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Connected and Distance in G $\otimes_2$ H

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Abstract : *The tensor product $G \otimes H$ of two graphs G and H is well-known graph product and studied in detail in the literature. This concept has been generalized by introducing 2-tensor product $G \otimes_2 H$ and it has been discussed for special graphs like P_n and C_n [5]. In this paper, we discuss $G \otimes_2 H$, where G and H are connected graphs. Mainly, we discuss connectedness of $G \otimes_2 H$ and obtained distance between two vertices in it.*

Keywords: *Bipartite graph, Connected graph, Non-bipartite graph, 2-tensor product of graphs.*

I. Introduction

The tensor product $G \otimes H$ of two graphs G and H is very well-known and studied in detail ([1], [2], [3], [4]). This concept has been extended by introducing $2-$tensor product $G \otimes_2 H$ of G and H and studied for special graphs [5]. In this paper, we discuss connectedness of $G \otimes_2 H$ for any connected graphs G and H. We also obtained the results for the distance between two vertices in $G \otimes_2 H$.

If $G = (V(G), E(G))$ is finite, simple and connected graph, then $d_G(u,u')$ is the length of the shortest path between u and u' in G. For a graph G, a maximal connected subgraph is a component of G. For the basic terminology, concepts and results of graph theory, we refer to ([1], [6], [7]).

We recall the definition of $2-$tensor product of graphs.

Definition 1.1 *[5] Let G and H be two connected graphs. The $2-$tensor product of G and H is the graph with vertex set $\{(u,v) : u \in V(G), v \in V(H)\}$ and two vertices (u,v) and (u',v') are adjacent in $2-$tensor product if* $d_G(u,u') = 2$ and $d_H(v,v') = 2$. It is denoted by $G \otimes_2 H$.

Note that $G \otimes_2 H$ is a null graph, if the diameter $D(G) < 2$ or $D(H) < 2$. So, throughout this paper we assume that G and H are non-complete graphs.

II. Connectedness of $G \otimes_2 H$

this section, first we consider the graphs G and H, both connected and bipartite with $N^2(w) \neq \phi;\ \forall\ w \in V(G) \cup V(H)$, where $N^2(u) = \{u' \in V(G) : d_G(u,u') = 2\}$

In usual tensor product $G \otimes H$, the following result is known.

Proposition 2.1 *[4] Let G and H be connected bipartite graphs. Then $G \otimes H$ has two components.*

Note that in case of $G \otimes_2 H$, the similar result is not true. We discuss the number of components in $G \otimes_2 H$ with different conditions on G and H.

We fix the following notations

Let $V(G) = U_1 \cup U_2$ and $V(H) = V_1 \cup V_2$ with U_i and V_j, $(i, j = 1,2)$ are partite sets of G and H respectively. Then, $V(G \otimes_2 H) = W_{11} \cup W_{12} \cup W_{21} \cup W_{22}$ with $W_{ij} = U_i \times V_j$

Remark 2.2 *If (u,v) and (u',v') are from different W_{ij}, then (u,v) and (u',v') can not be adjacent in $G \otimes_2 H$ as $d_G(u,u') \neq 2$ or $d_H(v,v') \neq 2$. So, $G \otimes_2 H$ has at least four components. Suppose (u,v) and (u',v') are in the same W_{ij}. Then $d_G(u,u')$ and $d_H(v,v')$ are even.*

Proposition 2.3 *Let G and H be connected bipartite graphs. If $d_G(u,u')$ and $d_H(v,v')$ are of the same form, $4k$ or $4k+2$, $(k \in \mathbb{N} \cup \{0\})$ then (u,v) and (u',v') are in the same component of $G \otimes_2 H$.*

*Proof.*Let (u,v) & $(u',v') \in U_1 \times V_1$.Suppose, $P_1 : u = u_0 \to u_1 \to \ldots \to u_m = u'$ and $P_2 : v = v_0 \to v_1 \to \ldots \to v_n = v'$ are paths between u, u' and v, v' respectively.

