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Study on derivation of shape functions in global coordinates and exact computation of element matrices for quadrilateral finite elements

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Abstract: *This article includes a technique to derive shape functions in global co-ordinates for general quadrilateral finite elements. It identifies clearly the element geometry for which the shape functions in global co-ordinates cannot be derived and explains the way to overcome such situation. Finally, it presents formulae based on array multiplication for exact computation of different types of element matrices needed for the employment of the said element in finite element solution procedure. The computation process of element matrices require only: (1) the nodal co-ordinates of the element geometry to form a matrix G (say) and then it's inverse matrix H (say) and (2) the values of the integral of monomials over the element. All the components of element matrices are then computed by the product of components of H with the values of the integrals. Thus, the process reduces many time consuming steps of FEM solution procedure and that substantially reduces computational effort. The accuracy and efficiency of the formulae so presented are then demonstrated through application examples.*
Keywords: *Global co-ordinates, shape functions, Quadrilateral elements, inverse and monomials.*

I. Introduction

Generally, for the convenience shape functions are commonly derived in local co-ordinates in local spaces. Transformation equations are written in terms of shape functions and the original element (in global space) is transformed into its contiguous element in local space. Consequently, all the calculations needed to form the element matrices that is the evaluation of numerous integrals are carried out in local co-ordinate systems [1 − 10]. It is well known that for such transformations (isoparametric, sub-parametric and super parametric) the integrals so encountered to form the stiffness matrix for the general quadrilateral (convex, concave) finite elements are rational integral of bivariate polynomial numerators with bilinear or higher order bivariate expressions denominators [12 − 14, 17 − 21]. Evaluation of such integrals defies our analytical skills and we are resort to numerical integration schemes [1, 4, 17, 21]. It is astounding to note here that the pleasing advancements in regard to analytical evaluation of such rational integrals for straight sided quadrilateral elements are made by many researchers [3, 5, 7, 10, 12 − 14, 16 − 22]. They have identified the drawbacks of numerical integration techniques especially for the Gaussian quadrature schemes. It is also evident from numerous research articles that such analytical integration formulae are applicable for sub-parametric case only and not applicable for higher order isoparametric elements. Besides that these analytical formulae require lot of computational effort and inconvenience for computer coding. Hence, for such difficulties and short comings the numerical integration schemes are still the only instrument for its simplicity and easy incorporation.

Among all the numerical integration schemes, Gaussian quadrature scheme occupies a central role for such evaluations. Complications arise from two main sources, firstly the large number of integrations that need to be performed and secondly, in methods which use isoparametric/ subparametric/ superparametric elements, the presence of the determinant of the Jacobean matrix in the denominator of the stiffness matrix for which the integrands are rational functions. Many authors [10, 15 − 21] outlined clearly that the usual Gaussian quadrature cannot evaluate exactly such integrals of rational functions as it can evaluate exactly a polynomial of degree $2n$-1 by employing n Gaussian points and weights. Obviously, for the desired accuracy of evaluations the number of Gaussian points and weights are needed to be increased and that increases substantially the computing time. Hence, a proper balance between the accuracy and efficiency is an important task [10, 18, 19 − 22]. Further, an attention is always required to select the order of the integrating rule as it is not yet totally worked out.

A suitable alternative, the use of polynomial shape functions in global coordinates other than of [23] in the formulation give rise integrals of polynomials which can be exactly evaluated either by the selected order of the integrating rule or by analytical schemes. In this case the main barrier is the derivation of shape functions in global coordinates for the element under considerations. Especially it is very much difficult and so cumbersome in case of the quadrilateral elements. Considering all the facts and the popularity of the quadrilateral elements, we have concentrated to derive shape functions in global coordinates and to present all the components of element matrices in bivariate polynomial form in a systematic way. So that one can use the gaussian quadrature schemes or other numerical schemes easily for obtaining element matrices. The technique so implemented as: (1) formation of a matrix *G* (say) by the nodal co-ordinates of the element geometry and then its inverse matrix *H* (say), and (2) the values of the integral of monomials over the element. Finally, we have employed analytical schemes and presented all the components of element matrices as the expressions of products of components of matrix *H*. Thus, once the matrices *G*, *H* are formed and values of the integrals of monomials over the element are calculated the then computation of element matrices are computed by multiplication of components of matrix *H* only. It is clearly shown that the conventional derivation of polynomial shape functions in global co-ordinates and the other computations depends on non-singularity of matrix *G*. That is, if *G* is either singular or bad scaled then *H* cannot be computed and that leads impossibility of other computations. We have studied thoroughly and identified for the first time, two types of element geometry for which *G* is singular and shown geometrical reasons for which *G* is bad scaled. It has been found that slight change in the mesh may overcome such difficulties. One can easily apply the technique to derive polynomial shape functions in local co-ordinates by forming *G* matrix by local nodal co-ordinates.

The present technique reduces many time consuming steps of FEM solution procedure and that substantially reduces computational effort. For such substantiation, the Saint-Venant torsion problem studied by many researchers [11, 16, 18 − 22] is considered. The accuracy and efficiency of the presented formulae are then demonstrated through the calculation of Prandtl stress function values and torsional constant of different type of cross sections. Through comparison of computed results with the results of other researchers the importance of exact computation of element matrices are established. We believe that this study includes all the explicit expressions of element matrices in terms of components of matrix *H* which will be useful for solving other engineering problems. Therefore, we firmly believe that the technique of this basic study will be more contributing and attractive in the realm of application of the finite element method.

II. Straight Sided Quadrilateral Elements

We consider here the straight sided 4-noned (Linear), 8-noded (quadratic) and 12-noded (cubic) serendipity quadrilateral elements as shown in Figs. 2.1 − 2.3.

Usually, the field variable *u* (say) governing the physical problem is expressed as

$$u = \sum_{i=1}^{NP} N_i(x, y)u_i$$

where N_i is the shape function refer to the node *i* and *NP* is the number of points in the element and u_i is the functional values at node *i*.

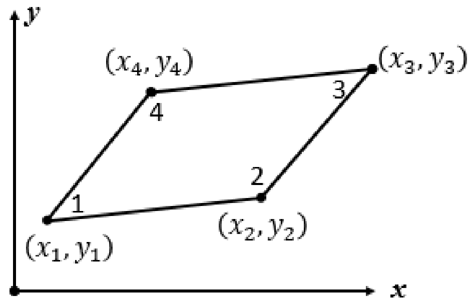


Fig. 2.1: A 4-noded quadrilateral elements

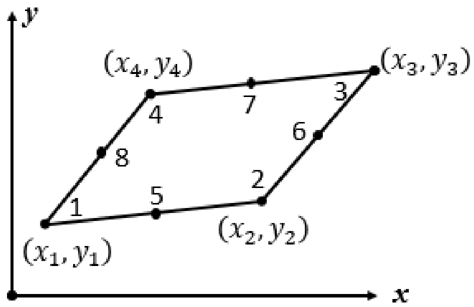


Fig. 2.2: A 8-noded quadrilateral elements

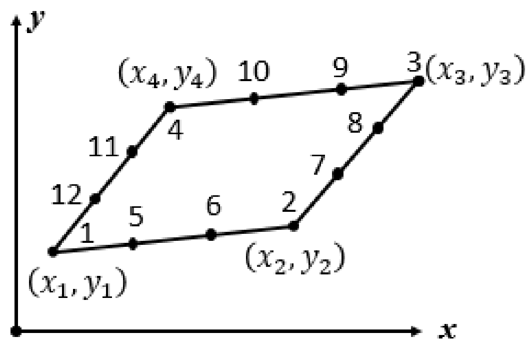


Fig. 2.3: A 12-noded quadrilateral elements

2.1 General form of shape functions:
For convenience, we write shape functions for 4-noded, 8-noded and 12-noded quadrilateral elements respectively as

$$N_i(x,y) = a_i + b_i x + c_i y + d_i xy\ , \quad i = 1,2,3,4 \tag{2.1}$$

$$N_i(x,y) = a_i + b_i x + c_i y + d_i xy + e_i x^2 + f_i y^2 + g_i x^2 y + h_i xy^2,\ i = 1,2,\cdots,8 \tag{2.2}$$

and

$$N_i(x,y) = a_i + b_i x + c_i y + d_i xy + e_i x^2 + f_i y^2 + g_i x^2 y + h_i xy^2 + l_i x^3 + m_i y^3 + n_i x^3 y + p_i xy^3\ , \quad i = 1,2,\cdots,12 \tag{2.3}$$

where $a_i, b_i, c_i, \cdots, p_i$ are needed to determine by satisfying the properties of shape functions.

