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On Numerical Treatment for Volterra - Nonlinear QuadraticIntegral Equation in Two-Dimensions

S. A. Raad
Department of Mathematics, Faculty of Science, Umm Al - Qura University, Makkah, Saudi Arabia

Abstract:*In this paper, Volterra - nonlinear quadratic integral equation (V-NQIE) of the second kind with continuous kernels in two-dimensions is considered. Then, under certain conditions, and fixed-point theorem, the existence of a unique solution of V-NQIEis proved. After that, a suitable numerical method is considered to transfer V-NQIE to a system of nonlinear quadratic integral equations (SNQIEs) of the second kind. To solve the nonlinear system, the modified Adomian decomposition method (MADM), with the aid of modified Simpson's rule (MSR) are used and considered, to obtain the nonlinear algebraic system (NAS). Finally, many applications are treated; the numerical results are computed, and the estimated error, in each case, is calculated.*

Key Word and Phrases: *Volterra-Quadratic integral equation, continuous kernels, Adomian decomposition method, Simpson's rule, nonlinear algebraic system.*

I. Introduction

There have been dramatic developments in integral equations during the last century. This isdue to its linear and nonlinear applications in numerous diverse fields, such as; processes engineering, contact problems,theory of elasticity, potential theory, mathematical physics problems, biology, chemistry, mixed problems and in solving the most of boundary value problems in ordinary and partial differential equations, [1]-[8].

Many researchers have interested in quadratic integral equations. Quadratic integral equations have been applied to improve real-life problems. It have played an important role in modeling the theory of traffic theory, neutron transport, modeling of radiative transfer, kinetic theory of gases, queuing theory and many others phenomena,[9]-[15].

Recently, several studies have been focused on the effective properties of quadratic integral equations; the existence of solutions for several classes, uniqueness, monotonic and positive solutions. The measure of non-compactness, the theory of compact operators and the Banach contraction fixed-point theorem have all utilized the major mission of the existence theory for integral equations [16]-[21]. Some numerical and analytical methods can be applied to estimate the solutions of the quadratic integral equations. However, **ADM** is the most common method used to obtain numerical solutions for quadratic integral equations,[22]-[27], beside some other usefulnumerical methods,[28] and [34].

Assume**V-NQIE** as the following formula

$$\mu\,\phi(x,t) = f(x,t) + \int_0^t G(t,\tau)\,\phi(x,\tau)d\tau + \int_0^1 \upsilon(x,y)\,\gamma\big(y,t,\phi(y,t)\big)dy \int_0^t\int_0^1 F(t,\tau)\,k(x,y)\psi\big(y,\tau,\phi(y,\tau)\big)dyd\tau \quad ,(1.1)$$

In (1.1) the functions $\psi\big(y,\tau,\phi(y,\tau)\big)$and $\gamma\big(y,t,\phi(y,t)\big)$, are known continuous nonlinear functions. The kernels$k(x,y)$,$\upsilon(x,y)$,$F(t,\tau)G(t,\tau)$and the function$f(x,t)$ are known linear continuous functions,while $\phi(x,t)$ is unknown, will be determined.

In the current paper, the **V-NQIE** of the second kind with continuous kernels in two-dimensions is considered. Fixed-point theorem is used to prove the existence of a unique solution of **V-NQIE**. A suitable numerical method is used to transfer this equation to a**SNQIEs** of the second kind. **MADM** and **MSR** are used, to obtain the numerical solution of the **SV-NQIEs**. In the remainder of this work. Many applications are treated; Maple18 software is used to obtain the numerical results and the estimated errors. Finally, the conclusion is included a comparison between the numerical solutions of the two methods and their respective errors.

II. The existence of a unique solution of V-NQIE

In order to proof the existence of a unique solution of (1.1), we assume the following conditions:
(i) The given function $f(x,t)$ with its partial derivatives with respect to position and time belong to $C([0,1] \times$ *0,T*, and its norm is defined by:

$$\|f(x,t)\|_{C([0,1]\times[0,T])} = \max_{x,t}|f(x,t)| \le M.$$

(ii) The kernels of position $k(x,y), \upsilon(x,y) \epsilon\, C([0,1] \times [0,1])$ and satisfy

$$|k(x,y)| \le K_1 \ , \qquad |\upsilon(x,y)| \le K_2 \quad , \quad (K_1 , K_2 \text{ are constants }) .$$

(iii) The kernels of time $F(t,\tau), G(t,\tau)$ are continuous;$0 \le \tau \le t \le T$, and satisfy

$$|F(t,\tau)| \le L_1 \ , \qquad |G(t,\tau)| \le L_2 \quad , \quad (L_1 , L_2 \text{ are constants }) .$$

(iv) For the constants A andA_0, the nonlinear function $\psi\big(x,t,\phi(x,t)\big)$ satisfies the conditions:
$(a)\big|\psi\big(x,t,\phi_1(x,t)\big) - \psi\big(x,t,\phi_2(x,t)\big)\big| \le A|\phi_1(x,t) - \phi_2(x,t)|$.
$(b)|\psi(x,t,0)| \le A_0$.
(v) The function $\gamma\ \big(x,t,\phi(x,t)\big)$ satisfies for the constants B andB_0, the following conditions:
$(a)\big|\gamma\ \big(x,t,\phi_1(x,t)\big) - \gamma\big(x,t,\phi_2(x,t)\big)\big| \le B|\phi_1(x,t) - \phi_2(x,t)|$.
$(b)|\gamma(x,t,0)| \le B_0$.
Rewrite(1.1) in the integral operator form

$$\bar{\mathcal{H}}\phi(x,t) = \frac{1}{\mu}\, f(x,t) + \mathcal{H}\phi(x,t) \qquad , \tag{2.1}$$

where

$$\mathcal{H}\phi(x,t) = \frac{1}{\mu}\int_0^t G(t,\tau)\,\phi(x,\tau)d\tau + \frac{1}{\mu}\int_0^1 \upsilon(x,y)\ \gamma\ \big(y,t,\phi(y,t)\big)dy \int_0^t\int_0^1 F(t,\tau)\ k(x,y)\psi\big(y,\tau,\phi(y,\tau)\big)dyd\tau. \tag{2.2}$$

Theorem 1: In view of the conditions (i)-(v),**V-NQIE**(1.1) has a unique solution in the Banach space $C([0,1] \times$ 0,*T*, under the condition:

$$[L_2 + K_2L_1K_1\ B\ (Ar_0 + A_0) + K_2(Br_0 + B_0)L_1K_1A]\ T < |\mu| . \tag{2.3}$$

Where $0 < \int_0^1 |\phi(x,t)|\, dx \le r_0 < 1$.

