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On Double Integrals Involving Generalized H–Function of Two Variables

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Abstract: *The aim of this paper is to derive a double integrals involving generalized H–function of two variables.*

I. Introduction

The generalized H–function of two variables is given by Shrivastava, H. S. P. [4] and defined as follows:

$$H^{m_1,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2,q_2;p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}\end{matrix}\right]$$

$$=\frac{-1}{4\pi^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\theta_2(\xi)\theta_3(\eta)x^\xi y^\eta\,d\xi d\eta \qquad (1)$$

where

$$\phi_1(\xi,\eta)=\frac{\prod_{j=1}^{n_1}\Gamma(1-a_j+\alpha_j\xi+A_j\eta)\prod_{j=1}^{m_1}\Gamma(b_j-\beta_j\xi-B_j\eta)}{\prod_{j=n_1+1}^{p_1}\Gamma(a_j-\alpha_j\xi-A_j\eta)\prod_{j=1}^{q_1}\Gamma(1-b_j+\beta_j\xi+B_j\eta)},$$

$$\theta_2(\xi)=\frac{\prod_{j=1}^{m_2}\Gamma(d_j-\delta_j\xi)\prod_{j=1}^{n_2}\Gamma(1-c_j+\gamma_j\xi)}{\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+\delta_j\xi)\prod_{j=n_2+1}^{p_2}\Gamma(c_j-\gamma_j\xi)}$$

$$\theta_2(\xi)=\frac{\prod_{j=1}^{m_3}\Gamma(f_j-F_j\eta)\prod_{j=1}^{n_3}\Gamma(1-e_j+E_j\eta)}{\prod_{j=m_3+1}^{q_3}\Gamma(1-f_j+F_j\eta)\prod_{j=n_3+1}^{p_3}\Gamma(e_j-E_j\eta)}$$

x and y are not equal to zero, and an empty product is interpreted as unity p_i, q_i, n_i and m_j are non negative integers such that $p_i \ge n_i \ge 0$, $q_i \ge 0$, $q_j \ge m_j \ge 0$, (i = 1, 2, 3; j = 2, 3). Also, all the A's, α's, B's, β's, γ's, δ's, E's, and F's are assumed to the positive quantities for standardization purpose.

The contour L_1 is in the ξ-plane and runs from – i∞ to + i∞, with loops, if necessary, to ensure that the poles of $\Gamma(d_j - \delta_j\xi)$ (j = 1, ..., m_2) lie to the right, and the poles of $\Gamma(1 - c_j + \gamma_j\xi)$ (j = 1, ..., n_2), $\Gamma(1 - a_j + \alpha_j\xi + A_j\eta)$ (j = 1, ..., n_1) to the left of the contour.

The contour L_2 is in the η-plane and runs from – i∞ to + i∞, with loops, if necessary, to ensure that the poles of $\Gamma(f_j - F_j\eta)$ (j = 1, ..., m_3) lie to the right, and the poles of $\Gamma(1 - e_j + E_j\eta)$ (j = 1, ..., n_3), $\Gamma(1 - a_j + \alpha_j\xi + A_j\eta)$ (j = 1, ..., n_1) to the left of the contour.

The generalized H-function of two variables given by (1) is convergent if

$$U=\sum_{j=1}^{n_1}\alpha_j+\sum_{j=1}^{m_1}\beta_j+\sum_{j=1}^{n_2}\gamma_j+\sum_{j=1}^{m_2}\delta_j$$

$$-\sum_{j=n_1+1}^{p_1}\alpha_j-\sum_{j=m_1+1}^{q_1}\beta_j-\sum_{j=n_2+1}^{p_2}\gamma_j-\sum_{j=m_2+1}^{q_2}\delta_j; \qquad (2)$$

$$V=\sum_{j=1}^{n_1}A_j+\sum_{j=1}^{m_1}B_j+\sum_{j=1}^{n_3}E_j+\sum_{j=1}^{m_3}F_j$$

$$-\sum_{j=n_1+1}^{p_1}A_j-\sum_{j=m_1+1}^{q_1}B_j-\sum_{j=n_3+1}^{p_3}E_j-\sum_{j=m_3+1}^{q_3}F_j, \qquad (3)$$

where | arg x | < ½ Uπ, | arg y | < ½ Vπ.

