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A Note on Nonsplitdomination number of a graph

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Abstract: *Let $G = (V, E)$ be any graph.A dominating set D of a graph G is a nonsplit dominat- ing if $< V - D >$ is connected. The minimum cardinality of a nonsplit dominating set is called nonsplit domination number $\gamma_{ns}(G)$. In this paper, we investigate several properties of this parameter.*

Keywords: *Domination, Domination number, Nonsplit Domination, Nonsplit Domination number.*
AMS subject classification: 05C 69

I. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph without loops or multiple edges.The order and size of G are denoted by p and q respectively. For graph theoretical terms we refer to Harary [6] and for terms related to domination we refer Haynes et al.[8] A subset D of V is said to be a dom- inating set in G if every vertex in $V - D$ is adjacent to atleast one vertex in D. Kulli and Janakiram introduced the concept of nonsplit domination in graphs [10]. A dominating set D of a graph G is a nonsplit dominating set if $< V - D >$ is connected. The nonsplit domination number $\gamma_{ns}(G)$ of G is the minimum cardinality of a nonsplit dominating set. A nonsplit dominating set with cardinality $\gamma_{ns}(G)$ is called a γ_{ns}-set. In this paper, we investigate several properties of this parameter.

II. Main Results

Theorem 2.1 [6] For any graph G, $\chi(G) \leq 1 + \Delta(G)$.

Proposition 2.2 For any connected graph G, $\gamma_{ns}(G) \leq p-1$. Further equality holds if and only if G is a star.

Proof.Every set $S \subseteq V(G)$ with $|S| = p - 1$ is a nonsplit dominating set of G and so $\gamma_{ns}(G) \leq p - 1$.
If G is a star, clearly $\gamma_{ns}(G) = p-1$. Suppose $\gamma_{ns}(G) = p-1$. If G is not a star, then G has an edge $e = uv$ such that both u and v are non - pendent vertices. Now $V(G) - \{u, v\}$ is a nonsplit dominating set of G and so $\gamma_{ns}(G) \leq p - 2$ which is a contradiction. Hence G is a star.

Remark2.3 1. If H is a spanning subgraph of G ,then $\gamma_{ns}(G) \leq \gamma_{ns}(H)$.
2. If H is any spanning subgraph of complete graph K_p with $\Delta(H) = p - 1$ and $|E(H)| = 2p - 3$, then $\gamma_{ns}(H) = \gamma_{ns}(K_p) = 1$.

Remark 2.4 1.For any graph G, $\gamma_{ns}(G) = 1$ if and only if $G \cong K_1 + H$ where H is a connected graph or a trivial graph.

2. For any graph G, $\gamma_{ns}(G) = p$ if and only if $G \cong \overline{K_p}$.

Theorem 2.5 For a non- trivial tree T, $\gamma_{ns}(T) \geq \Delta(T)$ and $\gamma_{ns}(T) = \Delta(T)$ if and only if $T \cong$ star or wounded spider.

Proof. Since T is a tree, T has at least $\Delta(T)$ pendent vertices. If $T \cong$ star then $\gamma_{ns}(T) = \Delta(T)$. If $T \ncong$ star then every nonsplit dominating set must contain all the pendent vertices and so $\gamma_{ns}(T) \geq \Delta(T)$.
Suppose $\gamma_{ns}(T) = \Delta(T)$ and $T \ncong$ star.Let v be a vertex of T such that $\deg v = \Delta(T)$. Let S be a γ_{ns}-set. S contains every pendent vertex of T .As $\gamma_{ns}(T) = \Delta(T)$, every component of $T - \{v\}$ must contain exactly one vertex of S. So v is adjacent to a pendent vertex. Since $T \ncong$ star there exists at least one vertex u in T such that $d(u, v) \geq 2$. If $d(u, v) = 3$ and u, u_1, u_2, v is

the path from u to v, then u, v_1, v_2 are in the same component of $T - \{v\}$ say $w_1(T)$. Then $|S \cap w_1(T)| \geq 2$ which is a contradiction. So every vertex of T is at a distance at most two from v. Every vertex except v in T must have degree

one or 2, otherwise $\Delta(T) <$ the number of pendents. So $T \cong$ wounded spider. If $G \cong$ star then $\gamma_{ns}(T) = \Delta(T)$. So when $\gamma_{ns}(T) = \Delta(T)$, then $T \cong$ star or a wounded spider. Converse is obvious.

Theorem 2.6 For any tree T not isomorphic to P_2, $\gamma_{ns}(\overline{T}) = 2$.

