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Totally geodesic submanifolds of (k, μ)- contact manifold

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***Abstract**: In this paper we study invariant submanifolds of (k, μ) -contact manifold.Here we investigate the conditions for invariant submanifolds of (k,μ)-contact manifold satisfying $Q(\sigma,R) = 0$, $Q(S,\sigma) = 0$ and $Q(\sigma,C) = 0$ to betotally geodesic, where S, R, C are the Ricci tensor, curvature tensor and concircularcurvature tensor respectively and σ is the second fundamental form.*
***Keywords:** Invariant submanifold, (k, μ) - contact manifold, totally geodesic.*

I. INTRODUCTION

The study of invariant submanifold of (k,μ)-contact manifold was initiated by Mukut ManiTripathi et al., [17]. They proved that, an odd dimensional invariant submanifold of a (k,μ)-contact manifold is a submanifold for which the structure tensor field ϕ maps tangent vectorsinto tangent vectors. This submanifold inherits a contact metric structure from the ambientspace and it is, in fact, a (k,μ)- contact manifold.

In general, an invariant submanifold of a Sasakian manifold is not totally geodesic. Forexample the circle bundle (S, Q_n) over an n-dimensional complex projective space $CP^{(n+1)}$isan invariant submanifold of a $(2n + 3)$-dimensional Sasakian space form with $c > -3$, whichis not totally geodesic [19]. Kon studied invariant submanifold of Sasakian manifold andobtained the well-known result that an invariant submanifold of a Sasakian manifold is totallygeodesic, provided that the second fundamental form of the immersion is covariantly constant[9]. Generalizing this Kon's result, the authors of [17] proved that if the second fundamentalform of an invariant submanifold in a (k,μ)-contact manifold is covariantly constant, theneither $k = 0$ or the submanifold is totally geodesic.

The authors Montano et al [11] have studied invariant submanifold of (k,μ)-contactmanifold and obtained the main result that every invariant submanifold of a non-Sasakian(k,μ)-contact manifold is totally geodesic, Conversely, every totally geodesic submanifold ofa non-Sasakian (k,μ)-contact manifold, with $\mu \neq 0$, and characteristic vector field is tangentto the submanifold is invariant. Recently, the authors of [2] and [14] find the necessaryand sufficient conditions for an invariant submanifold of a (k,μ)-contact manifold to betotally geodesic, when the second fundamental form is recurrent, 2-recurrent, generalized 2-recurrent, and when the submanifold is semiparallel, pseudoparallel, 2-pseudoparallel, Ricci-generalizedpseudoparallel, 2-Ricci-generalized pseudoparallel. Also in [7], the authors studiedinvariant submanifolds of Kenmotsu manifold satisfying $Q(\sigma,R) = 0$ and $Q(S,\sigma) = 0$. It isseen that invariant submanifolds of various types of contact manifolds have been studied byseveral authors like [1, 7, 9, 12, 15, 20].

Motivated by these works, in the present paper we consider invariant submanifold of (k,μ)-contact manifold satisfying $Q(\sigma,R) = 0$, $Q(S,\sigma) = 0$and $Q(\sigma,C) = 0$, where S, R and C arethe Ricci tensor, curvature tensor and concircular curvature tensor respectively and σ is thesecond fundamental form.
The paper is organized as follows:

In section 2, we give necessary details about submanifolds and the concircular curvaturetensor. In section 3, we recall the notion of (k,μ)-contact manifold and the related results.In section 4, we define invariant submanifold of (k,μ)-contactmanifold and review some basic results. Sections 5, 6, 7 deals with the study of invariant submanifolds of (k,μ)-contact manifold satisfying $Q(\sigma,R) = 0$, $Q(S,\sigma) = 0$and$Q(\sigma,C) = 0$, where S, R, C are the Ricci tensor, curvature tensor andconcircular curvature tensor respectively.

