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Properties (T) and (Gt) for f (T) Type Operators

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Abstract: *in this paper we establish the relation with property (t) and (gt) introduced by M. H. M. RACHID in [1] and sprectrale mapping theorem. We will special establish several sufficient and necessary conditions for which f(T) verify the (t) and (gt) as f an analytic function on the T spectrum and see the validity of these results for the semi- Browder operators. Analogously we ask question about the conditions for which the spectral theory hold for generalized a-weyl's theorem and generalized a-browder's theorem [18].*

Keywords: *generalized a-weyl's theorem. Generalized a-browder'stheorem . Semi-Browder operators. Property (t).Property (gt).Spectral theory.*

I. Introduction and Preliminary

Throughout this paper, X denote an infinite-dimentional complex space, L(X) the algebra of all bounded linear operators on X. For $T \in L(X)$, let $T^*, \ker(T), R(T), \sigma(T), \sigma_a(T)$ and $\sigma_s(T)$denote the adjoint, the null space, the range, the spectrum, the approximate point spectrum and the surjectivity spectrum of T respectively. Let $\alpha(T)$and $\beta(T)$be the nullity and deficiency of T defined by $\alpha(T) = \dim \ker(T)$ and $\beta(T) = \operatorname{codim} R(T)$. Let $SF_+(X) = \{T \in L(X): \alpha(T) < \infty \text{ and } R(T) \text{ is closed}\}$and $SF_-(X) = \{T \in L(X): \beta(T) < \infty\}$denote the semi-group of upper semi-Fredholm and lower semi-Fredholm operators on X respectively. If both $\alpha(T)$and $\beta(T)$ are finite , then T is called Fredholm operator. If T is semi-Fredholm then the index of T is definedby $ind(T) = \alpha(T) - \beta(T)$.

A bounded linear operator T acting on a Banach space X is Weyl if it is Fredholm of index zero and Browder if T is Fredholm of finite ascent and descent. Let $\mathbb{C}$ denote the set of complex numbers. The Weyl spectrum and Browder spectrum of T are defined by $\sigma_w(T) = \{\lambda \in \mathbb{C}: T - \lambda \text{ is not Weyl}\}$and $\sigma_b(T) = \{\lambda \in \mathbb{C}: T - \lambda \text{ is not Browder}\}$respectively. For $T \in L(X)$, $SF_+^-(X) = \{T \in SF_+(X): ind(T) \le 0\}$. Then theupper Weyl spectrum of T is defined by
$\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C}: T - \lambda \notin SF_+^-(X)\}$. Let $\Delta(T) = \sigma(T) \backslash \sigma_w(T)$and $\Delta_a(T) = \sigma_a(T) \backslash \sigma_{SF_+^-}(T)$. Following Cuburn [10], we say that Weyl's theorem holds for $T \in L(X)$if $\Delta(T) = E^0(T)$, where $E^0(T) = \{\lambda \in \operatorname{iso} \sigma(T): 0 < \alpha(T - \lambda) < \infty\}$. Here and elsewhere in this paper, for $K \subset \mathbb{C}$, $isoK$ is the set of isoleted points of K.
According to Rakocevic [20], an operator $T \in L(X)$is said to satisfy a-Weyl's theorem if $E_a^0(T) = \sigma_a(T) \backslash \sigma_{SF_+^-}(T)$, where $E_a^0(T) = \{\lambda \in \operatorname{iso} \sigma_a(T): 0 < \alpha(T - \lambda) < \infty\}$.
It is known from [20] that an operator satisfying a-Weyl's theorem satisfies Weyl's theorem, but the converse does not hold in general. For $T \in L(X)$and a non negative integer n define $T_{[n]}$to be the restriction$T$ to $R(T^n)$ viewed as a map for $R(T^n)$ to $R(T^n)$(in particular $T_{[0]} = T$). If for some integer n the range space $R(T^n)$is closed and $T_{[n]}$is an upper (resp., lower) semi-Fredholm operator, then T is called upper (resp., lower) semi-B-Fredholm operator.