Suppose $l(P_1) = 4k / 4k+2$ and $l(P_2) = 4t / 4t+2$ with $k \le t$. First assume that $k \neq 0 \neq t$, then there is a path P or P' between (u,v) and (u',v') in $G \otimes_2 H$ as follows:

$P:(u_0,v_0)\to(u_2,v_2)\to\ldots\to(u_{4k},v_{4k})\to(u_{4k-2},v_{4k+2})\to(u_{4k},v_{4k+4})\to\ldots\to(u_{4k},v_{4t})=(u',v')$.
$P':(u_0,v_0)\to(u_2,v_2)\to\ldots\to(u_{4k+2},v_{4k+2})\to(u_{4k},v_{4k+4})\to(u_{4k+2},v_{4k+6})\to\ldots\to(u_{4k+2},v_{4t+2})=(u',v')$.

Next, assume that $k=0$, i.e. $l(P_1)=0$ or 2, i.e. $u=u'$ or $u=u_0\to u_1\to u_2=u'$. Now as $N^2(u)\neq\phi$, $\exists\, a\in V(G)$ such that $d_G(u,a)=2$. So, in case of $l(P_1)=0$ and $l(P_2)=4t$, we get the path between (u,v) and (u,v') in $G\otimes_2 H$ as follows:
$(u,v)=(u,v_0)\to(a,v_2)\to(u,v_4)\to\ldots\to(u,v_{4t})$.

Next if $l(P_1)=2$ and $l(P_2)=4t+2$, then we get the path between (u_0,v) and (u_2,v') in $G\otimes_2 H$ as follows:
$(u,v)=(u_0,v_0)\to(u_2,v_2)\to(u_0,v_4)\to(u_2,v_6)\to\ldots\to(u_2,v_{4t+2})=(u_2,v')$.
Thus in all cases there is a path from (u,v) to (u',v') in $G\otimes_2 H$. Which completes the proof.

Remarks 2.4

[i] Suppose (u,v) and (u',v') are in same W_{ij}. But if $d_G(u,u')$ and $d_H(v,v')$ are not of the same form, then (u,v) and (u',v') may be in different components. So, $U_1\times V_1$ give at most two components. Thus $G\otimes_2 H$ has at most eight components.

[ii] Suppose $\Delta(G)\le 2$ and $\Delta(H)\le 2$, $\Delta(G)$ and $\Delta(H)$ are maximum degree of G and H respectively. Then G and H are either path or cycle. If the cycle is of the form C_{4l}, then in each of the cases, $P_m\otimes_2 P_n$, $P_m\otimes_2 C_{4n}$ and $C_{4m}\otimes_2 C_{4n}$ have eight components [5].

Next, we discuss the conditions on G and H under which $G\otimes_2 H$ has 4, 5 or 6 components.

Proposition 2.5 *Let G and H be connected bipartite graphs and at least one of the graphs contains a cycle C_{4l+2} $(l\in\mathbb{N})$. Then $G\otimes_2 H$ has exactly four components.*

Proof. Let (u,v) and (u',v') be in $U_1\times V_1$. As we have seen in Proposition 2.3, if $d_G(u,u')$ and $d_H(v,v')$ are of the same form $4k$ or $4k+2$, then $U_1\times V_1$ gives connected component.

Let P_1 and P_2 be two paths between $u-u'$ and $v-v'$ in G and H respectively, as we have considered in Proposition 2.3. Suppose $l(P_1)$ and $l(P_2)$ are of the different form.

Suppose H contains a cycle C_{4l+2} with $V(C_{4l+2})=\{x_1,x_2,\ldots,x_{4l+2}\}$. Then select a vertex from C_{4l+2}, which is nearest to $v_n=v'$ and also it is in V_1. Suppose this vertex is x_j. Since v_n and x_j both are in V_1, we get a path P_0 from v_n to x_j of even length. We consider a walk $W: v=v_0\xrightarrow{P_2}v_n\xrightarrow{P_0}x_j\xrightarrow{C_{4l+2}}x_j\xrightarrow{P_0}v_n=v'$ between v and v' in H. Then,
$l(W)=l(P_2)+2(l(P_0))+l(C_{4l+2})=l(P_2)+2(2t')+(4l+2)=l(P_2)+4t''+2$.