II. PRELIMINARIES

Let$M$ be an n-dimensional submanifold immersed in a m-dimensional Riemannian manifold$\widetilde{M}$, we denote by the same symbol g the induced metric on M. Let TMbe the set of all vector fields tangent to M and $T^{\perp}M$ is the set of all vector fields normal to M. Then Gauss and Weingarten formulae are given by [6]

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \qquad (2.1)$$
$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \qquad (2.2)$$

for all vector fields X, Y tangent to M and normal vector field N on M, where $\nabla$is theRiemannianconnection on M determined by the induced metric g and $\nabla^{\perp}$is the normalconnection on $T^{\perp}M$ of M. The second fundamental form σ and A_N are related by

$$\tilde{g}(\sigma(X,Y),N) = g(A_N X, Y).$$

If $\sigma = 0$ then the manifold is said to be totally geodesic.Now for a $(0, k)$-tensor T, $k \geq 1$ and a $(0,2)$-tensor B, $Q(B, T)$ is defined by [18]

$$Q(B,T)(X_1, X_2, \cdots, X_k; X, Y) = -T\big((X \wedge_B Y)X_1, X_2, \cdots, X_k\big) - -T(X_1, (X \wedge_B Y)X_2, \cdots, X_k) -T(X_1, X_2, \cdots, (X \wedge_B Y)X_k), \tag{2.3}$$

where $X \wedge_B Y$ is defined by

$$(X \wedge_B Y)Z = B(Y, Z)X - B(X, Z)Y. \tag{2.4}$$

For an n-dimensional, $(n \geq 3)$, Riemannian manifold (M, g), the concircular curvature tensor C of M is defined by [19]

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{2.5}$$

for all vector fields X, Y and Z on M, where r is the scalar curvature of M.

III. (k, μ) −CONTACT MANIFOLD

A manifold M^n (n-odd) is said to be a contact manifold if it is equipped with a global 1-form η such that $\eta \wedge (d\eta)^{(n-1)/2}$ everywhere on M^n. For a contact formη, it is well known that there exists a vector field ξ, called the characteristic vector field of η, such that $\eta(\xi) = 1$ and $d\eta(X, \xi) = 0$ for any vector field X on M^n. A Riemannian metric g is said to be associated metric if there exists a tensor field ϕ of type (1,1) such that

$$d\eta(X,Y) = g(X, \phi Y), \quad \eta(X) = g(X, \xi), \tag{3.1}$$

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta(X) = g(X, \xi), \tag{3.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(X, \phi Y) = -g(\phi X, \phi Y), \tag{3.3}$$

for all vector fields X, Y onM^n. The manifold equipped with a contact metric structure is called a contact metric manifold [4].

Given a contact metric manifold$M^n(\phi, \xi, \eta, g)$, we define a (1,1) tensor field h by$h = \frac{1}{2}\pounds_\xi \phi$, where £ denotes the Lie differentiation. Then h is symmetric and satisfies $h\phi = -\phi h$. Hence, if λ is an eigen value of h with eigen vector X, $-\lambda$ is also an eigen value with eigen vector ϕX. Also, we have $Tr \cdot h = Tr \cdot \phi h = 0$ and$h\xi = 0$. Moreover, if ∇ denotes the Riemannian connection of g, then the following relation holds:

$$\nabla_X \xi = -\phi X - \phi h X. \tag{3.4}$$

A contact metric manifold is Sasakian if and only if the relation $R(X,Y)\xi = \eta(Y)X - \eta(X)Y$ holds for allX, Y, where R denotes the curvature tensor of the manifold. It is well known that there exists contact metric manifolds for which the curvature tensor R and the direction of the characteristic vector field ξ satisfy $R(X,Y)\xi = 0$ for every vector fields X and Y.

As a generalization of both $R(X,Y)\xi = 0$ and the Sasakian case: Blair, Koufogiorgos and Papantoniou introduced the notion of (k, μ)-nullity distribution and is defined by

$$N(k,\mu): p \to N_p(k,\mu) = \{W \in T_pM | R(X,Y)W = (kI + \mu h)[g(Y,W)X - g(X,W)Y]\}$$

for all$X, Y \in TM$, where$(k, \mu) \in R^2$.