II. Result Required

The following result are required in our present investigation:

From Shrivastava [3, p.1, Eq. (1)]:

$$S_n^m[x] = \sum_{u=0}^{[n/m]} \frac{(-n)_{mu}}{u!} X^u A_{n,u}, \ (n = 0, 1, 2, \ldots), \tag{4}$$

where m is arbitrary positive integer, and the coefficients $A_{n,u}$ (n, u ≥ 0) are arbitrary constants, real or complex.

III. Main Result

In this paper we will establish the following integral:

$$\int_0^1 \int_0^1 x^{\lambda-1} (1-x)^{\mu-1} y^{\rho-1} (1-y)^{a-2\rho} (1+ty)^{\rho-a-1}$$

$$\cdot\ {}_2F_1\left[{}^{a,b;}_{1+a-b;} \frac{(1+t)y}{1+ty}\right] J_s^{(\alpha,\beta)}(1-2x;k) S_n^m[cy^\gamma (1+ty)^\gamma (1-y)^{-2\gamma}]$$

$$\cdot\ H^{m_1,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2,q_2;p_3,q_3}\left[{}^{z_1 x^\sigma (1-x)^\nu \{\frac{y(1+ty)}{(1-y)^2}\}^T}_{z_2} \Big|^{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3}}_{(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}}\right] dx\, dy$$

$$= \frac{2^a \{4(1+t)\}^{-\rho} \Gamma\left(1+\frac{a}{2}\right) \Gamma(1+a-b)(a+1)_{ks}}{\sqrt{\pi}\, \Gamma(1+a) \Gamma\left(1+\frac{a}{2}-b\right) s!} \sum_{u=0}^{[n/m]} \sum_{j=0}^{s} \frac{(-n)_{mu}}{u!}$$

$$\cdot \frac{(\alpha+\beta+s+1)_{kj} (-s)_j c^u}{(\alpha+1)_{kj}\ j!\, \{4(1+t)\}^{\gamma u}} A_{n,u} H^{m_1,n_1;m_2+2,n_2+3;m_3,n_3}_{p_1,q_1;p_2+4,q_2+3;p_3,q_3}\left[{}^{z_1\{4(1+t)\}^{-T}}_{z_2} \Big|^{(a_j,\alpha_j;A_j)_{1,p_1}:A:(e_j,E_j)_{1,p_3}}_{(b_j,\beta_j;B_j)_{1,q_1}:B:(f_j,F_j)_{1,q_3}}\right], \tag{5}$$

where

$$A = (1-\mu,\nu), (1-\lambda-kj,\sigma), (1-\rho-\gamma u,T), (c_j,\gamma_j)_{1,p_2}, (1+a-b-\rho-\gamma u,T)$$

$$B = \left(\frac{1}{2}+\frac{a}{2}-\rho-\gamma u,T\right), \left(1+\frac{a}{2}-b-\rho-\gamma u,T\right), (d_j,\delta_j)_{1,q_2}, (1-\lambda-\mu-kj,\sigma+\nu).$$

The integral (5) is valid if the following sets of (sufficient) conditions are satisfied:

(i) m is arbitrary positive integer, and the coefficient $A_{n,u}$ (n,u≥0) are arbitrary constants, real or complex.
(ii) $|\arg z_1| < \frac{1}{2} U\pi$, $|\arg z_2| < \frac{1}{2} V\pi$, where U and V are given in (2) and (3) respectively.
(iii) $Re(1+a-b) > 0, Re(\alpha) > -1, Re(\beta) > -1, t > -1,$
$\gamma \geq 0, Re(\lambda+\sigma\xi) > 0, Re(\rho+\nu\xi) > 0,$
$Re(\rho+\gamma u+T\xi) > 0, Re(1+a-2\rho-2\gamma u-2T\xi) > 0.$
(iv) $Re(1-2b) > 0$.