Proof: To prove the theorem, we must prove that the integral operator (2.1) is bounded and continuous. For this, in view of the formulas (2.1) and (2.2), we have

$$|\bar{\mathcal{H}}\phi(x,t)| \le \frac{1}{|\mu|}\left[|f(x,t)| + \int_0^t |G(t,\tau)||\phi(x,\tau)|d\tau + \int_0^1 |\upsilon(x,y)|\big|\gamma\ \big(y,t,\phi(y,t)\big)\big|dy \int_0^t\int_0^1 |F(t,\tau)||k(x,y)|\big|\psi\big(y,\tau,\phi(y,\tau)\big)\big|dyd\tau\right].$$

Then, use the conditions (i-v), to obtain

$$\|\bar{\mathcal{H}}\phi(x,t)\| \le \frac{1}{|\mu|}\left[\max_{x,t}|f(x,t)| + L_2\int_0^t \max_{x,\tau}|\phi(x,\tau)|d\tau + K_2\max_{y,t}\int_0^1 \big|\gamma\ \big(y,t,\phi(y,t)\big)\big|dy \times L_1K_1 \int_0^t\int_0^1 \big|\psi\big(y,\tau,\phi(y,\tau)\big)\big|dyd\tau\right.$$

$$\|\bar{\mathcal{H}}\phi(x,t)\| \le \frac{1}{|\mu|}\Big[M + L_2\, T\, \|\phi(x,t)\| + K_2L_1K_1\ B\|\phi(x,\tau)\| \int_0^t\int_0^1 \big|\psi\big(y,\tau,\phi(y,\tau)\big) - \psi(y,\tau,0) + \psi(y,\tau,0)\big|dyd\tau + K_2\int_0^1 \big|\gamma\ \big(y,t,\phi(y,t)\big) - \gamma\ (y,t,0) + \gamma\ (y,t,0)\big|dy \times L_1K_1A\ T\ \|\phi(x,t)\|\Big]$$

$$\|\bar{\mathcal{H}}\phi(x,t)\| \le \frac{1}{|\mu|}[\mathrm{M} + L_2\, T\, \|\phi(x,t)\| + K_2 L_1 K_1\, T\, B\|\phi(x,t)\|(Ar_0 + A_0) + K_2 L_1 K_1 A\, T\, (Br_0 + B_0)\|\phi(x,t)\|] \le \frac{1}{|\mu|}[\mathrm{M} + \alpha\|\phi(x,t)\|],$$

$$(\alpha = [L_2 + L_1 K_1 K_2\, B\, (Ar_0 + A_0) + L_1 K_1 K_2 A\, (Br_0 + B_0)]\, T). \qquad (2.4)$$

The previous inequality (2.4) shows that, the operator $\bar{\mathcal{H}}$ maps the ball S_ρ into itself, where

$$\rho = \frac{M}{|\mu| - [L_2 + L_1 K_1 K_2\, B\, (Ar_0 + A_0) + L_1 K_1 K_2 A\, (Br_0 + B_0)]\, T}, \qquad (2.5)$$

since $\rho > 0$ & $M > 0$, therefore we have $\alpha < 1$. Also, (2.4) involves that the operator $\mathcal{H}$ is bounded, where

$$\|\mathcal{H}\phi(x,t)\| \le \frac{\alpha}{|\mu|}\|\phi(x,t)\|. \qquad (2.6)$$

Moreover, (2.4) and (2.6) define that the operator $\bar{\mathcal{H}}$ is bounded.

For the continuity, consider the two functions $\phi_1(x,t)$ and $\phi_2(x,t)$ in the space $C([0,1]\times[0,T])$, then from (2.1) and (2.2), we find

$$|\bar{\mathcal{H}}\phi_1(x,t) - \bar{\mathcal{H}}\phi_2(x,t)| \le \frac{1}{|\mu|}\Bigg[\int_0^t |G(t,\tau)||\phi_1(x,\tau) - \phi_2(x,\tau)|d\tau + \int_0^1 |\upsilon(x,y)||\gamma(y,t,\phi_1(y,t)) - \gamma(y,t,\phi_2(y,t))|dy$$
$$\times \int_0^t\int_0^1 |F(t,\tau)||k(x,y)||\psi(y,\tau,\phi_1(y,\tau))|dy d\tau + \int_0^1 |\upsilon(x,y)||\gamma(y,t,\phi_2(y,t))|dy$$
$$\times \int_0^t\int_0^1 |F(t,\tau)||k(x,y)||\psi(y,\tau,\phi_1(y,\tau)) - \psi(y,\tau,\phi_2(y,\tau))|dy d\tau\Bigg],$$

which implies that

$$\|\bar{\mathcal{H}}\phi_1(x,t) - \bar{\mathcal{H}}\phi_2(x,t)\| \le \frac{1}{|\mu|}\Bigg[\int_0^t |G(t,\tau)|\max_{x,\tau}|\phi_1(x,\tau) - \phi_2(x,\tau)|d\tau + \int_0^1 |\upsilon(x,y)|\max_{y,t}|\gamma(y,t,\phi_1(y,t)) - \gamma(y,t,\phi_2(y,t))|dy$$
$$\times \int_0^t\int_0^1 |F(t,\tau)||k(x,y)||\psi(y,\tau,\phi_1(y,\tau))|dy d\tau + \int_0^1 |\upsilon(x,y)||\gamma(y,t,\phi_2(y,t))|dy$$
$$\times \int_0^t\int_0^1 |F(t,\tau)||k(x,y)|\max_{y,\tau}|\psi(y,\tau,\phi_1(y,\tau)) - \psi(y,\tau,\phi_2(y,\tau))|dy d\tau\Bigg].$$