A contact metric manifold M^n with $\xi \in N(k, \mu)$ is called a (k, μ)-contact metric manifold. Then, we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{3.5}$$

In a (k, μ)-contact metric manifold the following relations hold:

$$h^2 = (k-1)\phi^2, k \leq 1, \tag{3.6}$$

$$(\nabla_X \phi)Y = g(X + hX, Y) - \eta(Y)(X + hX), \tag{3.7}$$

$$S(X, \xi) = (n-1)k\eta(X), \tag{3.8}$$

$$r = (n-1)(n - 3 + k - \left(\frac{(n-1)}{2}\right)\mu), \tag{3.9}$$

where S is the Ricci tensor of type(0,2), Q is the Ricci operator, i.e., $g(QX, Y)$and r is the scalar curvature of the (k, μ)-contact manifold have been studied by several authors such as [5, 8, 13, 16] and many others.
From (2.5), we have

$$C(X,Y)\xi = \left(k - \frac{r}{n(n-1)}\right)[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{3.10}$$

IV. INVARIANT SUBMANIFOLD OF (k, μ)-CONTACT MANIFOLD

A submanifold M of is said to be invariant if the structure vector field ξ is tangent to M, at every point of M and ϕX is tangent to M for any vector field X tangent to M at every point on M, that is, $\phi(TM) \subset TM$ at every point on M .

Proposition-1:*[17] Let M be an invariant submanifold of a* (k,μ)*-contact manifold* $\widetilde{M}$*. Then the following equalities hold on* M.

$$\widetilde{\nabla}_X\xi = -\phi X - \phi hX, \quad (4.1)$$

$$\sigma(X,\xi) = 0, \quad (4.2)$$

$$R(\xi,X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX], \quad (4.3)$$

$$S(X,\xi) = (n-1)k\eta(X), \quad (4.4)$$

$$(\nabla_X\phi)Y = g(X+hX,Y)\xi - \eta(Y)(X+hX), \quad (4.5)$$

$$\sigma(X,\phi Y) = \phi\sigma(X,Y) \quad (4.6)$$

for all vector fields X,Y tangent to M.

So we can state the following:

Theorem-2:*[17] An invariant submanifold M of a* (k,μ)*-contact manifold* $\widetilde{M}$ *is a* (k,μ)*-contact manifold.*

V. INVARIANT SUBMANIFOLD OF $(\boldsymbol{k},\boldsymbol{\mu})$-CONTACT MINVARIANT SUBMANIFOLDS OF $(\boldsymbol{k},\boldsymbol{\mu})$-CONTACT MANIFOLDS SATISFYING $\boldsymbol{Q}(\boldsymbol{\sigma},\boldsymbol{R})=0$

This section is devoted with the study of invariant submanifolds of (k,μ)-contact manifolds satisfying $Q(\sigma,R)=0$. Therefore

$$0 = Q(\sigma,R)(X,Y,Z;U,V)$$
$$= \big((U\wedge_\sigma V)\cdot R\big)(X,Y)Z = -R\big((U\wedge_\sigma V)X,Y\big)Z - R(X,(U\wedge_\sigma V)Y)Z - R(X,Y)(U\wedge_\sigma V)Z, \quad (5.1)$$

where $U\wedge_\sigma Y$ is defined by

$$(U\wedge_\sigma V)W = \sigma(V,W)U - \sigma(U,W)V. \quad (5.2)$$

Using (5.2) in (5.1) we have

$$-\sigma(V,X)R(U,Y)Z + \sigma(U,X)R(V,Y)Z - \sigma(V,Y)R(X,U)Z$$
$$+\sigma(U,Y)R(X,V)Z - \sigma(V,Z)R(X,Y)U + \sigma(U,Z)R(X,Y)V = 0. \quad (5.3)$$

Putting $Z=V=\xi$ in (5.3) and in view of (4.2), we obtain

$$\sigma(U,X)R(\xi,Y)\xi + \sigma(U,Y)R(X,\xi)\xi = 0. \quad (5.4)$$

Using (4.3) in (5.4) we have

$$k\eta(Y)\sigma(U,X)\xi - k\sigma(U,X)Y - \mu\sigma(U,X)hY + k\sigma(U,Y)X - k\eta(X)\sigma(U,Y)\xi + \mu\sigma(U,Y)hX = 0. \quad (5.5)$$

Taking inner product with W yields

$$k\eta(Y)\sigma(U,X)\eta(W) - k\sigma(U,X)g(Y,W) - \mu\sigma(U,X)g(hY,W) + k\sigma(U,Y)g(X,W)$$
$$-k\eta(X)\sigma(U,Y)\eta(W) + \mu\sigma(U,Y)g(hX,W) = 0. \quad (5.6)$$