Proof:

To establish (5), we use the series representation of $S_n^m[x]$ as given by (4) and for the generalized H-function of two variables we use Mellin-Barnes types of contour integral as given in (1), on the left-hand side of (5), change the order of integration and summation (which is justified under the conditions given with (5)), we then obtain

Left-hand side of (5) = $\sum_{u=0}^{[n/m]} \frac{(-n)_{mu}}{u!} c^u A_{n,u} \frac{(-1)}{4\pi^2} \int_{L_1} \int_{L_2} \phi_1(\xi,\eta) \theta_2(\xi) \theta_3(\eta)$

$$\cdot \int_0^1 x^{\lambda+\sigma\xi-1} (1-x)^{\mu+\nu\xi-1} J_s^{(\alpha,\beta)}(1-2x;k) dx$$

$$\cdot \int_0^1 y^{\rho+\gamma u+T\xi-1} (1-y)^{a-2T\xi-2\gamma u-2\rho} (1+ty)^{\rho-a+\gamma u+T-1}$$

$$\cdot\ {}_2F_1\left[{}^{a,b;}_{1+a+b;} \frac{(1+t)y}{1+ty}\right] dy] x^\xi y^\eta d\xi d\eta$$

Now using known results [1, p.118, Eq.(3.3.1)] and [2, p.254, Eq.(2.1)], and interpreting the resulting contour integral as the generalized H-function of two variables, we once get the right-hand side of (5).

IV. Special Cases

On specializing the parameters in (5), we get following integral in terms of H-function:

$$\int_0^1\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}y^{\rho-1}(1-y)^{a-2\rho}(1+ty)^{\rho-a-1}$$

$$\cdot\ {}_2F_1\left[{}^{a,b;}_{1+a-b;}\frac{(1+t)y}{1+ty}\right] J_s^{(\alpha,\beta)}(1-2x;k)S_n^m\left[cy^{\gamma}(1+ty)^{\gamma}(1-y)^{-2\gamma}\right]$$

$$.\ H_{P,Q}^{M,N}\left[zx^{\sigma}(1-x)^{\nu}\left\{\frac{y(1+ty)}{(1-y)^2}\right\}^{T}\Big|{}^{(a_j,\alpha_j)_{1,P}}_{(b_j,\beta_j)_{1,Q}}\right]dx\,dy$$

$$=\frac{2^{a}\{4(1+t)\}^{-\rho}\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-b)(a+1)_{ks}}{\sqrt{\pi}\,\Gamma(1+a)\Gamma\left(1+\frac{a}{2}-b\right)s!}\sum_{u=0}^{[n/m]}\sum_{j=0}^{s}\frac{(-n)_{m_u}}{u!}$$

$$\cdot\frac{(\alpha+\beta+s+1)_{kj}(-s)_j c^{u}}{(\alpha+1)_{kj}\ j!\{4(1+t)\}^{\gamma u}}A_{n,u}\ H_{P+4,Q+3}^{M+2,N+3}\left[z\{4(1+t)\}^{-T}\Big|{}^{A}_{B}\right], \tag{6}$$

where

$$A=(1-\mu,\nu),(1-\lambda-kj,\sigma),(1-\rho-\gamma u,T),\left(a_j,\alpha_j\right)_{1,P},(1+a-b-\rho-\gamma u,T)$$

$$B=\left(\frac{1}{2}+\frac{a}{2}-\rho-\gamma u,T\right),\left(1+\frac{a}{2}-b-\rho-\gamma u,T\right),\left(b_j,\beta_j\right)_{1,Q},(1-\lambda-\mu-kj,\sigma+\nu).$$

The integral (6) is valid if the following sets of (sufficient) conditions are satisfied:

(i) m is arbitrary positive integer, and the coefficient $A_{n,u}$ ($n,u\geq 0$) are arbitrary constants, real or complex.

(ii) $\Omega=\sum_{j=1}^{N}\alpha_j-\sum_{j=n+1}^{P}\alpha_j+\sum_{j=1}^{M}\beta_j-\sum_{j=n+1}^{Q}\beta_j>0$, and $|\arg z| < (1/2)\Omega\pi$.

(iii) $\mathrm{Re}(1+a-b)>0, Re(\alpha)>-1, Re(\beta)>-1, t>-1,$
$\gamma\geq 0, \mathrm{Re}(\lambda+\sigma\xi)>0, Re(\rho+\nu\xi)>0,$
$\mathrm{Re}(\rho+\gamma u+T\xi)>0, \mathrm{Re}(1+a-2\rho-2\gamma u-2T\xi)>0.$

(iv) $\mathrm{Re}(1-2b) > 0$.

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