Thus, if $l(P_2)=4t+2$ or $4t$, then $l(W)=4q$ or $4q+2$. So, in any case, $l(P_1)$ and $l(W)$ are of the same form and therefore, as in Proposition 2.3, we can show that there is a path between (u,v) and (u',v') in $G\otimes_2 H$. So, $U_1\times V_1$ gives a connected components in $G\otimes_2 H$. Thus, $G\otimes_2 H$ has four components.

Next, we assume that the graphs G and H do not contain a cycle of the form C_{4l+2}. We prove that the number of components in $G\otimes_2 H$ is depending upon $\Delta(G)$ as well as $\Delta(H)$.

Let $\Delta(U_i)=\max\{d(u):u\in U_i\}$ and $\Delta(V_i)=\max\{d(v):v\in V_i\}$; $i=1,2$ and for $a\in V(G)$, $N(a)=\{b\in V(G):d_G(a,b)=1\}$.

Proposition 2.6 *Let G and H be connected bipartite graphs with $\Delta(G)\le 2$ and $\Delta(H)\ge 3$.*

(a) If $\Delta(V_1)\le 2$ and $\Delta(V_2)\ge 3$, then $G\otimes_2 H$ has six components.

(b) If $\Delta(V_1)\ge 3$ and $\Delta(V_2)\ge 3$, then $G\otimes_2 H$ has four components.

Proof. We know that $U_1\times V_1$, $U_1\times V_2$, $U_2\times V_1$ and $U_2\times V_2$ give disconnected subgraphs in $G\otimes_2 H$.

(a) Fixed $U_1\times V_1$. We shall show that $U_1\times V_1$ gives connected subgraph of $G\otimes_2 H$. Let $z\in V_2$ with $d(z)\ge 3$ and $N(z)=\{w_0,w_1,w_2,\ldots\}\subset V_1$.

Fixed $(u_0,v_0)\in U_1\times V_1$ with $v_0=w_0$. Let (u',v') be any vertex in $U_1\times V_1$. Suppose, P_1 and P_2 are paths between $u=u_0$ to u' and $v=v_0$ to v' in G and H, as we have considered in Proposition 2.3. If $l(P_1)=4k$ or $4k+2$ and $l(P_2)=4t$ or $4t+2$, then the result is clear.

Next, suppose $l(P_1) = 4k$ and $l(P_2) = 4t + 2$; $k \le t$ with $k \ne 0 \ne t$. First, we show that for $v_0 \to v_1 \to v_2$ there is a path between (u_0, v_0) and (u_0, v_2) in $G \otimes_2 H$.

Case (1) Suppose $z \ne v_1$ in V_2.

If $v_2 \ne w_1$ in V_1 as given in figure.1, then $(u_0,v_0) \to (u_2,w_1) \to (u_0,w_2) \to (u_2,v_0) \to (u_0,v_2)$ is a path between (u_0,v_0) and (u_0,v_2) in $G \otimes_2 H$.

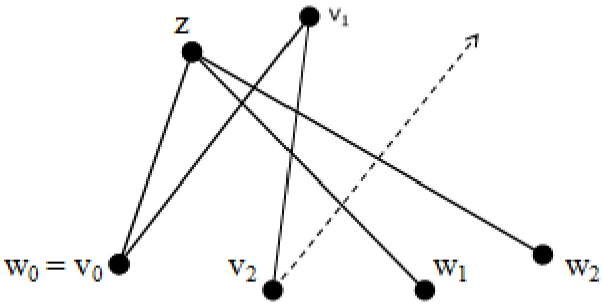

figure.1

If $v_2 = w_1$ in V_1, then, $(u_0,v_0) \to (u_2,w_2) \to (u_0,v_2)$ is the required path.

Case (2) Suppose $z = v_1$ in V_2.

If $w_1 \ne v_2 \ne w_2$ in V_1, then $(u_0,v_0) \to (u_2,w_2) \to (u_0,v_2)$ is the path and if $v_2 = w_2$, then consider the path $(u_0,v_0) \to (u_2,w_1) \to (u_0,v_2)$ in $G \otimes_2 H$.