Contracting Y and W we get

$$k\sigma(U,X) - kn\sigma(U,X) + k\sigma(U,X) + \mu\sigma(U,hX) = 0. \quad (5.7)$$

This implies

$$[k(2-n) \pm \mu\lambda]\sigma(U,X) = 0. \quad (5.8)$$

Hence $\sigma(U,X)=0$, provided $[k(2-n)\pm\mu\lambda]\neq 0$. Thus the manifold is totally geodesic. Conversely, if $\sigma(X,Y)=0$, then from (5.3), it follows that $Q(\sigma,R)=0$. Therefore in view of the above results we get

Theorem-3: *An invariant submanifold of a (k,μ)-contact manifold with $[k(2-n)\pm\mu\lambda]\neq 0$ satisfies $Q(\sigma,R)=0$ if and only it is totally geodesic.*

Take $k=1$ in (5.8) yields

$$(2-n)\sigma(U,X) = 0.$$

We know that (k,μ)-contact manifolds becomes Sasakian for $k=1$. Hence from Theorem-3, we have

Corollary-1:*An invariant submanifold of a Sasakian manifold satisfies $Q(\sigma,R)=0$ is always totally geodesic.*

VI. INVARIANT SUBMANIFOLDS OF $(\boldsymbol{k},\boldsymbol{\mu})$-CONTACT MANIFOLDS SATISFYING $\boldsymbol{Q}(\boldsymbol{S},\boldsymbol{\sigma})=0$

In this section we study invariant submanifolds of (k,μ)-contact manifold satisfying $Q(S,\sigma)=0$. Therefore

$$0 = Q(S,\sigma)(X,Y;U,V)$$
$$= -\sigma\big((U\wedge_S V)X,Y\big) - \sigma(X,(U\wedge_S V)Y), \quad (6.1)$$

where $U\wedge_S Y$ is defined by

$$(U\wedge_S V)W = S(V,W)U - S(U,W)V. \quad (6.2)$$

Using (6.2) in (6.1) yields

$$-S(V,X)\sigma(U,Y) + S(U,X)\sigma(V,Y) - S(V,Y)\sigma(X,U) + S(U,Y)\sigma(X,V) = 0. \quad (6.3)$$

Putting $U=Y=\xi$ in (6.3) we obtain

$$S(\xi,\xi)\sigma(X,V) = 0. \quad (6.4)$$

This implies

$$(n-1)k\sigma(X,V) = 0.$$

It follows that $\sigma(X,V)=0$, provided $k\neq 0$. Hence $M$is totally geodesic. Conversely, let M be totally geodesic, then from (6.2) we get$Q(S,\sigma)=0$.
Thus we can state the following:
Theorem-4: *An invariant submanifold of a (k,μ)-contact manifold with $k\neq 0$ satisfies$Q(S,\sigma)=0$ if and only it is totally geodesic.*
Corollary-2:*An invariant submanifold of a Sasakian manifold satisfies $Q(S,\sigma)=0$if and only it is totally geodesic.*

VII. INVARIANT SUBMANIFOLD OF (k,μ)-CONTACT MANIFOLDSSATISFYING$Q(\sigma,C)=0$

In this section we study invariant submanifolds of (k,μ)-contact manifold satisfying $Q(\sigma,C)=0$. Therefore

$$0=Q(\sigma,C)(X,Y,Z;U,V)$$

$$=\big((U\wedge_\sigma V)\cdot C\big)(X,Y)Z=-C\big((U\wedge_\sigma V)X,Y\big)Z-C(X,(U\wedge_\sigma V)Y)Z-C(X,Y)(U\wedge_\sigma V)Z. \tag{7.1}$$

Using (5.2) in (7.1) we have

$$-\sigma(V,X)C(U,Y)Z+\sigma(U,X)C(V,Y)Z-\sigma(V,Y)C(X,U)Z$$
$$+\sigma(U,Y)C(X,V)Z-\sigma(V,Z)C(X,Y)U+\sigma(U,Z)C(X,Y)V=0. \tag{7.2}$$