In view of the conditions (ii-v), we have

$$\|\bar{\mathcal{H}}\phi_1(x,t) - \bar{\mathcal{H}}\phi_2(x,t)\| \le \frac{1}{|\mu|}\Bigg[L_2 \max_{x,\tau}\int_0^t |\phi_1(x,\tau) - \phi_2(x,\tau)|d\tau$$
$$+K_2\int_0^1 \max_{y,t}|\gamma(y,t,\phi_1(y,t)) - \gamma(y,t,\phi_2(y,t))|dy \times L_1 K_1 \int_0^t\int_0^1 |\psi(y,\tau,\phi_1(y,\tau))|dy d\tau$$
$$+K_2\int_0^1 |\gamma(y,t,\phi_2(y,t))|dy \times L_1 K_1 \int_0^t\int_0^1 \max_{y,\tau}|\psi(y,\tau,\phi_1(y,\tau)) - \psi(y,\tau,\phi_2(y,\tau))|dy d\tau\Bigg]$$

$$\|\bar{\mathcal{H}}\phi_1(x,t) - \bar{\mathcal{H}}\phi_2(x,t)\| \le \frac{1}{|\mu|}[L_2\, T\, \|\phi_1(x,t) - \phi_2(x,t)\|$$
$$+K_2 L_1 K_1\, B\|\phi_1(x,t) - \phi_2(x,t)\|(Ar_0 + A_0) + K_2(Br_0 + B_0)L_1 K_1 A\, T\, \|\phi_1(x,t) - \phi_2(x,t)\|]$$

$$\|\bar{\mathcal{H}}\phi_1(x,t) - \bar{\mathcal{H}}\phi_2(x,t)\| \le \frac{\alpha}{|\mu|}\|\phi_1(x,t) - \phi_2(x,t)\|. \qquad (2.7)$$

From (2.7), we see that the operator $\bar{\mathcal{H}}$ is continuous in the space $C([0,1]\times[0,T])$. Moreover, $\bar{\mathcal{H}}$ is a contraction operator under the condition $\alpha < |\mu|$. So, from Banach fixed point theorem, $\bar{\mathcal{H}}$ has a unique fixed point which is, of course, the unique solution of (1.1).

III. Nonlinear system of quadratic integral equations (NSQIEs)

In this section, a numerical method is used in **V-NQIE** (1.1) to obtain a **NSQIEs**,(see [7] and [8]).
If we divide the interval $[0,T]$ into l subintervals, by means of the points: $0=t_0<t_1<\cdots<t_l=T$, where $t=t_r$, $\tau=t_s$, $r,s=0,1,2,\dots,l$, then use the quadrature formula, the time integral terms of (1.1) becomes

$$\int_0^{t_r} G(t_r,\tau)\,\phi(x,\tau)d\tau+\int_0^1 \upsilon(x,y)\,\gamma\big(y,t,\phi(y,t)\big)dy\int_0^{t_r}\int_0^1 F(t_r,\tau)\,k(x,y)\psi\big(y,\tau,\phi(y,\tau)\big)dyd\tau$$
$$=\sum_{s=0}^{r}u_sG_{r,s}\phi_s(x)+\int_0^1\upsilon(x,y)\gamma_r\big(y,\phi_r(y)\big)dy\sum_{s=0}^{r}u_sF_{r,s}\int_0^1 k(x,y)\psi_s\big(y,\phi_s(y)\big)dy. \quad (3.1)$$

Next, use (3.1) in (1.1), to get

$$\mu\,\phi_r(x)=f_r(x)+\sum_{s=0}^{r}u_sG_{r,s}\phi_s(x)$$
$$+\int_0^1\upsilon(x,y)\gamma_r\big(y,\phi_r(y)\big)dy\sum_{s=0}^{r}u_sF_{r,s}\int_0^1 k(x,y)\psi_s\big(y,\phi_s(y)\big)dy. \quad (3.2)$$

We used the following notations

$$\phi(x,t_r)=\phi_r(x)\ ,\quad F(t_r,t_s)=F_{r,s}\ ,\quad f(x,t_r)=f_r(x),\quad G(t_r,t_s)=G_{r,s}$$
$$\gamma(x,t_r,\phi(x,t_r))=\gamma_r\big(x,\phi_r(x)\big),\quad \psi(y,t_s,\phi(y,t_s))=\psi_s\big(y,\phi_s(y)\big), \quad (3.3)$$

and u_s arethe weights

$$u_s=\begin{cases} h_s/2 & ;\quad s=0\ ,s=r \\ h_s & ;\quad 0<s<r \end{cases}. \quad (3.4)$$

The formula (3.2) representsa **NSQIEs** of the second kind.where h_s is the step-size of integration.