In this case index of T is defined as the index of semi-Fredholmoperator $T_{[n]}$. Moreover, if $T_{[n]}$is a Fredholm operator then T is called a B-Fredholm operator. An operator T is said to be B-Weyl operator if it is a B-Fredholm operator of index zero. Let $\sigma_{BW}(T) = \{\lambda \in \mathbb{C}: T - \lambda \text{ is not } B - Weyl\}$.Recall that the ascent, $a(T)$, of an operator$T \in L(X)$is the smallest non negative integer p such that $\ker(T^p) = \ker(T^{p+1})$and if such integer does not exist we put $a(T) = \infty$. Analogously the descent, $d(T)$, of an operator $T \in L(X)$is the smallest non negative integer q such that $R(T^q) = R(T^{q+1})$and if such integer does exist we put $d(T) = \infty$.According to Berkani [3], an operator $T \in L(X)$is said to be *Drazin invertible* if it has finite ascent and descent . The *Drazin spectrum* of T is defined by $\sigma_D(T) = \{\lambda \in \mathbb{C}: T - \lambda \text{ is not Drazin invertible}\}$. Define the set $LD(X) = \{T \in L(X): a(T) < \infty \text{ and } R(T^{a(T)+1}) \text{ is closed }\}and\sigma_{LD}(T) = \{\lambda \in \mathbb{C}: T - \lambda \notin LD(X)\}$. Following [4], an operator $T \in L(X)$is said to be left *Drazin invertible*if $T \in LD(X)$. We say that $\lambda \in \sigma_a(T)$is a left pole of T if $T - \lambda \in LD(X)$, and that $\lambda \in \sigma_a(T)$is a left pole of T of finite rank if $\lambda$is a left pole of T and $\alpha(T - \lambda) < \infty$[4, Definition 2.6]. Let $\pi_a(T)$denotes the set of all left poles of T and let $\pi_a^0(T)$denotes set of all left poles of finite rank. It follows from [4, theorem2.8] that if $T \in L(X)$is left Drazin invertible, then T is upper semi-B-

Fredholm of index less than or equal to 0.We say that *Browder's theorem* holds for $T \in L(X)$if $\Delta(T) = \pi^0(T)$, where $\pi^0(T)$is the set of all poles of T of finite rank and that *a-Browder's theorem* holds for T if $\Delta_a(T) = \pi_a^0(T)$. Let $\Delta^g(\mathrm{T}) = \sigma(\mathrm{T})\backslash\sigma_{\mathrm{BW}}(\mathrm{T})$. Following [3], we say that *generalized Weyl's theorem* holds for $T \in L(X)$if $\Delta^g(\mathrm{T}) = \mathrm{E}(\mathrm{T})$, $\mathrm{E}(\mathrm{T})$is the set of all eigenvalues of T which are isolated in $\sigma(T)$, and that *generalized Browder's Theorem* holds for T if $\Delta^g(\mathrm{T}) = \pi(T)$, where $\pi(T)$is the set of poles of T. It is proved in [8, theorem 2.1] that generalized Browder's theorem is equivalent to Browder's theorem.

Let $SBF_+^-(X)$denote the class of all upper semi-B-Fredholm operators such that $ind(T) \leq 0$. The upper *B-Weyl spectrum* of T is defined by $\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SBF_+^-(X)\}$. Let $\Delta_a^g(\mathrm{T}) = \sigma_a(T)\backslash \sigma_{SBF_+^-}(T)$. We say that $T \in L(X)$satisfies *generalized a-Weyl's theorem*, if $E_a(\mathrm{T}) = \sigma_a(T)\backslash \sigma_{SBF_+^-}(T)$, where $E_a(T)$is the set of all eigenvalues of T which are isolated in $\sigma_a(T)$and that $T \in L(X)$satisfies *generalized a-Browder's theorem* if $\Delta_a^g(\mathrm{T}) = \pi_a(T)[4, Definition\ 2.13]$. It is proved in [8, theorem 2.2] that generalized a-Browder's theorem is equivalent to a-Browder's theorem.

Following [20], we say that $T \in L(X)$satisfies property $(w) if\ \Delta_a(\mathrm{T}) = \mathrm{E}^0(\mathrm{T})$. The property (w) has been studied in [1, 5, 19]. In Theorem 2.8 of [5] , it is shown that property (w) implies Weyl's theorem, but the converse is not true in general. We say that $T \in L(X)$satisfies *property (gw)*if $\Delta_a^g(\mathrm{T}) = \mathrm{E}(\mathrm{T})$. Property *(gw)* extends *(w)* to the context of B-Fredholm theory, and it is proved in [9] that an operator possessing property *(gw)* satisfies property *(w)* but the converse is not true in general. According to [16] , an operator $T \in L(X)$is said to possess *property (gb)* if $\Delta_a^g(\mathrm{T}) = \pi(\mathrm{T})$, and is said to possess *property (b)* if $\Delta_a(\mathrm{T}) = \pi^0(\mathrm{T})$. it is shown in theorem 2.3 of [16] that an operator possessing property *(gb)* satisfies property *(b)* but the converse is not true in general. Following [10], we say an operator $T \in L(X)$, is said to be satisfies *property (R)* if $\pi_a^0(\mathrm{T}) = \mathrm{E}^0(\mathrm{T})$. In Theorem 2.4 of [8] , it is shown that T satisfies property *(w)* if and only if T satisfies a-Browder's theorem and T satisfies property *(R)* .