Putting $Z=V=\xi$ in (7.2) and in view of (4.2), we obtain

$$\sigma(U,X)C(\xi,Y)\xi+\sigma(U,Y)C(X,\xi)\xi=0. \tag{7.3}$$

Using (3.10) in (7.3) we have

$$\left(k-\frac{r}{n(n-1)}\right)[\eta(Y)\xi-Y]\sigma(U,X)-\mu\sigma(U,X)hY+\left(k-\frac{r}{n(n-1)}\right)[X-\eta(X)\xi]\sigma(U,Y)$$
$$+\mu\sigma(U,Y)hX=0. \tag{7.4}$$

Taking inner product with W yields

$$\left(k-\frac{r}{n(n-1)}\right)[\eta(Y)\eta(W)-g(Y,W)]\sigma(U,X)-\mu\sigma(U,X)g(hY,W)$$
$$+\left(k-\frac{r}{n(n-1)}\right)[g(X,W)-\eta(X)\eta(W)]\sigma(U,Y)+\mu\sigma(U,Y)g(hX,W)$$
$$=0. \tag{7.5}$$

Contracting Y and W, we get

$$\left(k-\frac{r}{n(n-1)}\right)\sigma(U,X)(1-n)+\left(k-\frac{r}{n(n-1)}\right)\sigma(U,X)+\mu\sigma(U,hX)=0. \tag{7.6}$$

This implies

$$\left[\left((2-n)k-\frac{(2-n)r}{n(n-1)}\right)\pm\mu\lambda\right]\sigma(U,X)=0, \tag{7.7}$$

and hence$\sigma(U,X)=0$, provided$r\neq\frac{n(n-1)}{(2-n)}[(2-n)k\pm\mu\lambda]$. Thus the manifold is totally geodesic. Conversely, if$\sigma(X,Y)=0$, then from (7.2), it follows that$Q(\sigma,C)=0$. Therefore in view of the above results we get
Theorem-5: *An invariant submanifold of a (k,μ)-contact manifold with$r\neq\frac{n(n-1)}{(2-n)}[(2-n)k\pm\mu\lambda]$ satisfies$Q(\sigma,C)=0$ if and only it is totally geodesic.*
Corollary 3 *An invariant submanifold of a Sasakian manifold with $r\neq n(n-1)$ satisfies$Q(\sigma,C)=0$ if and only it is totally geodesic.*

VIII. EXAMPLE

We consider five dimensional manifold $\widetilde{M}=\{(x_1,x_2,y_1,y_2,z)\in R^5: z\neq 0\}$, where (x_1,x_2,y_1,y_2,z) are standard coordinates inR^5. We choose the vector fields

$$e_1=2\frac{\partial}{\partial x^1},\quad e_2=2\frac{\partial}{\partial x^2},\quad e_3=2\left(\frac{\partial}{\partial y^1}+x^1\frac{\partial}{\partial z}\right),\quad e_4=2\left(\frac{\partial}{\partial y^2}+x^2\frac{\partial}{\partial z}\right),\quad e_5=2\frac{\partial}{\partial z},$$

which are linearly independent at each point of $\widetilde{M}$. Let g be the Riemannian metric defined by

$$g=\frac{1}{4}(dx^1\otimes dx^1+dx^2\otimes dx^2+dy^1\otimes dy^1+dy^2\otimes dy^2)+\eta\otimes\eta,$$

where $\eta$is the 1-form defined by $\eta(X)=g(X,e_5)$ for any vector field X on$\widetilde{M}$. Hence $(e_1,e,e_3,e_4,e_5)$is an orthonormal basis of $\widetilde{M}$. We define the (1,1) tensor field ϕ as

$$\phi(e_1)=e_3,\ \phi(e_2)=e_4,\quad \phi(e_3)=-e_1,\ \phi(e_4)=-e_3,\ \phi(e_5)=0.$$

The linear property of g and ϕ yields that

$$\eta(e_5)=1,\ \ \phi^2X=-X+\eta(X)e_5,\qquad g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y),$$

for any vector fields X,Y on $\widetilde{M}$. Thus for$e_5=\xi$, $\widetilde{M}(\phi,\xi,\eta,g)$defines an almost contact metric manifold.

Moreover, we get

$$[e_1, e_2] = 2e_5, \quad [e_2, e_4] = 2e_5,$$

and remaining $[e_i, e_j] = 0$ for all $1 \leq i, j \leq 5$.