Definition 1: The error of using quadratic method in (3.2) can be determine by

$$E_s=\Bigg|\int_0^{t} G(t,\tau)\,\phi(x,\tau)d\tau+$$
$$\int_0^1\upsilon(x,y)\,\gamma\,\big(y,t,\phi(y,t)\big)dy\int_0^{t}\int_0^1 F(t,\tau)\,k(x,y)\psi\big(y,\tau,\phi(y,\tau)\big)dyd\tau-\sum_{s=0}^{r}u_sG_{r,s}\phi_s(x)-$$
$$\int_0^1\upsilon(x,y)\gamma_r\big(y,\phi_r(y)\big)dy\sum_{s=0}^{r}u_sF_{r,s}\int_0^1 k(x,y)\psi_s\big(y,\phi_s(y)\big)dy\Bigg|. \quad (3.5)$$

IV. The Modified Adomian Decomposition Method

ADM is a semi-analytical method, it has been proven as a powerful and reliable scheme for solving a variety of linear and nonlinear problems; integral equations, boundary value problems, ordinary or partial differential equations, algebraic equations, and so on, [16] and[17]. The **ADM** involves separating the equation into linear and nonlinear portions. The nonlinear portion is decomposed into a series of Adomian polynomials. **ADM** includes generating the solution in the form of a series which terms are determined by a recurrence relationship using the Adomian polynomials. so, the solution can be determined by calculation of the Adomian polynomials which allow for solution convergence of the nonlinear portion of the equation, without simply linearizing the system. One can admit that it is practically difficult to find the exact sum of an Adomian series. Indeed, we only able to calculate a finite terms of the series. On the other hand, the Adomian series is quickly convergent and a truncation error can be easily calculated. In many researches, Fixed-point theorem was used to prove the **ADM** convergence,more than that the convergence was ensured with weak hypothesis.In the references [20-23] the convergence of ADM was discussed and proved by different methods.
Consider that the functions $\phi_r(x)$ in the system (3.2) can be expressed as an infinite series

$$\phi_r(x)=\sum_{n=0}^{\infty}\phi_{r,n}(x). \quad (4.1)$$

Alongside, the nonlinear terms $\psi_s\big(x,\phi_s(x)\big)$, $\gamma_r\big(x,\phi_r(x)\big)$ of (3.2) can be supposed in the form

$$\psi_s\big(x,\phi_s(x)\big)=\sum_{n=0}^{\infty}A_{s,n}\quad,\quad \gamma_r\big(x,\phi_r(x)\big)=\sum_{n=0}^{\infty}\bar{A}_{r,n}\quad, \tag{4.2}$$

where the Adomian polynomials $A_{r,n}$, $\bar{A}_{r,n}$ can be determined by

$$A_{s,n}=\frac{1}{n!}\left(\frac{d^n}{d\lambda^n}\psi_s\left(\sum_{i=0}^{n}\lambda^i\phi_{s,i}\right)\right)_{\lambda=0}\quad,\quad \bar{A}_{r,n}=\frac{1}{n!}\left(\frac{d^n}{d\lambda^n}\gamma_r\left(\sum_{i=0}^{n}\lambda^i\phi_{r,i}(x)\right)\right)_{\lambda=0}. \tag{4.3}$$

Another formula of Adomian polynomials, is given by

$$A_{s,n}=\psi_s\big(P_{s,n}\big)-\sum_{i=0}^{n-1}A_{s,i}\quad,\quad \bar{A}_{r,n}=\gamma_r\big(P_{r,n}\big)-\sum_{i=0}^{n-1}\bar{A}_{r,i}\quad, \tag{4.4}$$

where, the partial sum $P_{s,n}$ is

$$P_{r,n}=\sum_{i=0}^{n}\phi_{r,i}(x)\ , \tag{4.5}$$

after applying the **ADM** on (3.2), the Adomian decomposition method introduces the recurrence relation

$$\mu\,\phi_{r,0}(x)=f_r(x)\ ;\qquad \mu\,\phi_{r,i}(x)=\sum_{s=0}^{r}u_sG_{r,s}\phi_{s,i}(x)+$$
$$\int_0^1 \upsilon(x,y)\bar{A}_{r,i-1}(y)dy\sum_{s=0}^{r}u_sF_{r,s}\int_0^1 k(x,y)A_{s,i-1}(y)dy\,,(i\geq 1)\,. \tag{4.6}$$

For **MADM**, the modification of the free term is written in the form

$$f_r(x)=\sum_{n=0}^{\infty}f_{r,n}(x)\ , \tag{4.7}$$

in view of (4.7), the modification of the solution can be modified to

$$\mu\,\phi_{r,0}(x)=f_{r,0}(x)\ ;\qquad \mu\,\phi_{r,i}(x)=f_{r,i}(x)+\sum_{s=0}^{r}u_sG_{r,s}\phi_{s,i}(x)$$
$$+\int_0^1 \upsilon(x,y)\bar{A}_{r,i-1}(y)dy\sum_{s=0}^{r}u_sF_{r,s}\int_0^1 k(x,y)A_{s,i-1}(y)dy\,,(i\geq 1). \tag{4.8}$$

Where the Adomian polynomials $A_{s,n}$ and $\bar{A}_{r,n}$ can be evaluated for the nonlinear functions $\psi_s\big(x,\phi_s(x)\big)$ and $\gamma_r\big(x,\phi_r(x)\big)$, therefore the Adomian polynomials are given by