The single valued extension property plays an importants role in local spectral theory, see the recent monograph of Laursen and Neumann [21] and Aiena [1]. In this article we shall consider the following local version of this property, which has been studied in recent papers. [5,20'] and previously by Finch [19]. Following [19] we say that $T \in L(X)$has the *single-valued extension property (SVEP)* at point $\in \mathbb{C}$, if for every open neighborhood $U_\lambda$of λ, the only analytic function $f: U_\lambda \longrightarrow X$which satisfies the equation $(T-\mu)f(\mu) = 0$is the constant function $f \equiv 0$. It is well-known that $T \in L(X)$has SVEP at every point of the resolvent $\rho(T) = \mathbb{C}\backslash\sigma(T)$. Moreover, from the identity Theorem for analytic function it easily follow that $T \in L(X)$has SVEP at every point of the boundary $\partial\sigma(T)$of the spectrum. In particular, T has SVEP at every isolated point of $\sigma(T)$. In [20, Proposition 1.8], Laursen proved that if T is of finite ascent, then T has SVEP.

Theorem 1.1.[18, Theorem 1.3]$if\ T\ \in SF_\pm(X)\ the\ following\ statements\ ar\ equivalents$:
(i) $T\ has\ SVEP\ at\ \lambda_0$.
(ii) $a(T-\lambda_0 I) < \infty$.
(iii) $\sigma_a(T) does\ not\ cluster\ at\ \lambda_0$.
(iv) $H_0(T-\lambda_0 I)\ is\ finite\ dimentional$.
By duality we have
Theorem 1.2.$if\ T\ \in SF_\pm(X)\ the\ following\ statements\ ar\ equivalents$:
(i) $T^*\ has\ SVEP\ at\ \lambda_0$.
(ii) $d(T-\lambda_0 I) < \infty$.
(iii) $\sigma_s(T) does\ not\ cluster\ at\ \lambda_0$.

According to M. H. M. Rashid, we say that $T \in L(X)$satisfies *property (t) if* $\Delta_+(T) = E^0(T)$, where $\Delta_+(T) = \sigma(T)\backslash\sigma_{SF_+^-}(T)$, and T satisfies $property\ (gt)\ if\ \Delta_+^g(T) = E(T)$, where $\Delta_+^g(T) = \sigma(T)\backslash\sigma_{SBF_+^-}(T)$[18, Definition 2.1]. It shown in [18, Theorem 2.2] that if T satisfies property (gt), then T satisfies property (t). the converse of this Theorem does not hold in general. In seem reference it is proved that if T satisfies property (t), then T satisfies property (w) and we have equivalence if $\sigma(T) = \sigma_a(T)$. Analogously it is shown that if T satisfies property (gt), then T satisfies property (gw) and the equivalence hold is $\sigma(T) = \sigma_a(T)$. As a consequence in [2] it is proved that if T possesses property (t), then T satisfies Weyl's Theorem also T satisfies a-Browder's Theorem and $\pi_a^0(T) = E^0(T)$. More general if T possesses property (gt), then T satisfies generalized Weyl's Theorem also T satisfies generalized a-Browder's Theorem and $\pi_a(T) = E(T)$.

II. About generalized a- Weyl's and a-Browder's Theorems and SVEP

We start this part by this consequence
Corollary 2.1.if T obey generalized a-Browder's Theorem then T has the SVEP at $\lambda \notin \sigma_{SBF_+^-}(T)$
Proof. If T obey generalized a-Browder's Theorem, then$\Delta_a^g(\mathrm{T}) = \pi_a(T)$. Let $\lambda \in \pi_a(T)$then $\lambda$is isolated in $\sigma_a(T)$, so $T-\lambda$has SVEP en 0. (ie) T has SVEP at $\lambda \notin \sigma_{SBF_+^-}(T)$([19])

Recall that for each $T \in L(X)$and $f \in \mathcal{H}(\sigma(T))$, then we have $f\left(\sigma_{SBF_+^-}(T)\right) = \sigma_{SBF_+^-}(f(T))$.