2.2 Derivation of shape functions:
First, we consider the 4-noded quadrilateral element for deriving shape functions. By use of the properties

$$N_i(x_j, y_j) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \ , i, j = 1,2,3,4.$$

in Eq. (2.1), we have the following system of equations:

$$\begin{pmatrix} \delta_{i1} \\ \delta_{i2} \\ \delta_{i3} \\ \delta_{i4} \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ 1 & x_3 & y_3 & x_3 y_3 \\ 1 & x_4 & y_4 & x_4 y_4 \end{pmatrix} \begin{pmatrix} a_i \\ b_i \\ c_i \\ d_i \end{pmatrix} \text{ or, } \begin{pmatrix} \delta_{i1} \\ \delta_{i2} \\ \delta_{i3} \\ \delta_{i4} \end{pmatrix} = G \begin{pmatrix} a_i \\ b_i \\ c_i \\ d_i \end{pmatrix} \text{ or, } \begin{pmatrix} a_i \\ b_i \\ c_i \\ d_i \end{pmatrix} = G^{-1} \begin{pmatrix} \delta_{i1} \\ \delta_{i2} \\ \delta_{i3} \\ \delta_{i4} \end{pmatrix} \tag{2.1.1}$$

where $G = \begin{pmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ 1 & x_3 & y_3 & x_3 y_3 \\ 1 & x_4 & y_4 & x_4 y_4 \end{pmatrix}$ and croneker delta, $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

If we assume $\mathrm{H} = G^{-1}$, then (2.1.1) becomes

$$\begin{pmatrix} a_i \\ b_i \\ c_i \\ d_i \end{pmatrix} = \mathrm{H} \begin{pmatrix} \delta_{i1} \\ \delta_{i2} \\ \delta_{i3} \\ \delta_{i4} \end{pmatrix}$$

Then we obtain $a_i = H_{1i}, b_i = H_{2i}, c_i = H_{3i}, d_i = H_{4i}\ , \quad i = 1,2,3,4$

Finally, Eq.(2.1) is expressed as

$$N_i(x,y) = \mathrm{H}_{1i} + \mathrm{H}_{2i} x + \mathrm{H}_{3i} y + \mathrm{H}_{4i} xy,\ i = 1,2,3,4 \tag{2.4}$$

Proceeding in the similar way, we obtain shape functions for 8-noded and 12-noded quadrilateral elements respectively as:

$$N_i(x,y) = \mathrm{H}_{1i} + \mathrm{H}_{2i} x + \mathrm{H}_{3i} y + \mathrm{H}_{4i} xy + \mathrm{H}_{5i} x^2 + \mathrm{H}_{6i} y^2 + \mathrm{H}_{7i} x^2 y + \mathrm{H}_{8i} xy^2\ ,\ i = 1,2,\cdots,8 \tag{2.5}$$

and

$$\begin{aligned} N_i(\mathrm{x,y}) = {} & \mathrm{H}_{1i} + \mathrm{H}_{2i} x + \mathrm{H}_{3i} y + \mathrm{H}_{4i} xy + \mathrm{H}_{5i} x^2 + \mathrm{H}_{6i} y^2 + \mathrm{H}_{7i} x^2 y + \mathrm{H}_{8i} xy^2 + \mathrm{H}_{9i} x^3 \\ & + \mathrm{H}_{10i} y^3 + \mathrm{H}_{11i} x^3 y + \mathrm{H}_{12i} xy^3,\ i = 1,2,\cdots,12 \end{aligned} \tag{2.6}$$

Note here that all the shape functions are now explicitly written by the components of matrix $\mathrm{H} = G^{-1}$ and it is verified that $\sum_{i=1}^{NP} N_i(x,y) = 1$ i.e. the completeness property is satisfied.

2.3 Algorithm to form *G* matrix:
By the known nodal co-ordinates $(x_i, y_i), i = 1,2,3,\cdots,NP$ the matrix *G* easily can be formed by the algorithm:

for i = 1,2,3, … , NP
G_{i1} = 1; $G_{i2} = x_i$; $G_{i2} = y_i$; $G_{i3} = x_i\, y_i$;
If ((NP == 8) or (NP == 12)) then
$G_{i5} = x_i^2;\ G_{i6} = y_i^2;\ G_{i7} = x_i^3 y_i;\ G_{i8} = x_i y_i^3;$
If (NP ==12)) then
$G_{i9} = x_i^3;\ G_{i10} = y_i^3;\ G_{i7} = x_i^2 y_i;\ G_{i8} = x_i y_i^2;$
end if

end if

III. Singularity And Non-Singularity Conditions For G Matrix

Through Eqns. (2.4) – (2.6), it is clear that all the shape functions $N_i(x, y)$ may be derived if G is invertible i.e. $H = G^{-1}$ is computed. That is if *G* is non-singular then its inverse matrix *H* can be computed and shape functions can be derived. Otherwise the shape functions cannot be derived.

Therefore, it is an important task to investigate throughly the singularity of *G* formed by the nodal co-ordinates of the element geometry.

For detailed study, we consider first the 4-noded quadrilateral element for which

$$G = \begin{pmatrix} 1 & x_1 & y_1 & x_1y_1 \\ 1 & x_2 & y_2 & x_2y_2 \\ 1 & x_3 & y_3 & x_3y_3 \\ 1 & x_4 & y_4 & x_4y_4 \end{pmatrix}$$

Case-1: If $x_1 = x_3$ and $y_2 = y_4$ for coordinates $(x_i, y_i), i = 1,2,3,4$ of a quadrilateral as shown in fig.3.1 then $|G| = 0$ that is *G* is singular.

Case-2: If $x_2 = x_4$ and $y_1 = y_3$ for coordinates $(x_i, y_i), i = 1,2,3,4$ of a quadrilateral as shown in fig.3.2 then $|G| = 0$ that is *G* is singular.

Case-3: If $y_1 = y_3$ and $x_2 \neq x_4$ for coordinates $(x_i, y_i), i = 1,2,3,4$ of a quadrilateral as shown in fig.3.3 then $|G| \neq 0$ that is *G* is non-singular.

Case-4: If $x_1 = x_3$ and $y_2 \neq y_4$ for coordinates $(x_i, y_i), i = 1,2,3,4$ of a quadrilateral as shown in fig.3.4 then $|G| \neq 0$ that is *G* is non-singular.

Case-5: If $x_1 = x_4, x_2 = x_3, y_1 = y_2, y_3 = y_4$ (for rectangular shape) for coordinates $(x_i, y_i), i = 1,2,3,4$ of a quadrilateral as shown in fig.3.5 then $|G| \neq 0$ that is *G* is non-singular.

Case-6: If $x_1 = x_4, x_2 = x_3, y_1 = y_2, y_3 \neq y_4$ (for trapezoidal shape) for coordinates $(x_i, y_i), i = 1,2,3,4$ of a quadrilateral as shown in fig.3.6 then $|G| \neq 0$ that is *G* is non-singular.

Case-7: If $x_1 = x_4, x_2 \neq x_3, y_1 = y_2, y_3 = y_4$ (for other trapezoidal shape) for coordinates $(x_i, y_i), i = 1,2,3,4$ of a quadrilateral as shown in fig.3.7 then $|G| \neq 0$ that is *G* is non-singular.

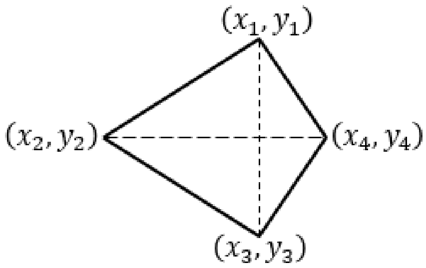


Fig. 3.1: $|G| = 0$ for the quadrilateral when $x_1 = x_3$ and $y_2 = y_4$

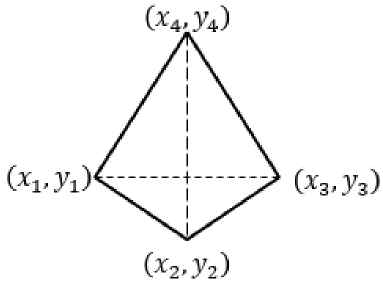


Fig. 3.2: $|G| = 0$ for the quadrilateral when $x_2 = x_4$ and $y_1 = y_3$

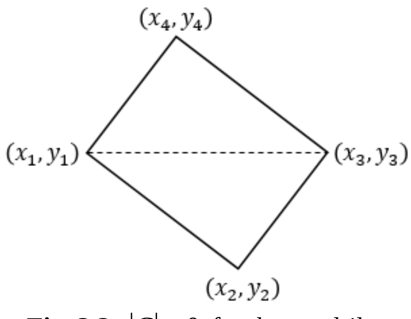


Fig. 3.3: $|G| \neq 0$ for the quadrilateral when $y_1 = y_3$ and $x_2 \neq x_4$

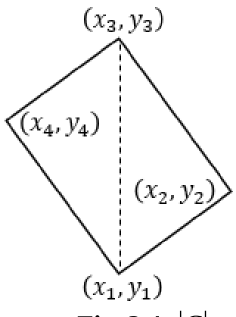


Fig. 3.4: $|G| \neq 0$ for the quadrilateral when $x_1 = x_3$ and $y_2 \neq y_4$

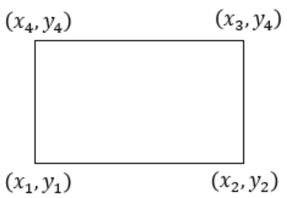


Fig. 3.5: $|G| \neq 0$ for the quadrilateral when $x_1 = x_4, x_2 = x_3, y_1 = y_2, y_3 = y_4$

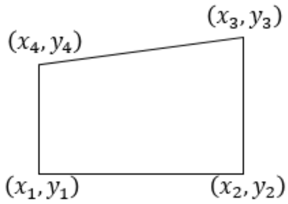


Fig. 3.6: $|G| \neq 0$ for the quadrilateral when $x_1 = x_4, x_2 = x_3, y_1 = y_2, y_3 \neq y_4$

Therefore, it can be concluded that only in two cases (case 1 and case 2) *G* is singular for which shape functions cannot be derived. But, in other cases *G* is non-singular and hence shape functions can be derived. For the case $|G| = 0$ (case-1, case-2) changes in mesh is necessary for which all the quadrilaterals are different from the quadrilaterals shown in Figs. 3.1 – 3.2. More specifically, we can overcome the situation of singularity of *G* by changing co-ordinates of the quadrilaterals remashing the domain of the problem.

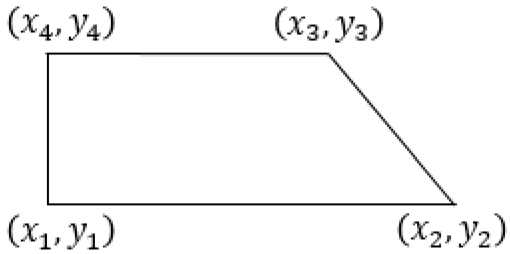


Fig. 3.7: $|G| \neq 0$ for the quadrilateral when $x_1 = x_4, x_2 \neq x_3, y_1 = y_2, y_3 = y_4$

Similarly, *G* matrix for 8-noded and 12-noded quadrilaterals are analyzed and it is found that for all straight sided element geometry *G* is always non-singular and hence shape functions can be derived. But if element contains curved side, then (*i*) $|G| \neq 0$ for 8-noded quadrilaterals, and (*ii*) $|G| = 0$ or *G* may be bad scaled for 12-noded quadrilaterals. For example, if we consider OAB of Fig.3.8 as one 12-noded (single) element then *G* is singular. Whereas for the models as shown in Fig.3.8.2 (a)-(b), *G* is non-singular for all 12-noded quadrilaterals.