$$A_{s,0}=\psi_s\left(\phi_{s,0}(x)\right)\quad,\quad \bar{A}_{r,0}=\gamma_r\left(\phi_{r,0}(x)\right),$$
$$A_{s,1}=\phi_{s,1}\acute{\psi}_s\left(\phi_{s,0}(x)\right)\quad,\quad \bar{A}_{r,1}=\phi_{r,1}(x)\acute{\gamma}_r\left(\phi_{r,0}(x)\right),$$
$$A_{s,2}=\frac{1}{2}\phi_{s,1}^2(x)\psi_s^{(2)}\left(\phi_{s,0}(x)\right)+\phi_{s,2}(x)\acute{\psi}_s\left(\phi_{s,0}(x)\right)\ ,$$
$$\bar{A}_{r,2}=\frac{1}{2}\phi_{r,1}^2(x)\gamma_r^{(2)}\left(\phi_{r,0}(x)\right)+\phi_{r,1}(x)\acute{\gamma}_r\left(\phi_{r,0}(x)\right),$$
$$A_{s,3}=\frac{1}{6}\phi_{s,1}^3(x)\psi_s^{(3)}\left(\phi_{s,0}(x)\right)+\phi_{s,1}(x)\phi_{s,2}(x)\psi_s^{(2)}\left(\phi_{s,0}(x)\right)+\phi_{s,3}(x)\acute{\psi}_s\left(\phi_{s,0}(x)\right)\ ,$$
$$\bar{A}_{r,3}=\frac{1}{6}\phi_{r,1}^3(x)\gamma_r^{(3)}\left(\phi_{r,0}(x)\right)+\phi_{r,1}(x)\phi_{r,2}(x)\gamma_r^{(2)}\left(\phi_{r,0}(x)\right)+\phi_{r,3}(x)\acute{\gamma}_r\left(\phi_{r,0}(x)\right),$$
$$A_{s,4}=\frac{1}{24}\phi_{s,1}^4(x)\psi_s^{(4)}\left(\phi_{s,0}(x)\right)+\frac{1}{2}\phi_{s,1}^2(x)\phi_{s,2}\psi_s^{(3)}\left(\phi_{s,0}(x)\right)+\left(\frac{1}{2}\phi_{s,2}^2(x)+\phi_{s,1}(x)\phi_{s,3}(x)\right)\psi_s^{(2)}\left(\phi_{s,0}(x)\right)$$
$$+\phi_{s,4}(x)\acute{\psi}_s\left(\phi_{s,0}(x)\right)\ ,$$
$$\bar{A}_{r,4}=\frac{1}{24}\phi_{r,1}^4(x)\gamma_r^{(4)}\left(\phi_{r,0}(x)\right)+\frac{1}{2}\phi_{r,1}^2(x)\phi_{r,2}\gamma_r^{(3)}\left(\phi_{r,0}(x)\right)$$
$$+\left(\frac{1}{2}\phi_{r,2}^2(x)+\phi_{r,1}(x)\phi_{r,3}(x)\right)\gamma_r^{(2)}\left(\phi_{r,0}(x)\right)+\phi_{r,4}(x)\acute{\gamma}_r\left(\phi_{r,0}(x)\right),$$

and so on … .

The determination of $\phi_{s,0}$ and $\phi_{s,1}$ leads to the determination of $A_{s,1}$, $\bar{A}_{s,1}$ that will allows us to determine $\phi_{s,2}$, and so on. This in turn will lead to the complete determination of the components of $\phi_{s,\mathrm{i}}$,$i\geq 1$, upon using the second part of (4.8). The series solution follows immediately after using (4.1). The applications of (4.8) aremore useful when the kernels arethe exponential or periodic functions, the obtained series converges to an exact solution of **V-QNIE** (1.1).

V. Modified Simpson's quadrature rule

In this section, **NSQIEs** (3.2) is approximated by using Modified Simpson's quadrature rule formula, (see [31] and [32]), to obtain

$$\mu\phi_{r,p} = f_{r,p} + \sum_{s=0}^{r} u_s G_{r,s}\phi_{s,p} + \frac{h}{3}\left[4\sum_{i=1}^{N/2} \upsilon_{p,2i-1}\gamma_{r,2i-1} + 2\sum_{i=1}^{(N/2)-1} \upsilon_{p,2i}\gamma_{r,2i} + \upsilon_{p,0}\gamma_{r,0} + \upsilon_{p,N}\gamma_{r,N}\right]$$
$$\times \sum_{s=0}^{r} u_s F_{r,s}\frac{h}{3}\left[4\sum_{j=1}^{N/2} k_{p,2j-1}\psi_{s,2j-1} + 2\sum_{j=1}^{(N/2)-1} k_{p,2j}\psi_{s,2j} + k_{p,0}\psi_{s,0} + k_{p,N}\psi_{s,N}\right], \quad (5.1)$$

which can be adapted in the form

$$\mu\phi_{r,p} = f_{r,p} + \sum_{s=0}^{r} u_s G_{r,s}\phi_{s,p} + \frac{h^2}{9}\sum_{i=1}^{N/2}\left[\upsilon_{p,2i-2}\gamma_{r,2i-2} + 4\upsilon_{p,2i-1}\gamma_{r,2i-1} + \upsilon_{p,2i}\gamma_{r,2i}\right]$$
$$\times \sum_{s=0}^{r} u_s F_{r,s}\sum_{j=1}^{N/2}\left[k_{p,2j-2}\psi_{s9,2j-2} + 4k_{p,2j-1}\psi_{s,2j-1} + k_{p,2j}\psi_{s,2j}\right], \quad (5.2)$$

The solution of the**NAS**system in (5.2), converges to the solution of (2.1) .

Definition 2: The following relation determines the estimate total error$R_{r,N}$ of (5.2)

$$R_{r,N} = \Bigg|\int_0^1 \upsilon(x,y)\gamma(y,\phi_r(y))dy \sum_{s=0}^{r} u_s F_{r,s}\int_0^1 k(x,y)\psi(y,\phi_s(y))dy$$
$$- \frac{h^2}{9}\sum_{i=1}^{N/2}\left[\upsilon_{p,2i-2}\gamma_{r,2i-2} + 4\upsilon_{p,2i-1}\gamma_{r,2i-1} + \upsilon_{p,2i}\gamma_{r,2i}\right]$$
$$\times \sum_{s=0}^{r} u_s F_{r,s}\sum_{j=1}^{N/2}\left[k_{p,2j-2}\psi_{s9,2j-2} + 4k_{p,2j-1}\psi_{s,2j-1} + k_{p,2j}\psi_{s,2j}\right]\Bigg|. \quad (5.4)$$

VI. Numerical Applications

In this section, some examplesis consideredin the form of**V-NQIE** (1.1).The numerical results are obtained by Maple 18 software, for $x \in [0,1]$, $\lambda = 9$and $t \in [0,T]$. The next tables give us the exact and the numerical solutions, which obtained by using modified Simpson's rule (**Num. MSR**) andmodified Adomian decomposition method (**Num. MAD**), and their corresponding errors (**Err. MSR**) and (**Err. MAD**), respectively,at the times $T = 0.008, T = 0.06$ and $T = 0.4$. The diagrams explain the difference between these results.The relation obtains the ratio between the two numerical solutions is:

$$Ratio = \frac{\text{Num. MAD}}{\text{Num. MSR}}$$

Example (1): Consider **V-NQIE**, in the form

$$\mu\,\phi(x,t) = f(x,t) + \int_0^t \frac{\tau}{5+t}\,\phi(x,\tau)d\tau$$
$$+ \int_0^1 x^2 y\cosh(\phi(y,t))\,dy \int_0^t\int_0^1 \exp(x(y-2))\,t\tau\sinh(\phi(y,\tau))\,dyd\tau\,. \quad (6.1)$$