Proof. If diam (T) = 2, then $T \cong K_{1,p-1}$. If u is the central vertex and $v_1, v_2, \ldots, v_{p-1}$ are the pendent vertices then $\{u, v_i\}$ $(1 \leq i \leq p-1)$ are non-Split dominating sets in $\overline{T}$.

If diam (T) = 3 and if u, v are the supports then $\{u, v\}$ is a nonsplit domi- nating set in $\overline{T}$.If diam (T) = 4, let $P = (v_1, v_2, v_3, v_4, v_5)$ be the diametrical path in T. Then $\{v_1, v_4\}$ is a nonsplit dominating set in $\overline{T}$. If diam(T) ≥ 5, let $P = (v_1, v_2, \ldots, v_n)(n \geq 6)$ be the diametrical path in T.Then $\{v_1, v_2\}$ is a nonsplit dominating set in $\overline{T}$. Thus $\gamma_{ns}(\overline{T}) = 2$.

Lemma 2.7 If G is an isolate - free disconnected graph with at least 2 com- ponents then $\gamma_{ns}(\overline{G}) = 2$.

Proof. If u and v are two vertices lying in two different components of G, then $\{u, v\}$ is a minimum nonsplit dominating set of $\overline{G}$ and so $\gamma_{ns}(\overline{G}) = 2$.

Theorem 2.8 If G is a connected graph with at least 2 pendents then $\gamma_{ns}(\overline{G}) \leq 3$. Further $\gamma_{ns}(\overline{G}) = 2$ if and only if $G \ncong G_1$ where G_1 is the graph given in Fig 1

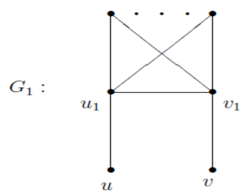

Fig 1

Proof. Claim 1: $\gamma_{ns}(\overline{G}) \leq 3$.

Suppose G has at least 3 pendent vertices. Let u, v and w be 3 pendent ver- tices with supports u_1, v_1 and w_1. If $u_1 = v_1 = w_1$ then $\{u, u_1\}$ is a nonsplit dominating set of $\overline{G}$. Since no subset of V (G) with cardinality 1 can be a nonsplit dominating set of $\overline{G}$, we have $\gamma_{ns}(\overline{G}) = 2$.

If $u_1 = w_1$ then also as above $\gamma_{ns}(\overline{G}) = 2$. If u_1, v_1, w_1 are distinct then obviously $\gamma_{ns}(\overline{G}) = 2$. Now let G contain exactly 2 pendent vertices. Let u and v be 2 pendents with supports u_1 and v_1 respectively.If $V(G) - \{u, v, u_1, v_1\} = \emptyset$ then $\gamma_{ns}(\overline{G}) = 2$. If $V(G) - \{u, v, u_1, v_1\} \neq \emptyset$ then $\{u, u_1, v_1\}$ is a nonsplit dominating set of G and so $\gamma_{ns}(\overline{G}) \leq 3$. So for a connected graph with at least 2 pendents, $\gamma_{ns}(\overline{G}) \leq 3$.

Claim 2:$\gamma_{ns}(\overline{G}) = 3$ if and only if $G \cong G_1$.

Since G is connected, $\overline{G} \ncong H + K_1$ for any connected graph and so by remark 2.4, $\gamma_{ns}(\overline{G}) \neq 1$. So $\gamma_{ns}(\overline{G}) = 2$ or 3.Let $\gamma_{ns}(\overline{G}) = 3$.From claim 1,we can Conclude that G contains exactly 2 pendent vertices.

Let u_1 and v_1 be the supports of the 2 pendents u and v respectively. If $u_1 = v_1$then $\{u, u_1\}$ is a γ_{ns}-set of $\overline{G}$.Let u_1 and v_1 be distinct and non-adjacent.

Then $\{u, x\}$ is a γ_{ns}-set of $\overline{G}$ where $x \in V(G) - \{u_1, v_1, v\}$, $x \notin N(u_1)$ and $N(u_1) \cap N(v_1) = \emptyset$. If $V(G) - \{u_1, v_1, u, v\} = N(u) \cap N(v)$ then $\{u_1, v\}$ is a γ_{ns}-set of $\overline{G}$.

Let u_1 and v_1 be adjacent. If there exists a vertex y such that $d(y, v_1) \geq 2$ (x

such that d(x, u_1) ≥ 2) then $\{u, u_1\}$ ($\{v, v_1\}$) is a γ_{ns}- set of $\bar{G}$. So d (v_1, y) = 1 for all y ∈ V (G) − $\{u\}$ and d(u_1 , y) = 1 for all y ∈ V (G) − $\{v\}$. So G ≅ G_1. If G ≅ G_1 obviously γ_{ns} $(\bar{G})$ = 3 as $\{u, u_1, v_1\}$ is a γ_{ns}-set of $\bar{G}$.
Thus $\gamma_{ns}(\bar{G})$ = 2 if and only if G is not isomorphic to G_1.

