tilde{S}_e^{(2)}(\alpha))[S_{t12}\tilde{S}_t^{(2)}(\eta)+S_{t22}\tilde{S}_t^{(1)}(\eta)] \\
&-(1-\tilde{S}_t^{(1)}(\eta)\tilde{S}_t^{(2)}(\eta))[S_{e12}\tilde{S}_e^{(2)}(\alpha)+S_{e22}\tilde{S}_e^{(1)}(\alpha)]] \\
&-6b\alpha\eta\lambda X1\times(\alpha[S_{t11}\tilde{S}_t^{(2)}(\eta)+S_{t21}\tilde{S}_t^{(1)}(\eta)]+\eta S_{e21}\tilde{S}_e^{(1)}(\alpha) \\
&+\eta[(1-\tilde{S}_e^{(2)}(\alpha))(S_{t11}\tilde{S}_t^{(2)}(\eta)+S_{t21}\tilde{S}_t^{(1)}(\eta))+(1-\tilde{S}_e^{(1)}(\alpha))(S_{t11}\tilde{S}_e^{(2)}(\alpha)+S_{e21}\tilde{S}_t^{(1)}(\eta)) \\
&+S_{e11}\tilde{S}_e^{(2)}(\alpha)(1-\tilde{S}_t^{(1)}(\eta))-S_{e21}\tilde{S}_t^{(1)}(\eta)\tilde{S}_t^{(2)}(\eta)]+6b\alpha\eta(\lambda X1)^2(1-\tilde{S}_t^{(1)}(\eta)\tilde{S}_t^{(2)}(\eta)) \\
&+\alpha\eta(1-\tilde{S}_e^{(1)}(\alpha))(1-\tilde{S}_e^{(2)}(\alpha))(1-\tilde{S}_t^{(1)}(\eta))(1-\tilde{S}_t^{(2)}(\eta))\times \\
&(3b(b-1)\lambda X1(\alpha+\eta)-[6\lambda^2X1.X2-(\alpha+\eta)\lambda X3]-3b[(\lambda X1)^2-(\alpha+\eta)\lambda X2]) \\
&+3b[2(\pi\alpha+\eta)(\lambda X1)^2-\alpha\eta\lambda X2-(b-1)\alpha\eta\lambda X1]\times \\
&[(\alpha+\eta)(1-\tilde{S}_t^{(1)}(\eta)\tilde{S}_t^{(2)}(\eta))+\alpha\tilde{S}_e^{(2)}(\alpha)[\tilde{S}_e^{(1)}(\alpha)(\tilde{S}_t^{(1)}(\eta)-1)+\tilde{S}_t^{(1)}(\eta)(\tilde{S}_t^{(2)}(\eta)-1)]] \\
&+(1-\tilde{S}_e^{(1)}(\alpha)\tilde{S}_e^{(2)}(\alpha))(1-\tilde{S}_t^{(1)}(\eta)\tilde{S}_t^{(2)}(\eta))(6\lambda X1[\alpha\eta\lambda X2-(\pi\alpha+\eta)(\lambda X1)^2] \\
&+(\alpha+\eta)[6(\pi\alpha+\eta)\lambda^2X1X2-6\pi(\lambda X1)^3-\alpha\eta\lambda X3])
\end{aligned}
$$