The exact solution is$\phi(x,t) = xt$.

x	Exact	Num. MSR	Err. MSR	Num. MAD	Err. MAD	Ratio
0	**0**	**0**	**0**	**0**	**0**	
0.2	**0.0016**	**0.001599999**	**5.3500E-10**	**0.0016**	**3.5500E-10**	**0.9999994**
0.4	**0.0032**	**0.003199999**	**1.0710E-09**	**0.003200001**	**7.1100E-10**	**0.9999994**
0.6	**0.0048**	**0.004799998**	**1.6060E-09**	**0.004800001**	**1.0690E-09**	**0.9999994**
0.8	**0.0064**	**0.006399998**	**2.1420E-09**	**0.006400001**	**1.4280E-09**	**0.9999994**
1	**0.008**	**0.007999997**	**2.6760E-09**	**0.008000002**	**1.7880E-09**	**0.9999994**

Table 1-1: $T = 0.008, N = 20$ and $l = 4$.

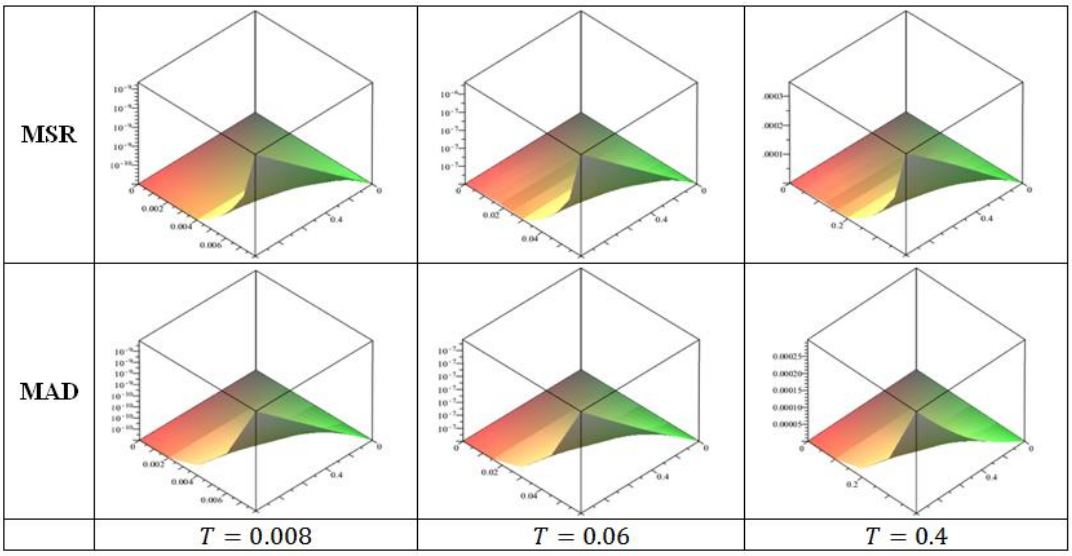
x	Exact	Num. MSR	Err. MSR	Num.MAD	Err. MAD	Ratio

0	**0**	**0**	**0**	**0**	**0**	
0.2	**0.012**	**0.011999774**	**2.2586E-07**	**0.012000147**	**1.4741E-07**	**0.99996889**
0.4	**0.024**	**0.023999546**	**4.5425E-07**	**0.024000299**	**2.9856E-07**	**0.99996863**
0.6	**0.036**	**0.035999318**	**6.8235E-07**	**0.036000456**	**4.5588E-07**	**0.99996838**
0.8	**0.048**	**0.047999091**	**9.0901E-07**	**0.048000618**	**6.1804E-07**	**0.99996819**
1	**0.06**	**0.059998866**	**1.13404E-06**	**0.060000783**	**7.8271E-07**	**0.99996805**

Table 1-2: $T = 0.06, N = 20$ and $l = 4$.

x	Exact	Num. MSR	Err. MSR	Num. MAD	Err. MAD	Ratio
0	**0**	**0**	**0**	**0**	**0**	
0.2	**0.08**	**0.079932005**	**6.79952E-05**	**0.080039339**	**3.93385E-05**	**0.99865899**
0.4	**0.16**	**0.159858718**	**1.41282E-04**	**0.160086135**	**8.61354E-05**	**0.99857941**
0.6	**0.24**	**0.239786158**	**2.13842E-04**	**0.240146246**	**1.46246E-04**	**0.99850055**
0.8	**0.32**	**0.319716797**	**2.83203E-04**	**0.320218456**	**2.18456E-04**	**0.99843339**
1	**0.4**	**0.399651283**	**3.48717E-04**	**0.400299352**	**2.99352E-04**	**0.99838104**

Table 1-3: $T = 0.4 N = 20$ and $l = 4$.



Dig. 1

Tables 1.1, 1.2 and 1.3, refer to that the accuracies of the numerical solution, of equation (6.1), by**MSR.**and the **MAD.**are$10^{-7}, 10^{-5}, 10^{-3}$, when$T = 0.008, 0.06$, and $T = 0.4$, respectively.

