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A Note on "$\alpha-\phi$ Geraghty contraction type mappings"

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Abstract: *In this paper, a fixed point result for $\alpha-\phi$ Geraghty contraction type mappings has been proved. Karapiner [2] assumes ϕ to be continuous. In this paper, the continuity condition of ϕ has been replaced by a weaker condition and fixed point result has been proved. Thus the result proved generalizes many known results in the literature [2-7].*

Keywords: *Fixed point, α - Geraghty contraction type map, α - ψ Geraghty contraction type, $\alpha-\phi$ Geraghty contraction type, metric space*

I. Introduction

The Banach contraction principle [1], which is a useful tool in the study of many branches of mathematics, is one of the earlier and fundamental results in fixed point theory. A number of authors have improved and extended this result either by defining a new contractive mapping or by investigating the existing contractive mappings in various abstract spaces, see, for e.g.,[2-10].

Geraghty [3] obtained a generalization of Banach contraction principle by considering an auxiliary function β :

Let $\Im$ denote the family of maps $\beta:[0,\infty) \to [0,1)$ satisfying the condition that $\beta(t_n) \to 1$ implies $t_n \to 0$.

He proved the following theorem:

Theorem 1.1: Let (X,d) be a metric space and let T: $X \to X$ be a map. Suppose there exists $\beta \in \Im$ such that for all x, y in X:

$$d(Tx,Ty) \le \beta(d(x,y))\, d(x,y).$$

Then T has a unique fixed point $x_* \in X$ and $\{T^n x\}$ converges to x_* for each $x \in X$.

Cho *et al*., [5] used the concept of α - admissible and triangular α - admissible maps to generalize the result of Geraghty [3].

Definition 1.1: Let T: $X \to X$ be a map and $\alpha : X\times X \to \mathbf{R}$ be a map. Then T is said to be α - admissible if $\alpha(x, y) \ge 1$ implies $\alpha(Tx, Ty) \ge 1$

Definition 1.2: An α - admissible map is said to be triangular α - admissible if

$$\alpha(x, z) \ge 1 \text{ and } \alpha(z, y) \ge 1 \text{ implies } \alpha(x, y) \ge 1$$

Definition 1.3: A map T: $X \to X$ is called a generalized α - Geraghty contraction type if there exists $\beta \in \Im$ such that for all x, y in X:

$$\alpha(x, y)\, d(Tx, Ty) \le \beta(M(x,y))\, M(x,y)$$

Where $M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty)\}$

Cho *et al*., [5] proved the following theorem:

Theorem 1.2: Let (X,d) be a complete metric space. $\alpha : X\times X \to \mathbf{R}$ be a map and let T: $X \to X$ be a map. Suppose the following conditions are satisfied:

1) T is generalized α - Geraghty contraction type map
2) T is triangular α - admissible
3) There exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \ge 1$
4) T is continuous

Then T has a fixed point fixed point $x_* \in X$ and $\{T^n x_1\}$ converges to x_*

Popescu [6] extended this result using concept of α-orbital admissible and triangular α-orbital admissible maps:

Definition 1.4: Let T: $X \to X$ be a map. $\alpha : X\times X \to \mathbf{R}$ be a map .T is said to be α-orbital admissible if

α (x, Tx) $\geq$ 1 implies α (Tx, T^2 x) $\geq$ 1

Definition 1.5: Let T: X$\rightarrow$X be a map. α:X$\times$X$\rightarrow$**R** be a map .T is said to be triangular α-orbital admissible if T is α-orbital admissible and α (x, y) $\geq$ 1 and α (y,Ty) $\geq$ 1implies α (x,Ty) $\geq$ 1

Popescu [6] proved the following theorem:

Theorem 1.3: Let (X,d) be a complete metric space. α:X$\times$X$\rightarrow$**R** be a function. Let T: X$\rightarrow$X be a map. Suppose the following conditions are satisfied:

1) T is generalized α - Geraghty contraction type map
2) T is triangular α - orbital admissible map
3) There exists $x_1 \in$ X such that α (x_1, Tx_1) $\geq$ 1
4) T is continuous

Then T has a fixed point fixed point $x_* \in$ X and { $T^n x_1$} converges to x_*

Karapinar [4], introduced the notion of α - ψ Geraghty contraction type map to extend the result:

Let Ψ denote the class of the functions ψ : [0,∞)→[0,∞) which satisfy the following conditions:

(a) ψ is non-decreasing.

(b) ψ is subadditive, that is, ψ (s+t) ≤ ψ (s)+ ψ (t) for all s, t

(c) ψ is continuous.

(d) ψ (t) = 0⇔t = 0.