Theorem 2.9 If G is a connected graph with at least 2 pendent vertices, then
$3 \leq \gamma_{ns}(G) + \gamma_{ns}(\bar{G}) \leq p + 1$. Further $\gamma_{ns}(G) + \gamma_{ns}(\bar{G}) = 3$ if and only if
G ≅ K_2 and $\gamma_{ns}(G) + \gamma_{ns}(\bar{G}) = p + 1$ if and only if G ≅ $K_{1,p-1}$ or H where
H is given in Fig 2.

H :

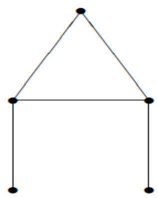

Fig 2

Proof. Clearly γ_{ns} (G) + $\gamma_{ns}(\bar{G}) \geq 3$. If $\gamma_{ns}(G) + \gamma_{ns}(\bar{G}) = 3$, then $\gamma_{ns}(G) = 1$ and $\gamma_{ns}(\bar{G}) = 2$. As G contains at least 2 pendent vertices and $\gamma_{ns}(G) = 1$, by

Remark 2.4(i), G ≅ K_2 .
By Proposition 2.2 and Theorem 2.8, $\gamma_{ns}(G) + \gamma_{ns}(\bar{G}) \leq p - 1 + 3 = p + 2$. If
$\gamma_{ns}(G) + \gamma_{ns}(\bar{G}) = p + 2$ then $\gamma_{ns}(G) = p - 1$ and $\gamma_{ns}(\bar{G}) = 3$. By Theorem
2.8 , $\gamma_{ns}(\bar{G}) = 3$ if and only if G ≅ G_1 where G_1 is in Figure 1. But for
G_1, $\gamma_{ns}(G) \neq p - 1$ and so there is no graph G with $\gamma_{ns}(G) + \gamma_{ns}(\bar{G}) = p + 2$.
Hence $\gamma_{ns}(G) + \gamma_{ns}(\bar{G}) \leq p + 1$.
If $\gamma_{ns}(G) + \gamma_{ns}(\bar{G}) = p + 1$, then either $\gamma_{ns}(G) = p - 1, \gamma_{ns}(\bar{G}) = 2$ or $\gamma_{ns}(G) =$
$p - 2, \gamma_{ns}(\bar{G}) = 3$. In the former case G ≅ star and in the latter case G ≅ H.
Converse is obvious.

Proposition 2.10 If T is a tree of order p ≥ 3 then $\gamma_{ns}(T)\gamma_{ns}(\bar{T}) = p$ if and only if $\gamma_{ns}(T) = \frac{p}{2}$.
Proof. Follows by Theorem 2.6.
Proposition 2.11 If T is a tree of order p ≥ 3, then $\gamma_{ns}(T) + \gamma_{ns}(\bar{T}) = p$ if and only if T has exactly two supports.
Proof. Follows from Theorem 2.6 and Theorem 2.2 of [14].
Theorem 2.12 Let G be a unicyclic graph with cycle C_p and δ(G) = 1. Then
1. $\gamma_{ns}(\bar{G}) = \chi(G) = 2$ if and only if p is even.

2. $\gamma_{ns}(\bar{G}) = \chi(G) = 3$ if and only if G ≅ G_1, G_2 where G_1 and G_2 are given in
Fig 3.

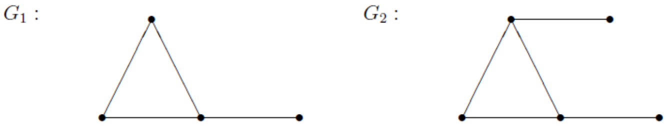

Fig 3

Proof. (1) If $\chi(G) = 2$ then p is even. Conversely, suppose that p is even. If G has two pendent vertices u, v with supports u_1, v_1 and $u_1 \neq v_1$, then for any other vertex $x \in C_p$, $\{u, x\}$ is a γ_{ns}- set of $\bar{G}$. If $u_1 = v_1$, then $\{u, u_1\}$ is a γ_{ns}- set of $\bar{G}$.

(2) If $\gamma_{ns}(\bar{G}) = \chi(G) = 3$ then p is odd and $C_p \cong C_3$ since otherwise $\gamma_{ns}(\bar{G}) =$ 2. If a tree rooted at a vertex of C_3 has diameter at least 2, then $\gamma_{ns}(\bar{G}) = 2$ and so every rooted tree is a P_2. If a vertex u of C_3 is of degree ≥ 4 then u with any pendent adjacent to u is a minimum nonsplit dominating set of G

and so every vertex of C_3 is of degree ≤ 3. If $G \cong K_3 \circ K_1$, then $\gamma_{ns}(\bar{G}) = 2$ and so $G \cong G_1$ or G_2. Converse is obvious.