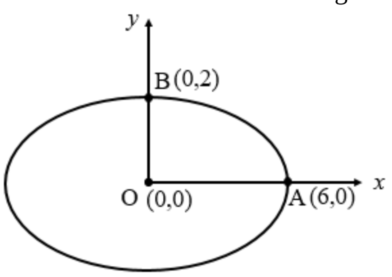


Fig. 3.8: An elliptic cross section

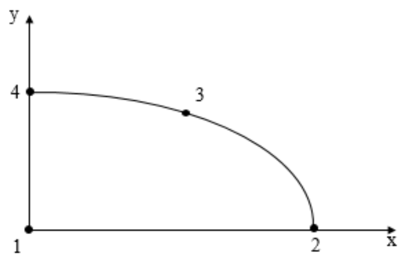


Fig. 3.8.1(a): Finite Element Model-1 with one 4-noded quadrilateral elements

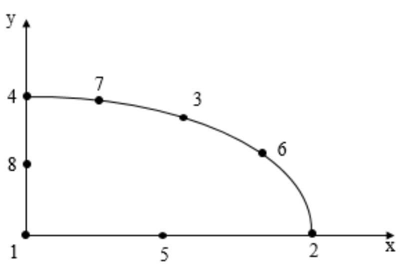


Fig. 3.8.1(b): Finite Element Model-2 with one 8-noded quadrilateral elements

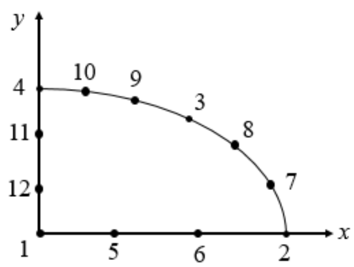


Fig. 3.8.1(c): Finite Element Model-3 with one 12-noded quadrilateral elements

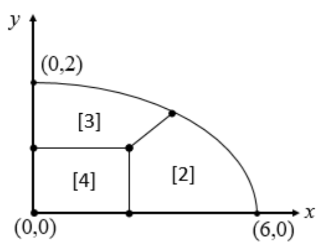


Fig. 3.8.2(a): Finite Element Model-4 with three 4-noded quadrilateral elements

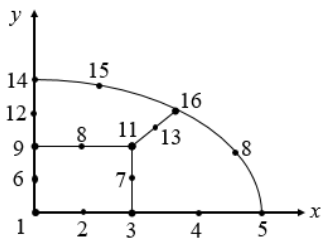


Fig. 3.8.2(b): Finite Element Model-5 with three 8-noded quadrilateral elements

We have investigated thoroughly the cases for which G is either singular or non-singular and it is found that G is always non-singular for 8-noded quadrilaterals.

IV. Global Derivatives, Product Of Global Derivatives And Components Of Element Matrices

This section deals with global derivatives, product of global derivatives and computation of various components of element matrices for quadrilateral finite elements. Step by step calculation process is shown for 4-noded quadrilateral element in details. Similar process is carried out for 8-noded and 12-noded (serendipity) quadrilateral elements. One can follow the process also for Lagrange type higher order quadrilateral elements.
For 4-noded quadrilateral element, using Eqn. (2.4), we obtain

$$\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} = \mathrm{H}_{2i}\mathrm{H}_{2j} + (\mathrm{H}_{2j}\mathrm{H}_{4i} + \mathrm{H}_{2i}\mathrm{H}_{4j})y + \mathrm{H}_{4i}\mathrm{H}_{4j}y^2 \tag{4.1}$$

4.1 Integration formula for $I^{\alpha,\beta}$:

It is necessary in FEM to integrate the product of shape functions, product of derivatives and other type of products e.g. product of shape functions and their derivatives over the element geometry, [6]. Since, such products are now in polynomial form one can exactly evaluate all these integrals by use of Gaussian quadrature schemes. Here, we intended to employ the following analytical integration formula.
Integral formula: If Γ is a polygonal boundary of a quadrilateral enclosed by the vertices $(x_i, y_i), i = 1,2,3,\cdots, NP$ with $x_{NP+1} = x_{NP}, y_{NP+1} = y_{NP}$. Then the integral of monomials over the quadrilateral i.e. $I^{\alpha,\beta} = \iint_{\partial} x^\alpha y^\beta dxdy$, where ∂ is the domain enclosed by Γ can be expressed as:

$$I^{\alpha,\beta} = \frac{1}{\alpha+1}\sum_{i=1}^{NP}\left[\sum_{p=0}^{\alpha+1}\sum_{q=0}^{\beta}\frac{\binom{\alpha+1}{p}\binom{\beta}{q}}{(p+q+1)} x_i^{\alpha+1-p}X_i^{\,p}y_i^{\beta-p}Y_i^{q+1}\right] \tag{4.2}$$

where α, β are non-negative positive integers and $X_i = x_{i+1} - x_i,\ Y_i = y_{i+1} - y_i$.

4.2 Integration of bivariate polynomials to compute element matrices of 4-noded quadrilateral:

Integrating Eqn. (4.1) by use of Eqn. (4.2), we have one type of components of element matrices

$$K_{ij}^{xx} = \iint \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}dxdy = \mathrm{H}_{2i}\mathrm{H}_{2j}I^{0,0} + (\mathrm{H}_{2j}\mathrm{H}_{4i} + \mathrm{H}_{2i}\mathrm{H}_{4j})I^{0,1} + \mathrm{H}_{4i}\mathrm{H}_{4j}I^{0,2}$$

Similarly, we have the other components of element matrices as in the following:

$$K_{ij}^{yy} = \iint \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}dxdy = \mathrm{H}_{3i}\mathrm{H}_{3j}I^{0,0} + (\mathrm{H}_{3j}\mathrm{H}_{4i} + \mathrm{H}_{3i}\mathrm{H}_{4j})I^{1,0} + \mathrm{H}_{4i}\mathrm{H}_{4j}I^{2,0}$$

$$K_{ij}^{xy} = \iint \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}dxdy = \mathrm{H}_{2i}\mathrm{H}_{3j}I^{0,0} + \mathrm{H}_{2i}\mathrm{H}_{4j}I^{1,0} + \mathrm{H}_{3j}\mathrm{H}_{4i}I^{0,1} + \mathrm{H}_{4i}\mathrm{H}_{4j}I^{1,1}$$

$$K_{ij}^{x} = \iint N_i(x,y)\frac{\partial}{\partial x}(N_j(x,y))dxdy = \mathrm{H}_{1i}\mathrm{H}_{2j}I^{0,0} + \mathrm{H}_{2i}\mathrm{H}_{2j}I^{1,0} + (\mathrm{H}_{2j}\mathrm{H}_{3i} + \mathrm{H}_{1i}\mathrm{H}_{4j})I^{0,1}$$
$$+ (\mathrm{H}_{2j}\mathrm{H}_{4i} + \mathrm{H}_{2i}\mathrm{H}_{4j})I^{1,1} + \mathrm{H}_{3i}\mathrm{H}_{4j}I^{0,2} + \mathrm{H}_{4i}\mathrm{H}_{4j}I^{1,2}$$

$$K_{ij}^{y} = \iint N_i(x,y)\frac{\partial}{\partial y}(N_j(x,y))dxdy = \mathrm{H}_{1i}\mathrm{H}_{3j}I^{0,0} + (\mathrm{H}_{2i}\mathrm{H}_{3j} + \mathrm{H}_{1i}\mathrm{H}_{4j})I^{1,0} + \mathrm{H}_{2i}\mathrm{H}_{4j}I^{2,0}$$
$$+ \mathrm{H}_{3i}\mathrm{H}_{3j}I^{0,1} + (\mathrm{H}_{3j}\mathrm{H}_{4i} + \mathrm{H}_{3i}\mathrm{H}_{4j})I^{1,1} + \mathrm{H}_{4i}\mathrm{H}_{4j}I^{2,1}$$

$$F_i = \int N_i(x,y)dxdy = \mathrm{H}_{1i}I^{0,0} + \mathrm{H}_{2i}I^{1,0} + \mathrm{H}_{3i}I^{0,1} + \mathrm{H}_{4i}I^{1,1}$$

$$B_{ij} = \int N_i(x,y)N_j(x,y)dxdy = \mathrm{H}_{1i}\mathrm{H}_{1j}I^{0,0} + (\mathrm{H}_{1j}\mathrm{H}_{2i} + \mathrm{H}_{1i}\mathrm{H}_{2j})I^{1,0} + \mathrm{H}_{2i}\mathrm{H}_{2j}I^{2,0} + (\mathrm{H}_{1j}\mathrm{H}_{3i} + \mathrm{H}_{1i}\mathrm{H}_{3j})I^{0,1}$$
$$+ (\mathrm{H}_{2j}\mathrm{H}_{3i} + \mathrm{H}_{2i}\mathrm{H}_{3j} + \mathrm{H}_{1j}\mathrm{H}_{4i} + \mathrm{H}_{1i}\mathrm{H}_{4j})I^{1,1} + (\mathrm{H}_{2j}\mathrm{H}_{4i} + \mathrm{H}_{2i}\mathrm{H}_{4j})I^{2,1}$$
$$+ \mathrm{H}_{3i}\mathrm{H}_{3j}I^{0,2} + (\mathrm{H}_{3j}\mathrm{H}_{4i} + \mathrm{H}_{3i}\mathrm{H}_{4j})I^{1,2} + \mathrm{H}_{4i}\mathrm{H}_{4j}I^{2,2}$$