$$
\begin{aligned}
K_3''(1) &= 2b\alpha^2\eta^2((S_{e11}(\tilde{S}_t^{(1)}(\eta)-1)+S_{t11}(\tilde{S}_e^{(1)}(\alpha)-1))(\tilde{S}_e^{(2)}(\alpha)+\tilde{S}_t^{(2)}(\eta)) \\
&+(S_{e21}(\tilde{S}_t^{(2)}(\eta)-1)+S_{t21}(\tilde{S}_e^{(2)}(\alpha)-1))(\tilde{S}_e^{(1)}(\alpha)+\tilde{S}_t^{(1)}(\eta)) \\
&+(S_{e11}+S_{t11})(\tilde{S}_e^{(2)}(\alpha)\tilde{S}_t^{(2)}(\eta))+(S_{e21}+S_{t21})(\tilde{S}_e^{(1)}(\alpha)\tilde{S}_t^{(1)}(\eta))) \\
&+(2b\alpha\eta\lambda X1(\pi\alpha+\eta)+b(b-1)\alpha^2\eta^2)\times \\
&(\tilde{S}_e^{(1)}(\alpha)[\tilde{S}_e^{(2)}(\alpha)+\tilde{S}_t^{(2)}(\eta)(1-\tilde{S}_t^{(1)}(\eta))(1-\tilde{S}_e^{(2)}(\alpha))] \\
&\tilde{S}_t^{(1)}(\eta)[\tilde{S}_t^{(2)}(\eta)+\tilde{S}_e^{(2)}(\alpha)(1-\tilde{S}_e^{(1)}(\alpha))(1-\tilde{S}_t^{(2)}(\eta))] \\
&+2[(1-\tilde{S}_e^{(1)}(\alpha))(1-\tilde{S}_t^{(1)}(\eta))-1][(1-\tilde{S}_e^{(2)}(\alpha))(1-\tilde{S}_t^{(2)}(\eta))-1]) \\
&+2\alpha\eta\lambda X1(\pi\alpha+\eta)(b(1-\tilde{S}_e^{(1)}(\alpha))(1-\tilde{S}_e^{(2)}(\alpha))(1-\tilde{S}_t^{(1)}(\eta))(1-\tilde{S}_t^{(2)}(\eta)) \\
&+b(\tilde{S}_e^{(1)}(\alpha)\tilde{S}_e^{(2)}(\alpha)-1)+b(\tilde{S}_t^{(1)}(\eta)\tilde{S}_t^{(2)}(\eta)-1) \\
&+(1-\tilde{S}_t^{(1)}(\eta))[(1-\tilde{S}_e^{(2)}(\alpha))(S_{e11}+S_{t21})-S_{e11}\tilde{S}_t^{(2)}(\eta)-S_{e21}\tilde{S}_e^{(1)}(\alpha)] \\
&+(1-\tilde{S}_t^{(2)}(\eta))[(1-\tilde{S}_e^{(1)}(\alpha))(S_{e21}+S_{t11})-S_{e11}\tilde{S}_e^{(2)}(\alpha)-S_{e21}\tilde{S}_t^{(1)}(\eta)] \\
&+(\tilde{S}_e^{(1)}(\alpha)-1)(S_{t11}\tilde{S}_e^{(2)}(\alpha)+S_{t21}\tilde{S}_t^{(1)}(\eta))
\end{aligned}
$$

$$+(\tilde{S}_e^{(2)}(\alpha)-1)(S_{t11}\tilde{S}_t^{(2)}(\eta)+S_{t21}\tilde{S}_e^{(1)}(\alpha)))$$
$$+2(\lambda X1)^2(2\pi\alpha\eta[1+\tilde{S}_t^{(1)}(\eta)\tilde{S}_t^{(2)}(\eta)\tilde{S}_e^{(1)}(\alpha)\tilde{S}_e^{(2)}(\alpha)]$$
$$+(\pi^2\alpha^2+\eta^2)(1-\tilde{S}_e^{(1)}(\alpha)\tilde{S}_e^{(2)}(\alpha))(1-\tilde{S}_t^{(1)}(\eta)\tilde{S}_t^{(2)}(\eta)))$$
$$+\alpha\eta[2\pi(\lambda X1)^2-\lambda X2(\pi\alpha+\eta)]((1+\tilde{S}_e^{(1)}(\alpha)\tilde{S}_e^{(2)}(\alpha))(\tilde{S}_t^{(1)}(\eta)+\tilde{S}_t^{(2)}(\eta))$$
$$+(\tilde{S}_e^{(1)}(\alpha)+\tilde{S}_e^{(2)}(\alpha))(1-\tilde{S}_t^{(1)}(\eta))(1-\tilde{S}_t^{(2)}(\eta)))$$
$$-[1-\tilde{S}_e^{(1)}(\alpha)\tilde{S}_e^{(2)}(\alpha)][1-\tilde{S}_t^{(1)}(\eta)\tilde{S}_t^{(2)}(\eta)]),$$