Thus in each case there is a path between (u_0,v_0) and (u_0,v_2) in $G \otimes_2 H$. Also as in Proposition 2.3 there is a path from (u_0,v_2) to (u_{4k}, v_{4t+2}) in $G \otimes_2 H$. Hence there is a path from (u_0,v_0) to (u', v') in $G \otimes_2 H$. By similar arguments if $l(P_1) = 4k + 2$ and $l(P_2) = 4t$, then also there is a path between (u_0,v_0) and (u', v') in $G \otimes_2 H$. So, $U_1 \times V_1$ gives a connected component in $G \otimes_2 H$.

Thus if $d(z) \ge 3$ with $z \in V_2$, then the other partite set V_1 contribute connected components $U_1 \times V_1$ and $U_2 \times V_1$ in $G \otimes_2 H$.

Here as $\Delta(U_i) \le 2$ and $\Delta(V_2) \le 2$, $U_1 \times V_2$ as well as $U_2 \times V_2$ each give two components in $G \otimes_2 H$. So, the graph $G \otimes_2 H$ has six components.

(b) In this case $\Delta(V_1) \ge 3$ and $\Delta(V_2) \ge 3$. So $U_1 \times V_1$, $U_1 \times V_2$ and $U_2 \times V_1$ and $U_2 \times V_2$ will give connected components in $G \otimes_2 H$. So, the graph $G \otimes_2 H$ has four components.

Corollary 2.7 *Let G and H be connected bipartite graphs with $\Delta(G) \ge 3$ and $\Delta(H) \ge 3$.*

(a) If $\Delta(U_1) \le 2$ and $\Delta(U_2) \ge 3$ as well as $\Delta(V_1) \le 2$ and $\Delta(V_2) \ge 3$, then $G \otimes_2 H$ has five components.

(b) If $\Delta(U_i) \ge 3$ and $\Delta(V_i) \ge 3$; $(i = 1, 2)$, then $G \otimes_2 H$ has four components.

Proof. (a) In this case $\Delta(U_2) \ge 3$ and $\Delta(V_2) \ge 3$. Since $\Delta(U_2) \ge 3$, the other partite set U_1 contribute connected components $U_1 \times V_1$ and $U_1 \times V_2$ in $G \otimes_2 H$. Similarly as $\Delta(V_2) \ge 3$, $U_1 \times V_1$ and $U_2 \times V_1$ give connected components in $G \otimes_2 H$.

Further as $\Delta(U_1) \le 2$ as well as $\Delta(V_1) \le 2$, corresponding to other partite set U_2 and V_2, we get two components for $U_2 \times V_2$ in $G \otimes_2 H$. Thus the graph $G \otimes_2 H$ has five components.

(b) By similar arguments as given in Proposition 1.6, for $\Delta(U_i) \ge 3$; $i = 1, 2$, we get connected components $U_1 \times V_1$, $U_1 \times V_2$, $U_2 \times V_1$ and $U_2 \times V_2$ in $G \otimes_2 H$. Thus, the graph $G \otimes_2 H$ has four components.

In general from Remarks 2.4, Proposition 2.6 and Corollary 2.7, we can summarize the number of components in $G \otimes_2 H$ as follows:

H \ G	$\Delta(V_1) \le 2$, $\Delta(V_2) \le 2$	$\Delta(V_1) \le 2$, $\Delta(V_2) \ge 3$	$\Delta(V_1) \ge 3$, $\Delta(V_2) \le 2$	$\Delta(V_1) \ge 3$, $\Delta(V_2) \ge 3$
$\Delta(U_1) \le 2$, $\Delta(U_2) \le 2$	8	6	6	4
$\Delta(U_1) \le 2$, $\Delta(U_2) \ge 3$	6	5	5	4
$\Delta(U_1) \ge 3$, $\Delta(U_2) \le 2$	6	5	5	4
$\Delta(U_1) \ge 3$, $\Delta(U_2) \ge 3$	4	4	4	4

Next, we discuss connectedness of $G \otimes_2 H$ for non-bipartite graphs. First we shall prove the following Proposition:

Proposition 2.8 *labelnon Let G be a non-bipartite connected graph with $N^2(u) \ne \phi, \forall u \in V(G)$. Assume that G contains C_{2l+1}, $l > 1$. Then between every pair of vertices, there exists a walk of length $4k$ as well as $4k+2$; $(k \in \mathbb{N} \cup \{0\})$ form in G.*

Proof. Since G is non-bipartite, it contains an odd cycle. Suppose G contains C_{2l+1} with $V(C_{2l+1}) = \{x_1, \ldots, x_{2l+1}\}$, $l > 1$. Let u and u' be in $V(G)$ with path $P: u = u_0 \to u_1 \to \ldots \to u_{2t+1} = u'$, where $l(P) = d_G(u, u') = 2t+1$.