$$R_{ijk}^{x}=\iint N_i\left(\mathrm{x},\mathrm{y}\right)N_j(x,y)\frac{\partial N_k}{\partial x}dxdy=\mathrm{H}_{1i}\mathrm{H}_{1j}\mathrm{H}_{2k}I^{0'0}+(\mathrm{H}_{1j}\mathrm{H}_{2i}\mathrm{H}_{2k}+\mathrm{H}_{1i}\mathrm{H}_{2j}\mathrm{H}_{2k})I^{1,0}+\mathrm{H}_{2i}\mathrm{H}_{2j}\mathrm{H}_{2k}I^{2,0}+(\mathrm{H}_{1j}\mathrm{H}_{2k}\mathrm{H}_{3i}+\mathrm{H}_{1i}\mathrm{H}_{2k}\mathrm{H}_{3j}$$
$$+\mathrm{H}_{1i}\mathrm{H}_{1j}\mathrm{H}_{4k})I^{0,1}+(\mathrm{H}_{2j}\mathrm{H}_{2k}\mathrm{H}_{3i}+\mathrm{H}_{2i}\mathrm{H}_{2k}\mathrm{H}_{3j}+\mathrm{H}_{1j}\mathrm{H}_{2k}\mathrm{H}_{4i}+\mathrm{H}_{1i}\mathrm{H}_{2k}\mathrm{H}_{4j}+\mathrm{H}_{1j}\mathrm{H}_{2i}\mathrm{H}_{4k}+\mathrm{H}_{1i}\mathrm{H}_{2j}\mathrm{H}_{4k})I^{1,1}+(\mathrm{H}_{2j}\mathrm{H}_{2k}\mathrm{H}_{4i}+\mathrm{H}_{2i}\mathrm{H}_{2k}\mathrm{H}_{4j}$$
$$+\mathrm{H}_{2i}\mathrm{H}_{2j}\mathrm{H}_{4k})I^{2,1}+(\mathrm{H}_{2k}\mathrm{H}_{3i}\mathrm{H}_{3j}+\mathrm{H}_{1j}\mathrm{H}_{3i}\mathrm{H}_{4k}+\mathrm{H}_{1i}\mathrm{H}_{3j}\mathrm{H}_{4k})I^{0,2}+(\mathrm{H}_{2k}\mathrm{H}_{3j}\mathrm{H}_{4i}+\mathrm{H}_{2k}\mathrm{H}_{3i}\mathrm{H}_{4j}+\mathrm{H}_{2j}\mathrm{H}_{3i}\mathrm{H}_{4k}+\mathrm{H}_{2i}\mathrm{H}_{3j}\mathrm{H}_{4k}+\mathrm{H}_{1j}\mathrm{H}_{4i}\mathrm{H}_{4k}$$
$$+\mathrm{H}_{1i}\mathrm{H}_{4j}\mathrm{H}_{4k})I^{1,2}+(\mathrm{H}_{2k}\mathrm{H}_{4i}\mathrm{H}_{4j}+\mathrm{H}_{2j}\mathrm{H}_{4i}\mathrm{H}_{4k}+\mathrm{H}_{2i}\mathrm{H}_{4j}\mathrm{H}_{4k})I^{2,2}+\mathrm{H}_{3i}\mathrm{H}_{3j}\mathrm{H}_{4k}I^{0,3}+(\mathrm{H}_{3j}\mathrm{H}_{4i}\mathrm{H}_{4k}+\mathrm{H}_{3i}\mathrm{H}_{4j}\mathrm{H}_{4k})I^{1,3}+\mathrm{H}_{4i}\mathrm{H}_{4j}\mathrm{H}_{4k}I^{2,3}$$

$$R_{ijk}^{y}=\int\int N_i\left(\mathrm{x},\mathrm{y}\right)N_j(x,y)\frac{\partial N_k}{\partial y}dxdy$$
$$=\mathrm{H}_{1i}\mathrm{H}_{1j}\mathrm{H}_{3k}I^{0,0}+(\mathrm{H}_{1j}\mathrm{H}_{2i}\mathrm{H}_{3k}+\mathrm{H}_{1i}\mathrm{H}_{2j}\mathrm{H}_{3k}+\mathrm{H}_{1i}\mathrm{H}_{1j}\mathrm{H}_{4k})I^{1,0}+(\mathrm{H}_{2i}\mathrm{H}_{2j}\mathrm{H}_{3k}+\mathrm{H}_{1j}\mathrm{H}_{2i}\mathrm{H}_{4k}+\mathrm{H}_{1i}\mathrm{H}_{2j}\mathrm{H}_{4k})I^{2,0}+\mathrm{H}_{2i}\mathrm{H}_{2j}\mathrm{H}_{4k}I^{3,0}$$
$$+(\mathrm{H}_{1j}\mathrm{H}_{3i}\mathrm{H}_{3k}+\mathrm{H}_{1i}\mathrm{H}_{3j}\mathrm{H}_{3k})I^{0,1}+(\mathrm{H}_{2j}\mathrm{H}_{3i}\mathrm{H}_{3k}+\mathrm{H}_{2i}\mathrm{H}_{3j}\mathrm{H}_{3k}+\mathrm{H}_{1j}\mathrm{H}_{3k}\mathrm{H}_{4i}+\mathrm{H}_{1i}\mathrm{H}_{3k}\mathrm{H}_{4j}+\mathrm{H}_{1j}\mathrm{H}_{3i}\mathrm{H}_{4k}+\mathrm{H}_{1i}\mathrm{H}_{3j}\mathrm{H}_{4k})I^{1,1}$$
$$+(\mathrm{H}_{2j}\mathrm{H}_{3k}\mathrm{H}_{4i}+\mathrm{H}_{2i}\mathrm{H}_{3k}\mathrm{H}_{4j}+\mathrm{H}_{2j}\mathrm{H}_{3i}\mathrm{H}_{4k}+\mathrm{H}_{2i}\mathrm{H}_{3j}\mathrm{H}_{4k}+\mathrm{H}_{1j}\mathrm{H}_{4i}\mathrm{H}_{4k}+\mathrm{H}_{1i}\mathrm{H}_{4j}\mathrm{H}_{4k})I^{2,1}+(\mathrm{H}_{2j}\mathrm{H}_{4i}\mathrm{H}_{4k}+\mathrm{H}_{2i}\mathrm{H}_{4j}\mathrm{H}_{4k})I^{3,1}$$
$$+\mathrm{H}_{3i}\mathrm{H}_{3j}\mathrm{H}_{3k}I^{0,2}+(\mathrm{H}_{3j}\mathrm{H}_{3k}\mathrm{H}_{4i}+\mathrm{H}_{3i}\mathrm{H}_{3k}\mathrm{H}_{4j}+\mathrm{H}_{3i}\mathrm{H}_{3j}\mathrm{H}_{4k})I^{1,2}+(\mathrm{H}_{3k}\mathrm{H}_{4i}\mathrm{H}_{4j}+\mathrm{H}_{3j}\mathrm{H}_{4i}\mathrm{H}_{4k}+\mathrm{H}_{3i}\mathrm{H}_{4j}\mathrm{H}_{4k})I^{2,2}+\mathrm{H}_{4i}\mathrm{H}_{4j}\mathrm{H}_{4k}I^{3,2}$$

Proceeding on the similar way, we obtain component of element matrices for 8-noded and 12-noded quadrilateral elements as the following:

4.3 Element matrices for 8-noded quadrilateral elements:

$$K_{ij}^{xx}=\mathrm{H}_{2i}\mathrm{H}_{2j}I^{0,0}+2(\mathrm{H}_{2j}\mathrm{H}_{5i}+\mathrm{H}_{2i}\mathrm{H}_{5j})I^{1,0}+4\mathrm{H}_{5i}\mathrm{H}_{5j}I^{2,0}+(\mathrm{H}_{2j}\mathrm{H}_{4i}+\mathrm{H}_{2i}\mathrm{H}_{4j})I^{0,1}+2(\mathrm{H}_{4j}\mathrm{H}_{5i}+\mathrm{H}_{4i}\mathrm{H}_{5j}+\mathrm{H}_{2j}\mathrm{H}_{7i}$$
$$+\mathrm{H}_{2i}\mathrm{H}_{7j})I^{1,1}+4(\mathrm{H}_{5j}\mathrm{H}_{7i}+\mathrm{H}_{5i}\mathrm{H}_{7j})I^{2,1}+(\mathrm{H}_{4i}\mathrm{H}_{4j}+\mathrm{H}_{2j}\mathrm{H}_{8i}+\mathrm{H}_{2i}\mathrm{H}_{8j})I^{0,2}+2(\mathrm{H}_{4j}\mathrm{H}_{7i}+\mathrm{H}_{4i}\mathrm{H}_{7j}+\mathrm{H}_{5j}\mathrm{H}_{8i}$$
$$+\mathrm{H}_{5i}\mathrm{H}_{8j})I^{1,2}+4\mathrm{H}_{7i}\mathrm{H}_{7j}I^{2,2}+(\mathrm{H}_{4j}\mathrm{H}_{8i}+\mathrm{H}_{4i}\mathrm{H}_{8j})I^{0,3}+2(\mathrm{H}_{7j}\mathrm{H}_{8i}+\mathrm{H}_{7i}\mathrm{H}_{8j})I^{1,3}+\mathrm{H}_{8i}\mathrm{H}_{8j}I^{0,4}$$