We show now the relationship between the spectral mapping Theorem and generalized a-Weyl's Theorem.

Proposition 2.2. $Let T \in L(X) obey to generalized a-Weyl's theorem$
$then, for each\ f \in \mathcal{H}(\sigma(T))$: $\sigma_a(f(T))\setminus E_a(f(T)) = f(\sigma_a(T)\setminus E_a(T))$.

Proof. For the direct inclusion we use the argument in [18, lemma 3. 89].

To prove the inverse inclusion, let $\lambda_0 \in f(\sigma_a(T)\setminus E_a(T))$. From the equality $f(\sigma_a(T)) = \sigma_a(f(T))$we know that $\lambda_0 \in \sigma_a(f(T))$. Suppose that $\lambda_0 \in E_a(T)$, so $\lambda_0$is isolated in $\sigma_a(f(T))$. Now we can write $\lambda_0 - f(T) = p(T)g(T), with\ g(T)\ invertible\ and$

(1) $p(T) = \prod_{i=1}^{k}(\lambda_i - T)^{n_i}$.

From the equality (1) and the fact that T satisfies generalized a-Weyl's Theorem it follows any of $\lambda_1, \dots, \lambda_k$must be an isolated point in $\sigma_a(T)$, hence an eigenvalue of T. Moreover, since $\lambda_0$is an eigenvalue isolated in $\sigma_a(f(T))$any $\lambda_i$must also be an eigenvalue isolated an $\sigma_a(T)$, so $\lambda_i \in E_a(T)$. This contradicts $\lambda_0 \in f(\sigma_a(T)\setminus E_a(T))$. Therefore $\lambda_0 \notin E_a(T)$, so we have the equality $\sigma_a(f(T))\setminus E_a(f(T)) = f(\sigma_a(T)\setminus E_a(T))$ ■.

Corollary 2.3. If $T \in L(X)$satisfies generalized a-Weyl's Theorem then $\sigma_{SBF_+^-}(f(T)) = \sigma_a(f(T))\setminus E_a(f(T))$, $\forall f \in \mathcal{H}(\sigma(T))$.

Proof. It's immediate from the preview result and the fact that $f\left(\sigma_{SBF_+^-}(T)\right) = \sigma_{SBF_+^-}(f(T))$. ■

We should recall some results due to [19].

Theorem 2.4 [Aiena, Theorem 1.62] If $T \in SF_{\pm}(X)$ then Tis essentially semi-regular.

Theorem 2.5. [Aiena, Theorem 1. 83] Suppose that T ∈ L(X)is upper semi B-Fredholm. Then there exists an open disc D(0,ε)centered at0such thatλI − Tis upper semi-Fredholm for all $\lambda \in D(0;\varepsilon)\setminus\{0\}$

andind(λI − T) = ind(T), ∀λ ∈ D(0,ε).

(T)AND(GT) FOR EACH f in $H(\sigma(T))$

Theorem 2.6*[1 ,Theorem 2.10 (i)]$T \in L(X)$ satisfies property (gt)equivalent to Tsatisfies generalized Weyl's theorem and*

$$\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$$

Theorem 2.7 *Let*T ∈ L(X)*, and* T*satisfies SVEP at* $\lambda \notin \sigma_{SBF_+^-}(T)$;

T satisfies property (gt) *if and only if* $\sigma_{SBF_+^-}(f(T)) = \sigma(f(T))\setminus\pi(f(T))$ for each $f \in H(\sigma(T))$.

Proof. Assumption T satisfies property (gt) imply that T satisfies generalized Weyl's theorem and $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$. Let f ∈ H(σ(T)), when T has SVEP thenf(T)has the SVEP at $\lambda \notin \sigma_{SBF_+^-}(T)$, so by [20, theoerem 10] f(T) satisfies generalized Browder's theorem , (i.e) $\sigma_{BW}(f(T)) = \sigma(f(T))\setminus\pi(f(T))$

then by $f\left(\sigma_{SBF_+^-}(T)\right) = \sigma_{SBF_+^-}(f(T))$.we have the result ■.

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