Example (2): Consider **QNIE**, in the form

$$\mu\,\phi(x,t) = f(x,t) + \int_0^1 \frac{t\,\tau}{2}\;\phi(y,\tau)d\tau$$

$$+\int_0^1 e^{x-y}\phi^2(y,t)dy \int_0^t\int_0^1 (2\,xy)\frac{\tau}{4-t}\phi^3(y,\tau)\,dyd\tau. \qquad (6.3)$$

The exact solution is $(x,t) = t\sin(\pi x/2)$.

x	Exact	Num. MSR	Err. MSR	Num. MAD	Err. MAD	Ratio
0	**0**	**0**	**0**	**0**	**0**	
0.2	**0.002472136**	**0.002472136**	**4.2067E-16**	**0.002472136**	**8.0004E-12**	**1.00000001**
0.4	**0.003631924**	**0.004702282**	**6.6021E-14**	**0.004702282**	**1.4660E-11**	**1.00000001**
0.6	**0.004702282**	**0.006472136**	**4.1980E-13**	**0.006472136**	**1.9000E-11**	**1.00000001**
0.8	**0.005656854**	**0.007608452**	**1.6388E-12**	**0.007608452**	**2.3639E-11**	**1.00000001**
1	**0.006472136**	**0.008**	**0**	**0.008**	**2.3000E-11**	**1.00000001**

Table 2-1: $T = 0.008$, $N = 20$ and $l = 4$.

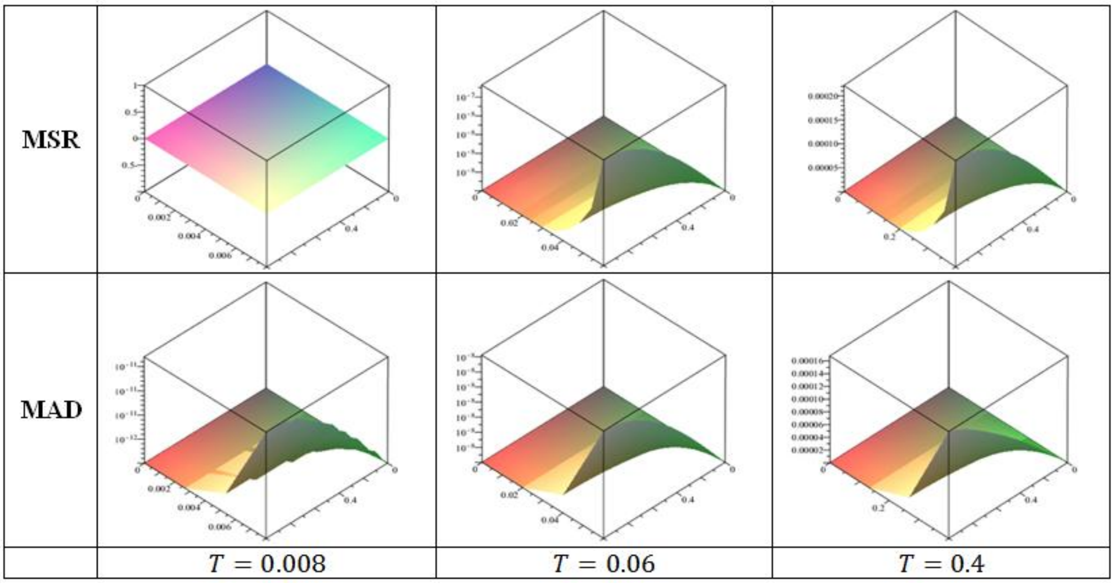
x	Exact	Num.MSR	Err. MSR	Num.MAD	Err. MAD	Ratio

0	**0**	**0**	**0**	**0**	**0**	
0.2	**0.009386068**	**0.018540985**	**3.4872E-08**	**0.018541042**	**2.1968E-08**	**0.99999693**
0.4	**0.01854102**	**0.035267049**	**6.6298E-08**	**0.035267157**	**4.1803E-08**	**0.99999693**
0.6	**0.02723943**	**0.048540928**	**9.1273E-08**	**0.048541077**	**5.7538E-08**	**0.99999693**
0.8	**0.035267115**	**0.057063284**	**1.0728E-07**	**0.057063459**	**6.7662E-08**	**0.99999693**
1	**0.042426407**	**0.059999887**	**1.128E-07**	**0.060000071**	**7.115E-08**	**0.99999693**

Table 2-2: T=0.06,$N = 20$ and $l = 4$.

x	Exact	Num. MSR	Err. MSR	Num.MAD	Err. MAD	Ratio
0	0	0	0	0	0	
0.2	0.062573786	0.12353763	6.9163E-05	0.12365007	4.3276E-05	0.99909067
0.4	0.123606798	0.23498244	1.3165E-04	0.23519677	8.2673E-05	0.99908873
0.6	0.1815962	0.32342576	1.8103E-04	0.32372385	1.1706E-04	0.99907918
0.8	0.235114101	0.38021067	2.1193E-04	0.38056995	1.4735E-04	0.99905595
1	0.282842712	0.39977812	2.2187E-04	0.40016789	1.6789E-04	0.99902599

Table 2-3: T=0.4,$N = 20$ and $l = 4$.



Dig. 2

Tables 2.1, 2.2 and 2.3, refer to that the accuracies of the numerical solution, of equation (6.2), by **MSR.**and the **MAD.**are $10^{-9}, 10^{-8}$, when$T = 0.008$, and$10^{-6}, 10^{-3}$, when$0.06, T = 0.4$, respectively.

Example (3): Consider**V-NQIE**, in the form

$$\mu\,\phi(x,t) = f(x,t) + \int_0^1 \sinh(t\,\tau)\quad \phi(y,t)d\tau$$

$$+\int_0^1 \frac{x\,y}{3+x}\exp(\phi(y,t)-1)\ dy \int_0^t\int_0^1 e^{-\tau}\,\frac{y-x}{y+x+5}\,\phi^2(y,\tau)\ dyd\tau. \qquad (6.5)$$

The exact solution is $(x,t) = t\ln|x+1|$.

x	Exact	Num. MSR	Err. MSR	Num. MAD	Err. MAD	Ratio
0	**0**	**0**	**0**	**0**	**0**	
0.2	**0.00145857**	**0.001458572**	**3.5164E-13**	**0.001458572**	**1.1648E-11**	**0.99999999**
0.4	**0.00269178**	**0.002691778**	**3.0297E-14**	**0.002691778**	**2.3030E-11**	**0.99999999**
0.6	**0.00376003**	**0.003760029**	**3.4115E-14**	**0.003760029**	**2.6034E-11**	**0.99999999**
0.8	**0.00470229**	**0.004702293**	**2.1695E-13**	**0.004702293**	**9.7831E-12**	**1.00000001**
1	**0.00554518**	**0.005545177**	**5.2044E-13**	**0.005545177**	**3.2479E-11**	**1.00000001**

Table 3-1: $T = 0.008, N = 20$ and $l = 4$.