$$K_{ij}^{yy}=\mathrm{H}_{3i}\mathrm{H}_{3j}I^{0,0}+(\mathrm{H}_{3j}\mathrm{H}_{4i}+\mathrm{H}_{3i}\mathrm{H}_{4j})I^{1,0}+(\mathrm{H}_{4i}\mathrm{H}_{4j}+\mathrm{H}_{3j}\mathrm{H}_{7i}+\mathrm{H}_{3i}\mathrm{H}_{7j})I^{2,0}+(\mathrm{H}_{4j}\mathrm{H}_{7i}+\mathrm{H}_{4i}\mathrm{H}_{7j})I^{3,0}+\mathrm{H}_{7i}\mathrm{H}_{7j}I^{4,0}$$
$$+2(\mathrm{H}_{3j}\mathrm{H}_{6i}+\mathrm{H}_{3i}\mathrm{H}_{6j})I^{0,1}+2(\mathrm{H}_{4j}\mathrm{H}_{6i}+\mathrm{H}_{4i}\mathrm{H}_{6j}+\mathrm{H}_{3j}\mathrm{H}_{8i}+\mathrm{H}_{3i}\mathrm{H}_{8j})I^{1,1}+2(\mathrm{H}_{6j}\mathrm{H}_{7i}+\mathrm{H}_{6i}\mathrm{H}_{7j}+\mathrm{H}_{4j}\mathrm{H}_{8i}+\mathrm{H}_{4i}\mathrm{H}_{8j})I^{2,1}$$
$$+2(\mathrm{H}_{7j}\mathrm{H}_{8i}+\mathrm{H}_{7i}\mathrm{H}_{8j})I^{3,1}+4\mathrm{H}_{6i}\mathrm{H}_{6j}I^{0,2}+4(\mathrm{H}_{6j}\mathrm{H}_{8i}+\mathrm{H}_{6i}\mathrm{H}_{8j})I^{1,2}+4\mathrm{H}_{8i}\mathrm{H}_{8j}I^{2,2}$$

$$K_{ij}^{xy}=\mathrm{H}_{2i}\mathrm{H}_{3j}I^{0,0}+(\mathrm{H}_{2i}\mathrm{H}_{4j}+2\mathrm{H}_{3j}\mathrm{H}_{5i})I^{1,0}+(2\mathrm{H}_{4j}\mathrm{H}_{5i}+\mathrm{H}_{2i}\mathrm{H}_{7j})I^{2,0}+2\mathrm{H}_{5i}\mathrm{H}_{7j}I^{,03}+(\mathrm{H}_{3j}\mathrm{H}_{4i}+2\mathrm{H}_{2i}\mathrm{H}_{6j})I^{0,1}$$
$$+(\mathrm{H}_{4i}\mathrm{H}_{4j}+4\mathrm{H}_{5i}\mathrm{H}_{6j}+2\mathrm{H}_{3j}\mathrm{H}_{7i}+2\mathrm{H}_{2i}\mathrm{H}_{8j})I^{1,1}+(2\mathrm{H}_{4j}\mathrm{H}_{7i}+\mathrm{H}_{4i}\mathrm{H}_{7j}+4\mathrm{H}_{5i}\mathrm{H}_{8j})I^{2,1}+2\mathrm{H}_{7i}\mathrm{H}_{7j}I^{3,1}$$
$$+(2\mathrm{H}_{4i}\mathrm{H}_{6j}+\mathrm{H}_{3j}\mathrm{H}_{8i})I^{0,2}+(4\mathrm{H}_{6j}\mathrm{H}_{7i}+\mathrm{H}_{4j}\mathrm{H}_{8i}+2\mathrm{H}_{4i}\mathrm{H}_{8j})I^{1,2}+(\mathrm{H}_{7j}\mathrm{H}_{8i}+4\mathrm{H}_{7i}\mathrm{H}_{8j})I^{2,2}+2\mathrm{H}_{6j}\mathrm{H}_{8i}I^{0,3}$$
$$+2\mathrm{H}_{8i}\mathrm{H}_{8j}I^{1,3}$$

$$K_{ij}^{x}=\mathrm{H}_{1i}\mathrm{H}_{2j}I^{0,0}+(\mathrm{H}_{2i}\mathrm{H}_{2j}+2\mathrm{H}_{1i}\mathrm{H}_{5j})I^{1,0}+(\mathrm{H}_{2j}\mathrm{H}_{5i}+2\mathrm{H}_{2i}\mathrm{H}_{5j})I^{2,0}+2\mathrm{H}_{5i}\mathrm{H}_{5j}I^{3,0}+(\mathrm{H}_{2j}\mathrm{H}_{3i}+\mathrm{H}_{1i}\mathrm{H}_{4j})I^{0,1}+(\mathrm{H}_{2j}\mathrm{H}_{4i}$$
$$+\mathrm{H}_{2i}\mathrm{H}_{4j}+2\mathrm{H}_{3i}\mathrm{H}_{5j}+2\mathrm{H}_{1i}\mathrm{H}_{7j})I^{1,1}+(\mathrm{H}_{4j}\mathrm{H}_{5i}+2\mathrm{H}_{4i}\mathrm{H}_{5j}+\mathrm{H}_{2j}\mathrm{H}_{7i}+2\mathrm{H}_{2i}\mathrm{H}_{7j})I^{2,1}+2(\mathrm{H}_{5j}\mathrm{H}_{7i}+\mathrm{H}_{5i}\mathrm{H}_{7j})I^{3,1}$$
$$+(\mathrm{H}_{3i}\mathrm{H}_{4j}+\mathrm{H}_{2j}\mathrm{H}_{6i}+\mathrm{H}_{1i}\mathrm{H}_{8j})I^{0,2}+(\mathrm{H}_{4i}\mathrm{H}_{4j}+2\mathrm{H}_{5j}\mathrm{H}_{6i}+2\mathrm{H}_{3i}\mathrm{H}_{7j}+\mathrm{H}_{2j}\mathrm{H}_{8i}+\mathrm{H}_{2i}\mathrm{H}_{8j})I^{1,2}+(\mathrm{H}_{4j}\mathrm{H}_{7i}+2\mathrm{H}_{4i}\mathrm{H}_{7j}$$
$$+2\mathrm{H}_{5j}\mathrm{H}_{8i}+\mathrm{H}_{5i}\mathrm{H}_{8j})I^{2,2}+2\mathrm{H}_{7i}\mathrm{H}_{7j}I^{3,2}+(\mathrm{H}_{4j}\mathrm{H}_{6i}+\mathrm{H}_{3i}\mathrm{H}_{8j})I^{0,3}+(2\mathrm{H}_{6i}\mathrm{H}_{7j}+\mathrm{H}_{4j}\mathrm{H}_{8i}+\mathrm{H}_{4i}\mathrm{H}_{8j})I^{1,3}+(2\mathrm{H}_{7j}\mathrm{H}_{8i}$$
$$+\mathrm{H}_{7i}\mathrm{H}_{8j})I^{2,3}+\mathrm{H}_{6i}\mathrm{H}_{8j}I^{0,4}+\mathrm{H}_{8i}\mathrm{H}_{8j}I^{1,4}$$

$$K_{ij}^{y}=\mathrm{H}_{1i}\mathrm{H}_{3j}I^{0,0}+(\mathrm{H}_{2i}\mathrm{H}_{3j}+\mathrm{H}_{1i}\mathrm{H}_{4j})I^{1,0}+(\mathrm{H}_{2i}\mathrm{H}_{4j}+\mathrm{H}_{3j}\mathrm{H}_{5i}+\mathrm{H}_{1i}\mathrm{H}_{7j})I^{2,0}+(\mathrm{H}_{4j}\mathrm{H}_{5i}+\mathrm{H}_{2i}\mathrm{H}_{7j})I^{3,0}+\mathrm{H}_{5i}\mathrm{H}_{7j}I^{4,0}$$
$$+(\mathrm{H}_{3i}\mathrm{H}_{3j}+2\mathrm{H}_{1i}\mathrm{H}_{6j})I^{0,1}+(\mathrm{H}_{3j}\mathrm{H}_{4i}+\mathrm{H}_{3i}\mathrm{H}_{4j}+2\mathrm{H}_{2i}\mathrm{H}_{6j}+2\mathrm{H}_{1i}\mathrm{H}_{8j})I^{1,1}+(\mathrm{H}_{4i}\mathrm{H}_{4j}+2\mathrm{H}_{5i}\mathrm{H}_{6j}+\mathrm{H}_{3j}\mathrm{H}_{7i}+\mathrm{H}_{3i}\mathrm{H}_{7j}$$
$$+2\mathrm{H}_{2i}\mathrm{H8j})I^{2,1}+(\mathrm{H}_{4j}\mathrm{H}_{7i}+\mathrm{H}_{4i}\mathrm{H}_{7j}+2\mathrm{H}_{5i}\mathrm{H}_{8j})I^{3,1}+\mathrm{H}_{7i}\mathrm{H}_{7j}I^{4,1}+(\mathrm{H}_{3j}\mathrm{H}_{6i}+2\mathrm{H}_{3i}\mathrm{H}_{6j})I^{0,2}+(\mathrm{H}_{4j}\mathrm{H}_{6i}+2\mathrm{H}_{4i}\mathrm{H}_{6j}$$
$$+\mathrm{H}_{3j}\mathrm{H}_{8i}+2\mathrm{H}_{3i}\mathrm{H}_{8j})I^{1,2}+(2\mathrm{H}_{6j}\mathrm{H}_{7i}+\mathrm{H}_{6i}\mathrm{H}_{7j}+\mathrm{H}_{4j}\mathrm{H}_{8i}+2\mathrm{H}_{4i}\mathrm{H}_{8j})I^{2,2}+(\mathrm{H}_{7j}\mathrm{H}_{8i}+2\mathrm{H}_{7i}\mathrm{H}_{8j})I^{3,2}+2\mathrm{H}_{6i}\mathrm{H}_{6j}I^{0,3}$$
$$+2(\mathrm{H}_{6j}\mathrm{H}_{8i}+\mathrm{H}_{6i}\mathrm{H}_{8j})I^{1,3}+2\mathrm{H}_{8i}\mathrm{H}_{8j}I^{2,3}$$

$$F_i=\mathrm{H}_{1i}I^{0,0}+\mathrm{H}_{2i}I^{1,0}+\mathrm{H}_{3i}I^{0,1}+\mathrm{H}_{4i}I^{1,1}+\mathrm{H}_{5i}I^{2,0}+\mathrm{H}_{6i}I^{0,2}+\mathrm{H}_{7i}I^{2,1}+\mathrm{H}_{8i}I^{1,2}$$