Definition 1.6: Let (X,d) be a metric space, and let α : X×X→R be a function. A mapping T : X→X is said to be a generalized α-ψ-Geraghty contraction if there exists β∈ $\Im$ such that

α(x,y) ψ (d(Tx,Ty)) ≤ β(ψ (M(x,y))) ψ (M(x,y)) for any x, y∈X

where M(x,y) = max{d(x,y),d(x,Tx),d(y,Ty)} and ψ ∈Ψ.

Karapinar, E. [4] proved the following theorem:

Theorem 1.4: Let (X,d) be a complete metric space, α : X×X → R be a function and let T : X → X be a map. Suppose that the following conditions are satisfied:

(1) T is generalized α-ψ Geraghty contraction type map

(2) T is triangular α-admissible

(3) there exists x_1 ∈X such that α(x_1,Tx_1)≥1

(4) T is continuous.

Then, T has a fixed point x∗ ∈X, and{$T^n x_1$} converges to x∗

Later Karapinar [2] observed that condition of subadditivity of ψ can be removed:

Let Φ denote the class of functions ϕ :[0,∞) →[0,∞) which satisfy the following conditions:

1) ϕ is nondecreasing
2) ϕ is continuous
3) ϕ (t)=0 iff t=0

Definition 1.7: Let (X,d) be a metric space. α:X$\times$X$\rightarrow$**R** be a map. A mapping T : X $\rightarrow$X is said to be generalized α - ϕ Geraghty contraction type map if there exists $\beta \in \Im$ such that

$$\alpha (x, y) \phi (d(Tx,Ty)) \leq \beta(\phi(M(x,y)))(\phi(M(x,y))) \text{ for all x,y in X}$$

Where M(x,y)=max{d(x,y),d(x,Tx),d(y,Ty)} and $\phi \in \Phi$

Karapinar [2] proved the following theorem:

Theorem 1.5: Let (X,d) be a complete metric space, α : X×X → R be a function, and let T : X → X be a map. Suppose that the following conditions are satisfied:

(1) T is generalized α-φ-Geraghty contraction type map

(2) T is triangular α-admissible

(3) There exists x_1 ∈X such that α(x_1,Tx_1)≥1

(4) T is continuous

Then T has a fixed point x∗ ∈X and {$T^n x_1$} converges to x∗.

In this paper we have shown that above result is true even if the continuity condition of ϕ is replaced by the following weaker condition:

$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = l\ (>0)$ implies $\lim_{n\to\infty} \phi(x_n) = \lim_{n\to\infty} \varphi(y_n) = m$ where $m \in R^+$. (1)

In this regard we have the following theorem:

Theorem 2.1: Let (X,d) be a complete metric space. $\alpha : X \times X \to R$ and $T:X \to X$ be such that :

(1) T is generalized α-φ-Geraghty contraction type map for some $\phi \in \Phi$

(2) T is triangular α-orbital admissible

(3) There exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq 1$

(4) T is continuous

Then T has a fixed point $x_* \in X$, and $\{T^n x_1\}$ converges to x_*

Where Φ denotes the class of functions $\phi : [0,\infty) \to [0,\infty)$ such that

(i) ϕ is non decreasing.

(ii) $\phi(t) = 0$ iff $t = 0$

(iii) $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = s\ (>0)$ implies $\lim_{n\to\infty} \phi(x_n) = \lim_{n\to\infty} \varphi(y_n) = m$ where $m \in \mathbf{R}^+$

Before proving the theorem, we need the following lemma:

Lemma 2.1: $T : X \to X$ be triangular α-orbital admissible. Suppose there exists such that $\alpha(x_1,Tx_1) \geq 1$. Define (x_n) by $x_{n+1} = T(x_n)$ then $\alpha(x_n,x_m) \geq 1$ for all n<m

Proof of lemma: Since T is α-orbital admissible and $\alpha(x_1,Tx_1) \geq 1$. We deduce

$\alpha(x_2,x_3) = \alpha(Tx_1,Tx_2) \geq 1$.

Continuing this way, we get, $\alpha(x_n,x_{n+1}) \geq 1$ for all n. suppose $\alpha(x_n,x_m) \geq 1$ where m>n. Since T is triangular α-orbital admissible and $\alpha(x_m,x_{m+1}) \geq 1$, we get $\alpha(x_n,x_{m+1}) \geq 1$.Thus lemma is proved.

Proof of main theorem 2.1: let $x_1 \in X$ be such that $\alpha(x_1,Tx_1) \geq$. Define (x_n) by $x_{n+1} = T(x_n)$

Now we will prove $\lim d(x_n,x_{n+1}) = 0$.