$$A'(1)=K_1'(1),\quad A''(1)=2.V1.(K_1'(1)-K_3'(1))+K_1''(1)$$
$$A'''(1)=3[V2.(K_1'(1)-K_3'(1))+V1.(K_1''(1)-K_3''(1))]+K_1'''(1)$$
$$B'(1)=-\eta.K_2'(1),\quad B''(1)=2.\lambda.X1.(K_2'(1)-K_3'(1))-\eta.K_2''(1)$$
$$B'''(1)=3[\lambda.X2.(K_2'(1)-K_3'(1))+\lambda.X1.(K_2''(1)-K_3''(1))]-\eta.K_2'''(1)$$
$$X1=E(X),\ X2=E(X^2),\ C1=\lambda.X1.E(C),\ V1=\lambda.X1.E(V)$$
$$C2=\lambda.X2.E(C)+(\lambda.X1)^2.E(C^2),\ \ V2=\lambda.X2.E(V)+(\lambda.X1)^2.E(V^2)$$
$$S_{e11}=(\lambda X1)\tilde{S}_e^{(1)'}(\alpha),\ S_{t11}=(\lambda X1)\tilde{S}_t^{(1)'}(\eta)\ S_{e21}=(\lambda X1)\tilde{S}_e^{(2)'}(\alpha),\ S_{t21}=(\lambda X1)\tilde{S}_t^{(2)'}(\eta),$$
$$S_{e12}=-(\lambda X1)^2\tilde{S}_e^{(1)''}(\alpha)-(-\lambda X2)\tilde{S}_e^{(1)'}(\alpha),\ S_{t12}=-(\lambda X1)^2\tilde{S}_t^{(1)''}(\eta)-(-\lambda X2)\tilde{S}_t^{(1)'}(\eta)$$
$$S_{e22}=-(\lambda X1)^2\tilde{S}_e^{(2)''}(\alpha)-(-\lambda X2)\tilde{S}_e^{(2)'}(\alpha),\ S_{t22}=-(\lambda X1)^2\tilde{S}_t^{(2)''}(\eta)-(-\lambda X2)\tilde{S}_t^{(2)'}(\eta)$$

5.2 Expected Length of Idle Period

Let I be the idle period random variable, then the expected length of idle period is given by $E(I)=E(C)+E(I_1)$, where $E(I_1)$ is the idle period due to multiple vacation process, E(C) is the expected closedown time.
Define a random variable Y as

$$Y=\begin{cases}0 & \text{if the server finds atleast 'a' customers after the first vacation}\\ 1 & \text{if the server finds less than 'a' customers after the first vacation.}\end{cases}$$

Now,

$$E(I_1)=E(I_1/Y=0)P(Y=0)+E(I_1/Y=1)P(Y=1)$$
$$=E(V)P(Y=0)+(E(V)+E(I_1))P(Y=1).$$

Solving for $E(I_1)$, we, have

$$E(I_1)=\frac{E(V)}{(1-P(Y=1))}=\frac{E(V)}{\left(1-\sum_{n=0}^{a-1}\sum_{i=0}^{n}\left[\sum_{j=0}^{n-i}\left[\gamma_j\beta_{n-i-j}\right]\right](\eta I_i+p_i)\right)}$$

where $\gamma_i's$ and $\beta_i's$ are the probabilities that i customers arrive during vacation and closedown time respectively.

5.3 Expected Length of Idle Period When The Main Server Under Repair

Let M be the random variable for idle period. Let $\tau_n,\ n=0,1,2,...,a-1,$ is the probability that the system state (number of customers in the system) visits $'n'$ during an idle period.