$$B_{ij}=\mathrm{H}_{1i}\mathrm{H}_{1j}I^{0,0}+(\mathrm{H}_{1j}\mathrm{H}_{2i}+\mathrm{H}_{1i}\mathrm{H}_{2j})I^{1,0}+(\mathrm{H}_{2i}\mathrm{H}_{2j}+\mathrm{H}_{1j}\mathrm{H}_{5i}+\mathrm{H}_{1i}\mathrm{H}_{5j})I^{2,0}+(\mathrm{H}_{2j}\mathrm{H}_{5i}+\mathrm{H}_{2i}\mathrm{H}_{5j})I^{3,0}+\mathrm{H}_{5i}\mathrm{H}_{5j}I^{4,0}+(\mathrm{H}_{1j}\mathrm{H}_{3i}$$
$$+\mathrm{H}_{1i}\mathrm{H}_{3j})I^{0,1}+(\mathrm{H}_{2j}\mathrm{H}_{3i}+\mathrm{H}_{2i}\mathrm{H}_{3j}+\mathrm{H}_{1j}\mathrm{H}_{4i}+\mathrm{H}_{1i}\mathrm{H}_{4j})I^{1,1}+(\mathrm{H}_{2j}\mathrm{H}_{4i}+\mathrm{H}_{2i}\mathrm{H}_{4j}+\mathrm{H}_{3j}\mathrm{H}_{5i}+\mathrm{H}_{3i}\mathrm{H}_{5j}+\mathrm{H}_{1j}\mathrm{H}_{7i}+\mathrm{H}_{1i}\mathrm{H}_{7j})I^{2,1}$$
$$+(\mathrm{H}_{4j}\mathrm{H}_{5i}+\mathrm{H}_{4i}\mathrm{H}_{5j}+\mathrm{H}_{2j}\mathrm{H}_{7i}+\mathrm{H}_{2i}\mathrm{H}_{7j})I^{3,1}+(\mathrm{H}_{5j}\mathrm{H}_{7i}+\mathrm{H}_{5i}\mathrm{H}_{7j})I^{4,1}+(\mathrm{H}_{3i}\mathrm{H}_{3j}+\mathrm{H}_{1j}\mathrm{H}_{6i}+\mathrm{H}_{1i}\mathrm{H}_{6j})I^{0,2}+(\mathrm{H}_{3j}\mathrm{H}_{4i}+\mathrm{H}_{3i}\mathrm{H}_{4j}$$
$$+\mathrm{H}_{2j}\mathrm{H}_{6i}+\mathrm{H}_{2i}\mathrm{H}_{6j}+\mathrm{H}_{1j}\mathrm{H}_{8i}+\mathrm{H}_{1i}\mathrm{H}_{8j})I^{1,2}+(\mathrm{H}_{4i}\mathrm{H}_{4j}+\mathrm{H}_{5j}\mathrm{H}_{6i}+\mathrm{H}_{5i}\mathrm{H}_{6j}+\mathrm{H}_{3j}\mathrm{H}_{7i}+\mathrm{H}_{3i}\mathrm{H}_{7j}+\mathrm{H}_{2j}\mathrm{H}_{8i}+\mathrm{H}_{2i}\mathrm{H}_{8j})I^{2,2}$$
$$+(\mathrm{H}_{4j}\mathrm{H}_{7i}+\mathrm{H}_{4i}\mathrm{H}_{7j}+\mathrm{H}_{5j}\mathrm{H}_{8i}+\mathrm{H}_{5i}\mathrm{H}_{8j})I^{3,2}+\mathrm{H}_{7i}\mathrm{H}_{7j}I^{4,2}+(\mathrm{H}_{3j}\mathrm{H}_{6i}+\mathrm{H}_{3i}\mathrm{H}_{6j})I^{0,3}+(\mathrm{H}_{4j}\mathrm{H}_{6i}+\mathrm{H}_{4i}\mathrm{H}_{6j}+\mathrm{H}_{3j}\mathrm{H}_{8i}+\mathrm{H}_{3i}\mathrm{H}_{8j})I^{1,3}$$
$$+(\mathrm{H}_{6j}\mathrm{H}_{7i}+\mathrm{H}_{6i}\mathrm{H}_{7j}+\mathrm{H}_{4j}\mathrm{H}_{8i}+\mathrm{H}_{4i}\mathrm{H}_{8j})I^{2,3}+(\mathrm{H}_{7j}\mathrm{H}_{8i}+\mathrm{H}_{7i}\mathrm{H}_{8j})I^{3,3}+\mathrm{H}_{6i}\mathrm{H}_{6j}I^{0,4}+(\mathrm{H}_{6j}\mathrm{H}_{8i}+\mathrm{H}_{6i}\mathrm{H}_{8j})I^{1,4}+\mathrm{H}_{8i}\mathrm{H}_{8j}I^{2,4}$$

Similarly, R_{ijk}^{x} and R_{ijk}^{y} may be expressed.

4.4 Element matrices for 12-noded quadrilateral elements:

$$
\begin{aligned}
K_{ij}^{xx} = {} & H_{2i}H_{2j}I^{0,0} + 2(H_{2j}H_{5i} + H_{2i}H_{5j})I^{1,0} + (4H_{5i}H_{5j} + 3H_{2j}H_{9i} + 3H_{2i}H_{9j})I^{2,0} + 6(H_{5j}H_{9i} + H_{5i}H_{9j})I^{3,0} \\
& + 9H_{9i}H_{9j}I^{4,0} + (H_{2j}H_{4i}y + H_{2i}H_{4j})I^{0,1} + 2(H_{4j}H_{5i} + H_{4i}H_{5j} + H_{2j}H_{7i} + H_{2i}H_{7j})I^{1,1} + (3H_{11j}H_{2i} \\
& + 3H_{11i}H_{2j} + 4H_{5j}H_{7i} + 4H_{5i}H_{7j} + 3H_{4j}H_{9i} + 3H_{4i}H_{9j})I^{2,1} + 6(H_{11j}H_{5i} + H_{11i}H_{5j} + H_{7j}H_{9i} + H_{7i}H_{9j})I^{3,1} \\
& + 9(H_{11j}H_{9i} + H_{11i}H_{9j})I^{4,1} + (H_{4i}H_{4j} + H_{2j}H_{8i} + H_{2i}H_{8j})I^{0,2} + 2(H_{4j}H_{7i} + H_{4i}H_{7j} + H_{5j}H_{8i} + H_{5i}H_{8j})I^{1,2} \\
& + (3H_{11j}H_{4i} + 3H_{11i}H_{4j} + 4H_{7i}H_{7j} + 3H_{8j}H_{9i} + 3H_{8i}H_{9j})I^{2,2} + 6(H_{11j}H_{7i} + H_{11i}H_{7j})I^{3,2} + 9H_{11i}H_{11j}I^{4,2} \\
& + (H_{12j}H_{2i} + H_{12i}H_{2j} + H_{4j}H_{8i} + H_{4i}H_{8j})I^{0,3} + 2(H_{12j}H_{5i} + H_{12i}H_{5j} + H_{7j}H_{8i} + H_{7i}H_{8j})I^{1,3} + 3(H_{11j}H_{8i} \\
& + H_{11i}H_{8j} + H_{12j}H_{9i} + H_{12i}H_{9j})I^{2,3} + (H_{12j}H_{4i} + H_{12i}H_{4j} + H_{8i}H_{8j})I^{0,4} + 2(H_{12j}H_{7i} + H_{12i}H_{7j})I^{1,4} \\
& + 3(H_{11j}H_{12i} + H_{11i}H_{12j})I^{2,4} + (H_{12j}H_{8i} + H_{12i}H_{8j})I^{0,5} + H_{12i}H_{12j}I^{0,6}
\end{aligned}
$$

$$
\begin{aligned}
K_{ij}^{yy} = {} & H_{3i}H_{3j}I^{0,0} + (H_{3j}H_{4i} + H_{3i}H_{4j})I^{1,0} + (H_{4i}H_{4j} + H_{3j}H_{7i} + H_{3i}H_{7j})I^{2,0} + (H_{11j}H_{3i} + H_{11i}H_{3j} + H_{4j}H_{7i} + H_{4i}H_{7j})I^{3,0} \\
& + (H_{11j}H_{4i} + H_{11i}H_{4j} + H_{7i}H_{7j})I^{4,0} + (H_{11j}H_{7i} + H_{11i}H_{7j})I^{5,0} + H_{11i}H_{11j}I^{6,0} + 2(H_{3j}H_{6i} + 2H_{3i}H_{6j})I^{0,1} + 2(H_{4j}H_{6i} \\
& + H_{4i}H_{6j} + H_{3j}H_{8i} + H_{3i}H_{8j})I^{1,1} + 2(H_{6j}H_{7i} + H_{6i}H_{7j} + H_{4j}H_{8i} + H_{4i}H_{8j})I^{2,1} + 2(H_{11j}H_{6i} + H_{11i}H_{6j} + H_{7j}H_{8i} \\
& + H_{7i}H_{8j})I^{3,1} + 2(H_{11j}H_{8i} + H_{11i}H_{8j})I^{4,1} + (3H_{10j}H_{3i} + 3H_{10i}H_{3j} + 4H_{6i}H_{6j})I^{0,2} + (3H_{12j}H_{3i} + 3H_{12i}H_{3j} + 3H_{10j}H_{4i} \\
& + 3H_{10i}H_{4j} + 4H_{6j}H_{8i} + 4H_{6i}H_{8j})I^{1,2} + (3H_{12j}H_{4i} + 3H_{12i}H_{4j} + 3H_{10j}H_{7i} + 3H_{10i}H_{7j} + 4H_{8i}H_{8j})I^{2,2} + 3(H_{10j}H_{11i} \\
& + H_{10i}H_{11j} + H_{12j}H_{7i} + H_{12i}H_{7j})I^{3,2} + 3(H_{11j}H_{12i} + H_{11i}H_{12j})I^{4,2} + 6(H_{10j}H_{6i} + H_{10i}H_{6j})I^{0,3} + 6(H_{12j}H_{6i} + H_{12i}H_{6j} \\
& + H_{10j}H_{8i} + H_{10i}H_{8j})I^{1,3} + 6(H_{12j}H_{8i} + H_{12i}H_{8j})I^{2,3} + 9H_{10i}H_{10j}I^{0,4} + 9(H_{10j}H_{12i} + H_{10i}H_{12j})I^{1,4} + 9H_{12i}H_{12j}I^{2,4}
\end{aligned}
$$