By lemma , $\alpha(x_n,x_{n+1}) \geq 1$ for all n. (2)

$$\phi(d(x_{n+1},x_{n+2})) = \phi(d(Tx_n,Tx_{n+1})) \leq \alpha(x_n,x_{n+1})\phi(d(Tx_n,Tx_{n+1}))$$
$$\leq \beta(\phi(M(x_n,x_{n+1})))\phi(M(x_n,x_{n+1})) \quad (3)$$

Where $M(x_n,x_{n+1}) = \max\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})\}$

Now $M(x_n,x_{n+1}) = d(x_{n+1},x_{n+2})$ is not possible.

Since if $M(x_n,x_{n+1}) = d(x_{n+1},x_{n+2})$ we will have

$$\phi(d(x_{n+1},x_{n+2})) \leq \beta(\phi(M(x_n,x_{n+1})))\phi(M(x_n,x_{n+1}))$$
$$\leq \beta(\phi(d(x_{n+1},x_{n+2})))\phi(d(x_{n+1},x_{n+2}))$$
$$< \phi(d(x_{n+1},x_{n+2}))$$

which is a contradiction.

Thus $M(x_n,x_{n+1}) = d(x_n,x_{n+1})$

Using Eq. (3), we get,

$$\phi(d(x_{n+1},x_{n+2})) < \phi(d(x_n,x_{n+1})) \Rightarrow d(x_{n+1},x_{n+2}) < (d(x_n,x_{n+1}) \text{ for all n}$$

Thus the sequence $\{d(x_n,x_{n+1})\}$ is non-negative and monotonically decreasing

This implies that $\lim_{n\to\infty} d(x_n,x_{n+1})) = r\ (\geq 0)$

Claim r = 0

If r > 0, from Eq. (3),

$$\frac{\phi(d(x_{n+1},x_n))}{\phi(M(x_n,x_{n+1}))} \le \beta(\phi(M(x_n,x_{n+1})) < 1$$

$\Rightarrow \lim \beta(\phi(M(x_n,x_{n+1})) = 1$

$\Rightarrow \lim \phi(M(x_n,x_{n+1})) = 0$

$\Rightarrow r = \lim d(x_n,x_{n+1}) = 0$ (4)

Now let (x_n) be not Cauchy .Thus , there exists $\in > 0$ such that
Given k there exists m(k) > n(k) > k such that

$d(x_{n(k)},x_{m(k)}) \ge \in$ but $d(x_{n(k)},x_{m(k)-1}) < \in$

$\in \le d(x_{n(k)},x_{m(k)}) \le d(x_{n(k)},x_{m(k)-1}) + d(x_{m(k)-1},x_{m(k)}) < \in + d(x_{m(k)-1},x_{m(k)})$

This implies $\lim_{k\to\infty} d(x_{n(k)},x_{m(k)}) = \in$

$\lim_{k\to\infty} \phi(d(x_{n(k)},x_{m(k)})) > 0$

Also $\lim_{k\to\infty} d(x_{m(k)-1},x_{n(k)-1}) = \in$

Now $\phi(d(x_{m(k)},x_{n(k)})) = \phi(d(Tx_{m(k)-1},Tx_{n(k)-1})) \le \alpha(x_{m(k)-1},x_{n(k)-1})\phi(d(Tx_{m(k)-1},Tx_{n(k)-1}))$

$\le \beta(\phi(M(x_{m(k)-1},x_{n(k)-1})))\,\phi(M(x_{m(k)-1},x_{n(k)-1}))$

$$\Rightarrow \frac{\phi(d(x_{m(k)},x_{n(k)}))}{\phi(M(x_{m(k)-1,}x_{n(k)-1}))} \le \beta(\phi(M(x_{m(k)-1},x_{n(k)-1}))) \quad (5)$$

Now $d(x_{m(k)},x_{n(k)}) \to \in$ and $M(x_{m(k)-1},x_{n(k)-1}) \to \in$
Thus by assumption:

$\lim \phi(d(x_{m(k)},x_{n(k)})) = \lim \phi(M(x_{m(k)-1},x_{n(k)-1}))$ and it will be +ve.

Thus by (5), $\lim \beta(\phi(M(x_{m(k)-1},x_{n(k)-1}))) = 1$

$\Rightarrow \phi(M(x_{m(k)-1},x_{n(k)-1})) \to 0$

$\Rightarrow M(x_{m(k)-1},x_{n(k)-1}) \to 0$

$\Rightarrow d(x_{m(k)-1},x_{n(k)-1}) \to 0$ which is a contradiction.

Thus the sequence (x_n) is Cauchy.
Hence the result.

Example: define a map, $\phi : \mathbf{R} \to \mathbf{R}$ as follows:

$\phi(x) = 1$ if $x > 0$ & $\phi(x) = 0$ if $x \le 0$

Clearly, ϕ is discontinuous but it satisfies the condition given in Eq. (1)
Thus our result applies to a wider class of mappings.

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