Theorem 2.13 If G is a graph with a $\chi(G)$-colouring where every colour is used at least for 3 vertices then $\gamma_{ns}(\bar{G}) \leq \chi(G)$.

Proof. Let $\chi(G) = m$ and let $\{V_1, V_2, \ldots, V_m\}$ be the $\chi(G)$ partition of V (G). For each $1 \leq i \leq m$, $u_i \in V_i$, $S = \{u_1, u_2, \ldots u_m\}$ is a dominating set in $\bar{G}$. As $|V_i| \geq 3$ $\forall i$, $< V - S >$ has no isolated vertices in $\bar{G}$. Also for $1 \leq i \leq m$, every vertex of V_i is adjacent to at least one vertex of every $V_j, j \neq i$ and so S is a nonsplit dominating set of $\bar{G}$. Hence $\gamma_{ns}(\bar{G}) \leq m = \chi(G)$.

Theorem 2.14 Let G be any connected bipartite graph. Then $\gamma_{ns}(G)+\chi(G) = p + 1$ if and only if $G \cong K_{1,p-1}$.

Proof. Since $\chi(G) = 2$, the result follows by Proposition 2.2.

Theorem 2.15 For any connected graph G, $\gamma_{ns}(G) + \chi(G) \leq p + \Delta(G)$ and equality holds if G is a star.

Proof. Follows from Proposition 2.2 and Theorem 2.1.

Theorem 2.16 For any connected graph G, $\gamma_{ns}(G) +$ diam (G) $\leq 2p - 2$.Further (i) $\gamma_{ns}(G) +$ diam (G) $= 2p - 2$ if and only if $G \cong K_{1,2}$.

(ii) $\gamma_{ns}(G) +$ diam (G) $= 2p - 3$ if and only if $G \cong K_{1,3}$ or G_1, where G_1 is given in Fig 4.

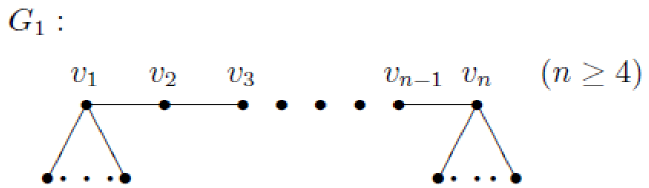

Fig 4

Proof.(i) Since a single vertex is assumed to be connected, γ_{ns} (G) $\leq p - 1$. Since G is connected, diam(G) $\leq p-1$. Hence $\gamma_{ns}(G)$+diam(G) $\leq 2p-2$. Sup- pose $\gamma_{ns}(G)$+diam (G) $= 2p-2$. Then $\gamma_{ns}(G) = p-1$ and diam(G) $= p-1$. By Proposition 2.2, $\gamma_{ns}(G) = p - 1$ if and only if $G \cong K_{1,p-1}$ and diam($K_{1,p-1}$) = 2 so that p = 3. Hence $G \cong K_{1,2}$. Converse is obvious. So (i) is proved.

(ii) Suppose $\gamma_{ns}(G) +$ diam(G) $= 2p - 3$. We have $\gamma_{ns}(G) = p - 1$ and Diam (G) $= p - 2$ or $\gamma_{ns}(G) = p - 2$ and diam(G) $= p - 1$. In the former case p = 4 and $G \cong K_{1,3}$. In the latter case, Theorem 2.2 of [14], $G \cong G_1$ where G_1 is given in Fig 4. Hence (ii) is proved.

Theorem 2.17 For any graph G, $\gamma_{ns}(G) + \kappa(G) \leq p + \Delta(G) - 1$, where $\kappa(G)$ is the connectivity of G and equality holds if and only if $G \cong K_2$.

Proof. For any graph G, $\gamma_{ns}(G) \leq p - 1$ and $\kappa(G) \leq \Delta(G)$ so that $\gamma_{ns}(G) + \kappa(G) \leq p + \Delta(G) - 1$. Suppose $\gamma_{ns}(G) + \kappa(G) = p + \Delta(G) - 1$. Then $\gamma_{ns}(G) = p - 1$ and $\kappa(G) = \Delta(G)$. By proposition 2.2, G $\cong K_{1,p-1}$. But now $\kappa(G) = 1$ and so $\Delta(G) = 1$. Hence G $\cong K_{1,1} = K_2$. Converse is obvious.

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