$$
\begin{aligned}
K_{ij}^{xy} = {} & H_{2i}H_{3j}I^{0,0} + (H_{2i}H_{4j} + 2H_{3j}H_{5i})I^{1,0} + (2H_{4j}H_{5i} + H_{2i}H_{7j} + 3H_{3j}H_{9i})I^{2,0} + (H_{11j}H_{2i} + 2H_{5i}H_{7j} + 3H_{4j}H_{9i})I^{3,0} \\
& + (2H_{11j}H_{5i} + 3H_{7j}H_{9i})I^{4,0} + 3H_{11j}H_{9i}I^{5,0} + (H_{3j}H_{4i} + 2H_{2i}H_{6j})I^{0,1} + (H_{4i}H_{4j} + 4H_{5i}H_{6j} + 2H_{3j}H_{7i} + 2H_{2i}H_{8j})I^{1,1} \\
& + (3H_{11i}H_{3j} + 2H_{4j}H_{7i} + H_{4i}H_{7j} + 4H_{5i}H_{8j} + 6H_{6j}H_{9i})I^{2,1} + (H_{11j}H_{4i} + 3H_{11i}H_{4j} + 2H_{7i}H_{7j} + 6H_{8j}H_{9i})I^{3,1} \\
& + (2H_{11j}H_{7i} + 3H_{11i}H_{7j})I^{4,1} + 3H_{11i}H_{11j}I^{5,1} + (3H_{10j}H_{2i} + 2H_{4i}H_{6j} + H_{3j}H_{8i})I^{0,2} + (3H_{12j}H_{2i} + 6H_{10j}H_{5i} + 4H_{6j}H_{7i} \\
& + H_{4j}H_{8i} + 2H_{4i}H_{8j})I^{1,2} + (6H_{12j}H_{5i} + 6H_{11i}H_{6j} + H_{7j}H_{8i} + 4H_{7i}H_{8j} + 9H_{10j}H_{9i})I^{2,2} + (H_{11j}H_{8i} + 6H_{11i}H_{8j} \\
& + 9H_{12j}H_{9i})I^{3,2} + (H_{12i}H_{3j} + 3H_{10j}H_{4i} + 2H_{6j}H_{8i})I^{0,3} + (3H_{12j}H_{4i} + H_{12i}H_{4j} + 6H_{10j}H_{7i} + 2H_{8i}H_{8j})I^{1,3} + (9H_{10j}H_{11i} \\
& + 6H_{12j}H_{7i} + H_{12i}H_{7j})I^{2,3} + (H_{11j}H_{12i} + 9H_{11i}H_{12j})I^{3,3} + (2H_{12i}H_{6j} + 3H_{10j}H_{8i})I^{0,4} + (3H_{12j}H_{8i} + 2H_{12i}H_{8j})I^{1,4} \\
& + 3H_{10j}H_{12i}I^{0,5} + 3H_{12i}H_{12j}I^{1,5}
\end{aligned}
$$

$$
F_i = H_{1i}I^{0,0} + H_{2i}I^{1,0} + H_{3i}I^{0,1} + H_{4i}I^{1,1} + H_{5i}I^{2,0} + H_{6i}I^{0,2} + H_{7i}I^{2,1} + H_{8i}I^{1,2} + H_{9i}I^{3,0} + H_{10i}I^{0,3} + H_{11i}I^{3,1} + H_{12i}I^{1,3}
$$

Similarly, B_{ij}, R_{ijk}^{x} and R_{ijk}^{y} may be expressed.

4.5 Algorithm to compute element matrices:

1. Input for (x_i, y_i) for $i = 1, 2, 3, \cdots, NP$.
2. Formation of G matrix.
3. Computation of $H = G^{-1}$.
4. Calculation of integrals $I^{\alpha,\beta}$.
5. Computation of $K_{ij}^{xx}, K_{ij}^{yy}, K_{ij}^{xy}, \cdots, F_i$ etc. as mentioned in subsections $4.2 - 4.4$.

V. Application Examples

To show the application of the derived formulae of this work, we consider the following two dimensional boundary value (torsion) problem:

$$
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 2 = 0 \quad \text{within } A
$$

$$
u = 0 \quad \text{on } C_1^* \text{ and } \frac{\partial u}{\partial n} = 0 \quad \text{on } C_2^*
$$

where C_1^* and C_2^* constitute the cross-section boundaries.

5.1 Finite Element Equation:
The field variable *u* (say) governing the physical problem is

$$u = \sum_{i=1}^{NP} u_i N_i(x, y)$$

where $N_i(x, y)$ are shape functions (as given) and

$$NP = \begin{cases} 4 & \text{for the 4-noded quadrilateral} \\ 8 & \text{for the 8-noded quadrilateral} \\ 12 & \text{for the 12-noded quadrilateral} \end{cases}$$

By using the Galerkin weighted residual FE procedure, we achieve the following finite element equations,

$$[K]\{u\} = \{F\}$$

where the components of matrix $[K]$ and $\{F\}$ are

$$K_{ij} = K_{xx}^{ij} + K_{yy}^{ij} \text{ and } F_i = 2\iint_A N_i(x, y)dxdy$$

5.2 Finite Element Procedure:

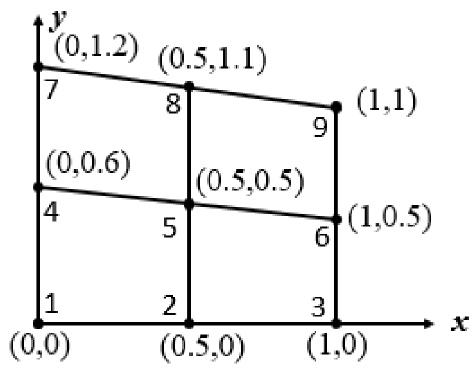
The calculation process consists of the following steps
(*i*) For each element obtain components K_{ij} and F_i
(*ii*) Obtain the global FE equations for the whole system by assembling element equations.
(*iii*) Impose boundary conditions and solve for the generalized stress vectors of the whole system.
(*iv*) Calculate the torsional constant k for which $k = 2\iint_A udxdy$

5.3 Test Problems:

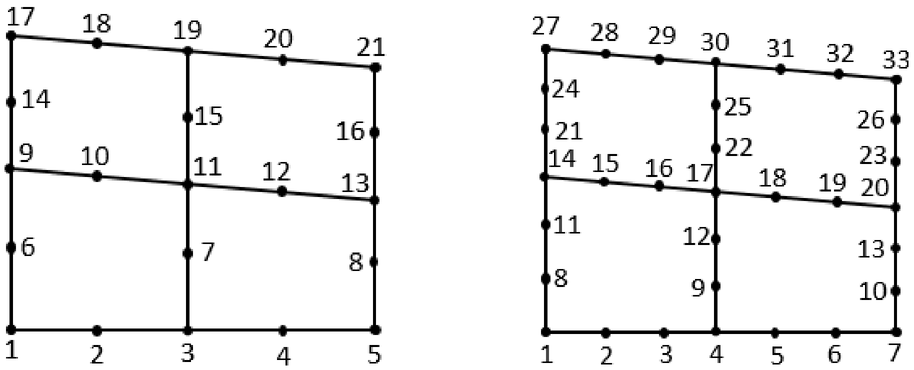
Three examples of solid cross-sections studied in [11] for which either exact or approximate and also FE solutions exists are presented. A measure of error, E_k is provided when an exact solution of the torsional constant k is available. Where

$$E_k = 100\left|1 - \frac{k}{k_{\text{exact}}}\right|$$

Example-1: A trapezoidal cross-section:
The cross-section is modeled by 4-noded, 8-noded and 12-noded quadrilaterals as shown in Fig. 5.1. Computed stress function values and the torsional constant k are tabulated in Table-1. Calculated value of k is in agreement with the result of [11, 17−22].



(a) The FE model-1 with four 4-noded elements



(b) The FE model-2 with four 8-noded elements

(c) The FE model-1 with four 12-noded elements

Fig. 5.1: A trapezoidal cross-section and the FE models with four 4-noded, 8-noded and 12-noded element and global nodes

Table-1

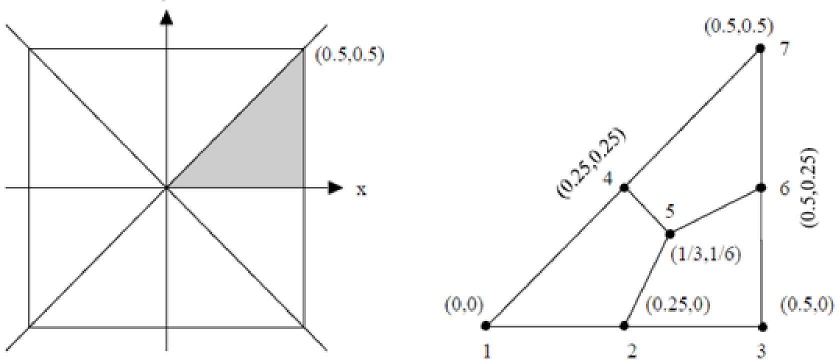
Computed stress function value(s), torsional constant k for example-1.