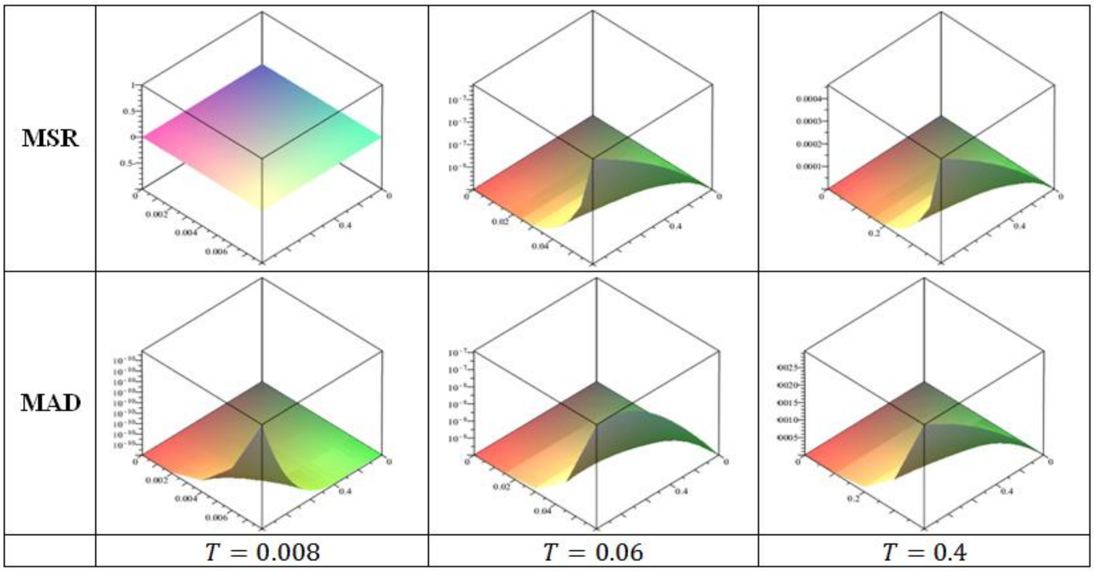
x	Exact	Num. MSR	Err. MSR	Num. MAD	Err. MAD	Ratio

0	**0**	**0**	**0**	**0**	**0**	
0.2	**0.01093929**	**0.01093923**	**6.2938E-08**	**0.010939334**	**4.0442E-08**	**0.99999055**
0.4	**0.02018833**	**0.020188219**	**1.1521E-07**	**0.02018841**	**7.5353E-08**	**0.99999056**
0.6	**0.02820022**	**0.028200058**	**1.5987E-07**	**0.02820032**	**1.0253E-07**	**0.9999907**
0.8	**0.0352672**	**0.035267001**	**1.9901E-07**	**0.035267318**	**1.1859E-07**	**0.99999099**
1	**0**	**0.041588597**	**2.3409E-07**	**0.041588951**	**1.2059E-07**	**0.99999147**

Table 3-2: T=0.06, $N = 20$ and $l = 4$.

x	Exact	Num. MSR	Err. MSR	Num. MAD	Err. MAD	Ratio
0	0	0	0	0	0	
0.2	0.07292862	0.07280743	1.2119E-04	0.073009584	8.0961E-05	0.9972311
0.4	0.13458889	0.134365493	2.2340E-04	0.134738526	1.4963E-04	0.9972314
0.6	0.18800145	0.187689694	3.1176E-04	0.188209637	2.0818E-04	0.9972374
0.8	0.23511467	0.234725051	3.8962E-04	0.235371918	2.5725E-04	0.9972517
1	0.27725887	0.276799585	4.5929E-04	0.277555708	2.9684E-04	0.9972757

Table 3-3: T=0.4,$N = 20$ and $l = 4$.



Dig. 3

Tables 3.1, 3.2 and 3.3, refer to that the accuracies of the numerical solution, of equation (6.2), by **MSR.**and the **MAD.**are $10^{-9}, 10^{-8}$, when$T = 0.008$, and$10^{-5}, 10^{-3}$, when$0.06, T = 0.4$, respectively.

VII. Conclusions

1- In the current research, a **V-NQIE** of the second kind with continuous kernels is considered. Banach Fixed-Point Theorem has been used to prove the existence of a unique solution of **V-NQIE**. Using quadratic numerical method,**SNIEs** of the second kind has been obtained. Then,**MADM**hasbeen used to solve this system. The **MSR** has been applied on the system to obtain a **NAS**. Finally, some examples are solved to obtain numerical results.

2-The previous numerical results of Tables (1-1) to (3-3), have shown:

2.1. The convergence of the approximate solutions of **MADM** and **MSR**to the exact solution.

2.2The results provide further confirmation of the effectiveness of **MADM** and **MSR** for obtaining the numerical solutions for linear and nonlinear problems.

2.3. From the ratio between **Num. MAD** and **Num. MSR**solutions, it obviously that the numerical solutions of the two method are too close.

2.4.The solutions of**Num. MAD**are more accurate than their corresponding to**Num. MSR**.

2.5.The effect of time factor is evidently on the numerical solutions.

2.6. Error values increase as we get closer to $= 1$.

Future Works: In future works, we can suppose and solve a Fredholm nonlinear quadratic integral with a singular kernel in position or time.

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