The Riemannian connection $\widetilde{\nabla}$of the metric tensor g is given by Koszula formula which is given by,

$$2g(\widetilde{\nabla}_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y])$$

Using Koszul's formula we get the following:

$$\widetilde{\nabla}_{e_1} e_3 = e_5, \ \widetilde{\nabla}_{e_1} e_5 = -e_3, \ \widetilde{\nabla}_{e_2} e_4 = e_5, \quad \widetilde{\nabla}_{e_2} e_5 = -e_4,$$
$$\widetilde{\nabla}_{e_3} e_1 = -e_5, \ \widetilde{\nabla}_{e_3} e_5 = e_1, \ \widetilde{\nabla}_{e_4} e_2 = -e_5, \quad \widetilde{\nabla}_{e_4} e_5 = e_2,$$
$$\widetilde{\nabla}_{e_5} e_1 = -e_3, \ \widetilde{\nabla}_{e_5} e_2 = -e_4, \ \widetilde{\nabla}_{e_5} e_3 = e_1, \quad \widetilde{\nabla}_{e_5} e_4 = e_2,$$

and the remaining $\widetilde{\nabla}_{e_i} e_j = 0$, for all$1 \leq i, j \leq 5$.

From the above results it is easy to verify that $\widetilde{M}$ is a (k, μ)-contact manifold with $k = 1$ and$\mu = 0$.

Let M be a subset of $\widetilde{M}$ and consider the isometric immersion $f: M \to \widetilde{M}$ defined by

$$f(x^1, y^1, z) = f(x^1, 0, y^1, 0, z).$$

It can be easily prove that$M = \{(x^1, y^1, z) \in R^3 : (x^1, y^1, z) \neq 0\}$, where (x^1, y^1, z) are standard coordinates in R^3 is a 3-dimensional submanifold of the 5-dimensional (k, μ)-contact manifold$\widetilde{M}$.

We choose the vector fields

$$e_1 = 2\frac{\partial}{\partial x^1}, \ \ e_3 = 2\left(\frac{\partial}{\partial y^1} + x^1 \frac{\partial}{\partial z}\right), \quad e_5 = 2\frac{\partial}{\partial z},$$

which are linearly independent at each point of$M$. Let g be the Riemannian metric defined by

$$g = \tfrac{1}{4}(dx^1 \otimes dx^1 + dy^1 \otimes dy^1) + \eta \otimes \eta,$$

where $\eta$is the 1-form defined by $\eta(X) = g(X, e_5)$ for any vector field X on$M$. Hence $(e_1, e_3, e_5)$is an orthonormal basis of M. We define the $(1,1)$ tensor field ϕ as

$$\phi(e_1) = e_3, \quad \phi(e_3) = -e_1, \quad \phi(e_5) = 0.$$

The linear property of g and ϕ yields that

$$\eta(e_5) = 1, \ \ \phi^2 X = -X + \eta(X) e_5, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields X, Y on M. Thus for$e_5 = \xi$, $M(\phi, \xi, \eta, g)$defines an almost contact metric manifold.

Taking$e_5 = \xi$, and using Koszul's formulae for the metric$g$, it can be easily calculated that

$$\nabla_{e_1} e_3 = e_5, \quad \nabla_{e_1} e_5 = -e_3, \quad \nabla_{e_5} e_1 = -e_3,$$
$$\nabla_{e_3} e_1 = -e_5, \ \nabla_{e_3} e_5 = e_1, \quad \nabla_{e_5} e_3 = e_1,$$

and the remaining $\nabla_{e_i} e_j = 0$, for all$1 \leq i, j \leq 5$ and $i, j \neq 2,4$.

Let us consider,

$$TM = D \oplus D^{\perp} \oplus < \xi >,$$

where $D = < e_1 >$ and$D^{\perp} = < e_3 >$. Then we see that$\phi(e_1) = e_3$, for $e_1 \in D$ and$\phi(e_3) = -e_1 \in D$, for$e_3 \in D^{\perp}$. Hence the submanifold is invariant. Now from the values of $\widetilde{\nabla}_{e_i} e_j$ and $\nabla_{e_i} e_j$, we see that $\sigma(e_i, e_j) = 0$, for all $i, j = 1,3,5$.This means that the submanifold is totally geodesic. Thus the theorems 3-5 are verified.

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