FE Model No	U_i values				Computed Torsion Constant k
1	U_5	0.20411036036036			0.112106069137319
2	U_7	0.131915219488211	U_{12}	0.130011182721599	0.163774369530811
	U_{10}	0.128537807516107	U_{15}	0.128809621180074	
	U_{11}	0.145877253027271			
3	U_9	0.10503448562457	U_{18}	0.148355120939095	0.168497771701865
	U_{12}	0.149353737686098	U_{19}	0.103180046293444	
	U_{15}	0.101015314201091	U_{22}	0.15129104422434	
	U16	0.147387025495711	U_{25}	0.102964955448969	
	U_{17}	0.150753644844546			

Above table shows very good convergence in the calculation of Prandtle stress function values and the torsional constant. We wish to declare that the solution for torsional constant k may be accepted as the best approximate solution.

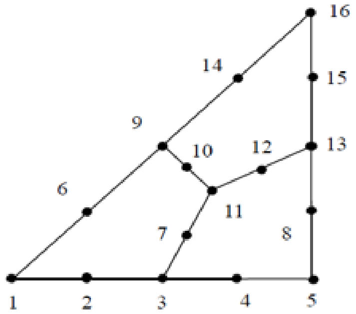
Example-2: **A square cross- section:**

The physical geometry and FE models are shown in Fig.5.2 This cross-section has four axes of symmetry; therefore, only one-eighth of the cross-section needs to be analyzed. This fractional portion is divided into three elements (Fig.5.2(a-c)).We wish to note that at least to the knowledge of present authors the octant of a square had been modeled first time by three quadrilateral elements in [17,20]. Computed results are tabulated in Table-2.



(a) Axes of symmetry of a square cross-section

(b) The FE model-1 with three 4-noded quadrilateral elements



(c) The FE model-1 with three 8-noded quadrilateral elements

Fig. 5.2: The octant of a square cross-section and element subdivision

Table-2

Computed stress function values, torsional constant k and error E_k for example-2

FE Model No	U_i values				Computed Torsion Constant ***k***	E_k
1	U_1	0.15768290	U_4	0.0900712	0.1303802725	7.28512%
	U_2	0.11872934	U_5	0.074753		
2	U_1	0.1471286244	U_9	0.0904859	0.14043709218	0.133623%
	U_2	0.139599417	U_{10}	0.0878646		
	U_3	0.114156447	U_{11}	0.07971913		
	U_4	0.0698718347	U_{12}	0.04339872		
	U_6	0.1317489	U_{14}	0.036225977		
	U_7	0.09980412	---	--------------		
Exact torsion constant ***k*** = 0.140625						

It is to be noted that the results are in good agreement compare to [17, 20] because element matrices are exactly obtained.

Example-3: An equilateral triangular cross-section:

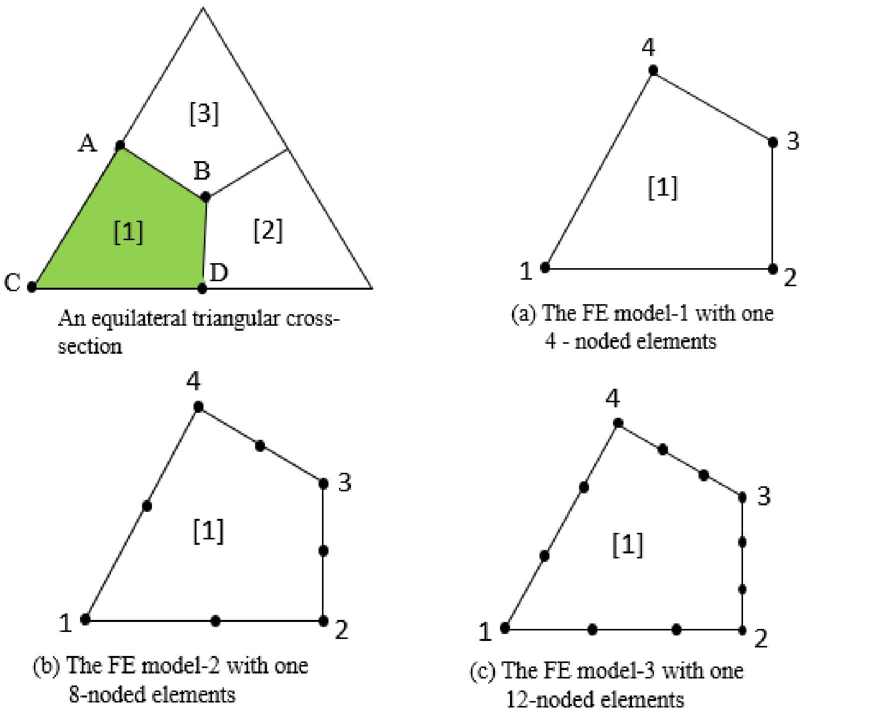


Fig. 5.3: The one third of an equilateral triangular cross-section and element subdivision

The cross- section and FE models are shown in Fig.5.3. Due to symmetry, only one third of the original model is used. u = 0 is specified on sides AC and CD and $\frac{\partial u}{\partial n} = 0$ enforced at the lines of symmetry: sides AB and BD. Calculated stress function values u_i and the torsional constant k are given in Table-3.

Table-3

Computed stress function values, torsional constant k and error E_k for example-3

FE Model No	U_i values				Computed Torsion Constant k	E_k
1	U_5	0.0234375	----		0.140625	549.519%
2	U_4	0.0008896	U_6	0.0240287	0.0193514660942148	10.6194%
	U_5	0.03883635	----			
3	U_5	0.03555065	U_8	0.02154	0.0220314297875825	1.7588%
	U_6	0.03914403	U_9	0.0039038		
	U_7	0.02178696	----			
Exact torsional constant is k = 0.021650635						

Example-4: An elliptic cross-section:

The cross-section and FE models are shown previously in Fig. 3.8 and Fig. 3.8.2. Due to symmetry, only one-fourth of the cross-section is considered for FE models. Computed results are given in Table-4.

Table -4

Computed Prandtl stress values U_i and torsional constant k and error E_k , for example-4

FE Model No	U_i values				Computed Torsion Constant $\boldsymbol{k}$	E_k
1	U_1	4.9755174	U_4	4.761976	142.827008323399	5.239%
	U_2	2.8884254	U_5	1.177573		
2	U_1	3.265994	U_9	1.906576	136.882566278808	0.8589%
	U_2	3.29038	U_{10}	2.527975		
	U_3	2.871832	U_{11}	1.90699		
	U_4	1.6818382	U_{12}	1.10534		
	U_6	2.870329	U_{13}	1.0914		
	U_7	2.5464496				
Exact torsion constant $\boldsymbol{k}$ = 135.716802635079						

We wish to remark here all the computed results tabulated in Tables (1)-(4) are more accurate comparing with the results reported in [11, 16, 18-22].

VI. Conclusion

The strong mathematical foundation of Finite Element Method based on shape functions. The derivations of polynomial shape functions in local co-ordinates are comparatively easier than that of in global co-ordinates. Common practice is the use of polynomial shape functions in local co-ordinates in transformation equations. In such instances complications arise from two main sources, firstly the large number of integrations that need to be performed and secondly, in methods which use isoparametric/ subparametric/ superparametric elements, the presence of the determinant of the Jacobean matrix in the denominator of the stiffness matrix for which the integrands are rational functions. Precisely, to form the element stiffness matrices we are required to evaluate numerous rational integrals. Analytical integration schemes are available only for the integral of bivariate polynomial numerators with bilinear denominators. In practical situation that employs higher order elements, all the integrals are rational with the denominator of higher order bivariate expressions. Such rational integrals cannot be evaluated analytically and we are bound to employ the Gaussian quadrature schemes. Obviously, for the desired accuracy of evaluations the number of Gaussian points and weights are needed to be increased and that increases substantially the computing time. Hence, a proper balance between the accuracy and efficiency is an important task. Further, an attention is always required to select the order of the integrating rule as it is not yet totally worked out.

A suitable alternative, the use of polynomial shape functions in global coordinates in the formulation give rise integrals of polynomials which can be exactly evaluated either by the selected order of the gauss quadrature rule or by analytical schemes. In this case the main barrier is the derivation of shape functions in global coordinates for the element under considerations. Especially it is very much difficult and so cumbersome in case of the quadrilateral elements. Considering all the facts and the popularity of the quadrilateral elements, we have concentrated to derive polynomial shape functions in global coordinates and to present all the components of element matrices in bivariate polynomial form in a systematic way. So that one can use the gaussian quadrature schemes or other numerical schemes easily for exact computing all the element matrices. The technique so developed in this study is as: (1) formation of a matrix G (say) by the global nodal co-ordinates of the element geometry and then its inverse matrix H (say), and (2) the values of the integral of monomials over the element. Finally, we have employed analytical schemes and presented all the components of element matrices as the expressions of products of components of matrix H. Thus, it can be stated shortly that once the matrices G, H are formed and values of the integrals of monomials over the element are evaluated then the computation of element matrices are simply done by the product of components of matrix H only. So, it reduces many time consuming steps of FEM solution procedure and finally that reduces substantially the computational effort.

It has been clearly shown for the first time that the matrix G is singular for two types of element geometry for which bilinear shape functions (for 4-noded quadrilateral) cannot be derived but that of higher order (for 8-noded, 12-noded quadrilaterals) shape functions can be derived. Furthermore, geometrical reasons are also shown for which the matrix G is bad scaled. Through demonstrations of different type of element geometry, it has been found that such difficulties in case of derivation of shape functions may be surmounted only by changing the mesh of the domain. Hence, sincere care is needed in case of deriving polynomial shape functions in global co-ordinates for quadrilateral elements that is to ensure the matrix G is nonsingular. One can easily apply the technique to derive polynomial shape functions in local co-ordinates by forming G matrix by the local nodal co-ordinates**.**

The present technique so presented to compute element matrices exactly is easy for computer coding, a computer code in MATLab® is developed which is not included here with. The accuracy and efficiency of the formulae so presented are then demonstrated through the calculation of Prandtl stress function values and torsional constant of different type of cross sections. Comparison of computed results with the results of other researchers clearly exhibits the best accuracy of the present technique. Since explicit expressions for all types of element matrices are presented, we believe that the technique of this basic study will be contributing and attractive in the realm of application of the finite element method.

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