Suppose u and u' are on C_{2l+1}. Then clearly there is a path between u and u' of even length.

Next, assume that u, $u' \notin V(C_{2l+1})$. Assume that u_i is the nearest vertex from the cycle C_{2l+1} and x_j is the corresponding nearest vertex of $V(C_{2l+1})$. Suppose P_0 is the path between u_i and x_j in G. Then there is a walk W' between u and u' in G as follows:

$W': u = u_0 \xrightarrow{\text{part of } P} u_i \xrightarrow{P_0} x_j \xrightarrow{C_{2l+1}} x_j \xrightarrow{P_0} u_i \xrightarrow{\text{part of } P} u'$.

Then $l(W') = l(P) + l(C_{2l+1}) + 2l(P_0) = (2t+1) + (2l+1) + 2l(P_0)$, which is of even length.

If necessary travelling on the cycle more than once, we get length of the walk in both the form $4k$ as well as $4k+2$ in G. If $l(P)$ is even, then by same arguments as above we get a walk of length $4k$ and $4k+2$ in G. Thus in all cases there is a walk between u and u' of length $4k$ as well as $4k+2$ form in G.

Note that since $l > 1$, in every walk $W': u = w_0 \to w_1 \to \ldots \to w_p = u'$ between u and u' in above cases, we get $d_G(w_i, w_{i+2}) = 2$.

Now onwards, whenever we consider a non-bipartite graph, we assume that it contain a cycle C_{2l+1}, $(l > 1)$.

Proposition 2.9 *Let G and H be two connected graphs. Suppose G is a non-bipartite graph. Then,*
(a) the graph $G \otimes_2 H$ has two components, if H is a bipartite graph.
(b) the graph $G \otimes_2 H$ is connected, if H is a non-bipartite graph.

Proof. (a) Suppose H is a bipartite graph. It is clear that $U \times V_1$ and $U \times V_2$ give two disconnected subgraphs in $G \otimes_2 H$.

Let (u, v) and (u', v') be in $U \times V_1$.

Let path P_1 between u and u' in G and path $P_2: v = v_0 \to v_1 \to \ldots \to v_n = v'$; ($n$ is an even integer) between v and v' in H be as follows:

Since G is non-bipartite graph, by Proposition 1.11, there are walks between u and u' of length of the form $4k$ as well as $4k+2$. Since n is an even integer, as we have discussed in Proposition 2.3, we get a path between (u, v) and (u', v') in $G \otimes_2 H$. Thus $U \times V_1$ gives a connected component, which proves (a).

(b) Let (u, v) and (u', v') be in $U \times V$. Since G and H both are non-bipartite graphs, there exist walks between v and v' of length $4k$ as well as $4k+2$ form. So, as above we get the result.

Corollary 2.10 *Let G and H be two connected graphs. Then $G \otimes_2 H$ is connected if and only if G and H both are non-bipartite graphs.*

Note that the similar result for usual Tensor product is as follows:

Proposition 2.11 *[7] Let G and H be connected graphs. Then $G \otimes H$ is connected graph if and only if either G or H is non-bipartite.*

III. Distance between two vertices in $G \otimes_2 H$

In this section, we discuss the distance between two vertices in $G \otimes_2 H$ for G and H both are connected and $N^2(w) \neq \phi; \forall w \in V(G) \cup V(H)$.

First we define $d_G^*(u,u')$ and $d_G^{**}(u,u')$ for u and u' in $V(G)$, where G is a connected graph.