Let

$$M_n = \begin{cases} 1 & \text{if the state n is visited during an idle period} \\ 0 & \text{otherwise.} \end{cases}$$

Conditioning on the first arrival size, we have

$$Pr(M_n = 1) = g_n + \sum_{k=1}^{n-1} g_k Pr(M_{n-k} = 1),$$

where, $Pr(M_n = 1) = \tau_n$ and $\tau_0 = 1$, we have

$$\tau_n = \left(\frac{\lambda}{\lambda+\eta}\right)\left[\sum_{l=0}^{n}\sum_{m=a}^{b} B_{m,l}^{(2)}(0) \sum_{p=0}^{n-l} \prod_{\sum_{h=1}^{k} i_h j_h = p} (g_{i_1}^{j_1} g_{i_2}^{j_2} g_{i_3}^{j_3} \ldots g_{i_k}^{j_k}) \left(\frac{\lambda}{\lambda+\eta}\right)^{\sum_{h=1}^{k} j_h} - n\sum_{m=a}^{b} B_{m,0}^{(2)}(0)\right].$$

Thus the expected length of the idle period is obtained as

$$E(M) = \frac{1}{\lambda}\sum_{n=0}^{a-1}\tau_n$$

5.4 Expected Waiting Time

The expected waiting time is obtained by using the Little's formula as;

$$E(W) = \frac{E(Q)}{\lambda E(X)}$$

where $E(Q)$ is given in equation (86).

VI. Numerical Example

A numerical example of our model is analysed for a particular model with the following assumptions:

1 Batch size distribution of the arrival is geometric with mean 2.
2 Service time distribution is 2-Erlang (Both main server and stand-by server).
3 Vacation time and Closedown time are exponential with parameters $\gamma = 10$ and $\beta = 7$ respectively.
4 Balking probability $\pi = 0.2$
5 Let m1, m2 be the service rate for first, second stage service by the main server respectively.
6 Let n1, n2 be the service rate for first, second stage service by the stand-by server respectively.

The unknown probabilities of the queue size distribution are computed using numerical techniques. Using Matlab, the zeros of the function $K_3(z)$ are obtained and simultaneous equations are solved.

The expected queue length $E(Q)$ and the expected waiting time $E(W)$ are calculated for various arrival rate and service rate and the results are tabulated.

From the **Tables 1 to 3** the following observations are made.

(1) As arrival rate λ increases, the expected queue size and expected waiting time are increases.

(2) When the main server service rate increases, the expected queue size and expected waiting time are decreases.

(3) When the stand-by server service rate increases, the expected queue size and expected waiting time are decreases.

TABLE 1: Arrival rate vs expected queue length and expected waiting time

$a = 2, b = 5, m1 = m2 = 7, n1 = n2 = 4, \alpha = 3, \eta = 1$ and $\pi = 0.2$

λ	ρ	$E(Q)$	$E(W)$
3	0.2180	0.002675	0.0004459
4	0.2906	0.02402	0.003002
5	0.3633	0.0506	0.00506

TABLE 2: Main server service rate vs expected queue length and expected waiting time
$a = 2, b = 5, \lambda = 4, n1 = n2 = 4, \alpha = 3, \eta = 1, \pi = 0.2$ and $m1 = m2 = m(say)$

m	ρ	$E(Q)$	$E(W)$
5	0.3273	0.02296	0.00287
6	0.3079	0.02164	0.002705
7	0.2906	0.01928	0.00241
8	0.2751	0.01821	0.002276
9	0.2612	0.01762	0.002202
10	0.2486	0.01625	0.002044

TABLE 3: Stand-by server service rate vs expected queue length and expected waiting time
$a = 2, b = 5, \lambda = 4, m1 = m2 = 7, \alpha = 3, \eta = 1, \pi = 0.2$ and $n1 = n2 = n(say)$

n	ρ	$E(Q)$	$E(W)$
6	0.2169	0.7149	0.08936
7	0.1925	0.5565	0.06956
8	0.1730	0.3342	0.04177
9	0.1571	0.2155	0.02693
10	0.1439	0.1345	0.01681

VII. Conclusion

In this paper, the behaviour of the main server's breakdown and repair in an $M^{[X]}/G(a,b)/1$ queueing system with multiple vacation, closedown, balking and a stand-by is analysed. Probability generating function of queue size distribution at an arbitrary time is obtained. Some important performance measures are obtained. Particular case of the model is also presented. From the numerical results, it is observed that due to server breakdown, the arrival rate increase then the expected queue length and waiting time of the customers are increases. It is also observed that if the service rate increase, then the expected queue length and expected waiting time decreases.

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