Definition 2.1 *Let $G = (U, E)$ be a connected graph and $u, v \in U$. Then,*

(1) $d_G^*(u,u')$ is defined as the length of a shortest walk $W : u = w_0 \to w_1 \to \ldots \to w_p = u'$ between u and u' of the form $4k$ $(k \in \mathrm{IN})$ in which $d_G(w_i, w_{i+2}) = 2$ for $i = 0, 2, 4, \ldots, 4k-2$. (2) $d_G^{**}(u,u')$ is defined as the length of a shortest walk $W : u = w_0 \to w_1 \to \ldots \to w_p = u'$ between u and u' of the form $4k+2$ $(k \geq 0)$ in which $d_G(w_i, w_{i+2}) = 2$ for $i = 0, 2, 4, \ldots, 4k$.

Note that $$d_G(u, u') = \begin{cases} d_G^*(u,u') < d_G^{**}(u,u'), & \text{if } d_G(u,u') = 4k \\ d_G^{**}(u,u') < d_G^*(u,u'), & \text{if } d_G(u,u') = 4k+2 \end{cases}$$

If there is no such shortest walk, then we write $d_G^*(u,u') = \infty$ $(d_G^{**}(u,u') = \infty)$.

Remark 3.2

[i] If G is a non-bipartite graph, then $d_G^*(u,u') < \infty$ and $d_G^{**}(u,u') < \infty$ for every $u, u' \in V(G)$, by Proposition 1.8.

[ii] If G is a bipartite graph and even, also if $d_G(u,u')$ is an even number $4k$, then $d_G^{**}(u,u')$ may not be finite.

For example, if $G = P_6 : u_1 \to u_2 \to \ldots \to u_5 \to u_6$, then $d_G(u_1, u_5) = 4$ but $d_G^{**}(u_1, u_5) = \infty$. However, if $G = C_{10}$, then $d_G(u_1, u_5) = 4$ but $d_G^{**}(u_1, u_5) = 6 < \infty$.

Now we fix the following notations:

Let $G = (U, E_1)$ and $H = (V, E_2)$ be connected graphs.Then $V(G \otimes_2 H) = U \times V$. Fix $u, u' \in U$, suppose $d_G = d_G(u,u') = m$ with path $P_1 : u = u_0 \to u_1 \to \ldots \to u_m = u'$ and $v, v' \in V$, $d_H = d_H(v,v') = n$ with path $P_2 : v = v_0 \to v_1 \to \ldots \to v_n = v'$. Denote $d = d_{G \otimes_2 H}((u,v),(u',v'))$. We assume that (u,v) and (u',v') are in the same component of $G \otimes_2 H$, i.e., $d < \infty$.

Proposition 3.3 *If d_G and d_H are of the same form $4k$ or $4k+2$, then $d = Max\left\{\frac{1}{2} d_G, \frac{1}{2} d_H\right\}$.*

Proof. Let $d_G = 4k$ and $d_H = 4t$; $k \leq t$. Then using paths P_1 and P_2 from u to u' and v to v' in G and H respectively, there is a path

$$P : (u,v) = (u_0, v_0) \to (u_2, v_2) \to \ldots \to (u_{4k}, v_{4k}) \to (u_{4k-2}, v_{4k+2}) \to (u_{4k}, v_{4k+4}) \to \ldots \to (u_{4k}, v_{4t}) = (u', v')$$

between (u,v) and (u',v') of length $2t$ in $G \otimes_2 H$. So, $d \leq 2t = \frac{1}{2} d_H = Max\left\{\frac{1}{2} d_G, \frac{1}{2} d_H\right\}$. Similarly if $d_G = 4k+2$ and $d_H = 4t+2$, then path

$$P' : (u,v) = (u_0, v_0) \to (u_2, v_2) \to \ldots \to (u_{4k+2}, v_{4k+2}) \to (u_{4k}, v_{4k+4}) \to (u_{4k+2}, v_{4k+6}) \to \ldots \to (u_{4k+2}, v_{4t+2}) = (u', v')$$

between (u,v) and (u',v') of length $2t+1$ in $G \otimes_2 H$. So, $d \leq 2t+1 = \frac{1}{2} d_H = Max\left\{\frac{1}{2} d_G, \frac{1}{2} d_H\right\}$.

Conversely suppose that $d < \infty$ with the path $(u,v) = (u_0, v_0) \to (u_1, v_1) \to \ldots \to (u_d, v_d) = (u', v')$ in $G \otimes_2 H$. Then $d_G(u_i, u_{i+1}) = 2 = d_H(v_i, v_{i+1}); \forall i$. So, there is a walk $W_G : u = u_0 \to a_0 \to u_1 \to \ldots \to u_d = u'$ of length $2d$ between u and u' in G with $u_i \neq u_{i+1}$. Similarly we get a walk W_H between v and v' in H. Hence $d_G \leq 2d$ and $d_H \leq 2d$. So, Max$\left\{\frac{1}{2} d_G, \frac{1}{2} d_H\right\} \leq d$. Thus we get Max$\left\{\frac{1}{2} d_G, \frac{1}{2} d_H\right\} = d$.

Next, we consider the case in which d_G and d_H are not in same form, but both are even.

Proposition 3.4 *If $d_G = 4k$ and $d_H = 4t+2$, then $d = Min\left\{Max\left\{\frac{1}{2} d_G, \frac{1}{2} d_H^*\right\}, Max\left\{\frac{1}{2} d_G^{**}, \frac{1}{2} d_H\right\}\right\}$.*

Proof. First we prove that $d \le Min\left\{Max\left\{\frac{1}{2}d_G, \frac{1}{2}d_H^*\right\}, Max\left\{\frac{1}{2}d_G^{**}, \frac{1}{2}d_H\right\}\right\}$.

If $d_H^* = \infty = d_G^{**}$, then it is clear.

Suppose $d_H^* < \infty$. Suppose $d_G = 4k$ and $d_H^* = 4t'; k \le t'$. Then there is a shortest walk $W_2 : v = w_0 \to w_1 \to \ldots \to w_{4t'} = v'$ such that $d_H(w_i, w_{i+2}) = 2$ for $i = 0, 2, 4, \ldots, 4t'-2$. So, using path P_1 and walk W_2, we get a path between (u,v) and (u',v') in $G \otimes_2 H$, as in Proposition 3.3. So, $d \le 2t' = \frac{1}{2}d_H^* = Max\left\{\frac{1}{2}d_G, \frac{1}{2}d_H^*\right\}$.Similarly, if $d_G^{**} = 4k'+2$, then $d \le Max\left\{\frac{1}{2}d_G^{**}, \frac{1}{2}d_H\right\}$.

Hence we get $d \le Min\left\{Max\left\{\frac{1}{2}d_G, \frac{1}{2}d_H^*\right\}, Max\left\{\frac{1}{2}d_G^{**}, \frac{1}{2}d_H\right\}\right\}$.

For the reverse inequality, as $d < \infty$, as we have seen in Proposition 3.3, there are walks W_G and W_H between $u-u'$ and $v-v'$ respectively with $l(W_G) = 2d = l(W_H)$. Also, as $d_G = 4k$ and $d_H = 4t+2$, we get $d_G = d_G^* < d_G^{**}$ and $d_H = d_H^{**} < d_H^*$.

Suppose d is even. Let $d = 2p$.Then $l(W_G) = 4p = l(W_H)$. So, $d_G^* \le 4p$ as well as $d_H^* \le 4p$. Thus $Max\left\{d_G, d_H^*\right\} = Max\left\{d_G^*, d_H^*\right\} \le 4p = 2d$. If $d = 2p+1$, then $l(W_G) = 4p+2 = l(W_H)$. So, $d_G^{**} \le 4p+2$ and $d_H^{**} \le 4p+2$ and therefore $Max\{d_G^{**}, d_H\} \le 4p+2 = 2d$. Hence

$$Min\left\{Max\left\{\frac{1}{2}d_G, \frac{1}{2}d_H^*\right\}, Max\left\{\frac{1}{2}d_G^{**}, \frac{1}{2}d_H\right\}\right\} = Min\left\{Max\left\{\frac{1}{2}d_G^*, \frac{1}{2}d_H^*\right\}, Max\left\{\frac{1}{2}d_G^{**}, \frac{1}{2}d_H^{**}\right\}\right\} \le d$$

Corollary 3.5 *Let d_H be an odd integer.*

(i) If d_G is odd, then $d = Min\left\{Max\left\{\frac{1}{2}d_G^*, \frac{1}{2}d_H^*\right\}, Max\left\{\frac{1}{2}d_G^{**}, \frac{1}{2}d_H^{**}\right\}\right\}$.

(ii) If $d_G = 4k$, then $d = Min\left\{Max\left\{\frac{1}{2}d_G, \frac{1}{2}d_H^*\right\}, Max\left\{\frac{1}{2}d_G^{**}, \frac{1}{2}d_H^{**}\right\}\right\}$.

(iii) If $d_G = 4k+2$, then $d = Min\left\{Max\left\{\frac{1}{2}d_G^*, \frac{1}{2}d_H^*\right\}, Max\left\{\frac{1}{2}d_G, \frac{1}{2}d_H^{**}\right\}\right\}$.

Proof. (i) Suppose If d_G, and d_H both are odd integers.

First we prove that $d \le Min\left\{Max\left\{\frac{1}{2}d_G^*, \frac{1}{2}d_H^*\right\}, Max\left\{\frac{1}{2}d_G^{**}, \frac{1}{2}d_H^{**}\right\}\right\}$.

Suppose $Max\left\{\frac{1}{2}d_G^*, \frac{1}{2}d_H^*\right\} = \infty = Max\left\{\frac{1}{2}d_G^{**}, \frac{1}{2}d_H^{**}\right\}$. Then it is clear.

Suppose $Max\left\{\frac{1}{2}d_G^*, \frac{1}{2}d_H^*\right\} < \infty$. Therefore $d_G^* = 4k'$ and $d_H^* = 4t'$; $k' \le t'$. Then using walks W_1 and W_2, we get a path between (u,v) and (u',v'), as in Proposition 3.4 by replacing P_1 by W_1. So, $d \le Max\left\{\frac{1}{2}d_G^*, \frac{1}{2}d_H^*\right\}$. Similarly, if $d_G^{**} = 4k'+2$ and $d_H^{**} = 4t'+2$, then $d \le Max\left\{\frac{1}{2}d_G^{**}, \frac{1}{2}d_H^{**}\right\}$. Hence we get $d \le Min\left\{Max\left\{\frac{1}{2}d_G^*, \frac{1}{2}d_H^*\right\}, Max\left\{\frac{1}{2}d_G^{**}, \frac{1}{2}d_H^{**}\right\}\right\}$.

Conversely, as $d < \infty$, as we have seen in Proposition 3.4, we get

$$Min\left\{Max\left\{\frac{1}{2}d_G^*, \frac{1}{2}d_H^*\right\}, Max\left\{\frac{1}{2}d_G^{**}, \frac{1}{2}d_H^{**}\right\}\right\} \le d.$$

(ii) If $d_G = 4k$, then $d_G = d_G^*$ and hence the result follows.

(iii) If $d_G = 4k+2$, then $d_G = d_G^{**}$ and hence the result follows.

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References

[1]. R. Balakrishnan and K. Ranganathan, *A Textbook of Graph Theory*, Universitext, Springer, New York, 2012.
[2]. I. H. Naga Raja Rao and K. V. S. Sarma, On Tensor Product of Standard Graphs, *Int. J. Comp., **8**(3)*, (2010), 99-103.
[3]. Sirous Moradi, A note on Tensor Product of Graphs, *Iranian J. of Math. Sci. and Inf., **7**(1)*, (2012), 73-81.
[4]. E. Sampathkumar, On Tensor Product of Graphs, *J. Austral. Math. Soc., **20**(Series-A)*, (1975), 268-273.
[5]. U. P. Acharya and H. S. Mehta, 2- Tensor Product of special Graphs, *Int. J. of Math. and Scietific Comp., **4**(1)*, (2014), 21-24.
[6]. Biggs. N. L., *Algebraic Graph Theory*, Cambridge University Press, Cambridge, 1993.
[7]. N. Deo, *Graph Theory with Application in Engineering and Computer Science*, Prentice-Hall of India, Delhi, 1989.