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Signal Change Solution for a Fourth-Order Nonlinear Biharmonic Problem

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Abstract: *In this paper, through the establishment of a new space, we discuss the existence of sign-changing solution of a fourth-order nonlinear elliptic equation with Hardy potential in the new Hilbert space. The existence of sign-changing solution for fourth-order nonlinear elliptic equation are obtained under a linking theorem.*

Keywords: *sign-changing solution; nonlinear elliptic problem; (PS) condition;*
MSC 35J40; 35J65

I. Introduction

This work on the nonlinear fourth-order elliptic equations involves:

$$\begin{cases} \triangle^2 u - \frac{u}{|x|^4(lnR/|x|)^2} = f(x,\ u) & ,\ x \in \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0 & ,\ x \in \partial\Omega. \end{cases} \tag{1}$$

where $\triangle^2$ denotes the biharmonic operator, $\Omega \subset R^4$ is a bounded domain with smooth boundary.

We assume that $f(x,\ t)$ satisfies the following hypotheses in problem (1):

(f_1) $f(x,\ t) \in C(\overline{\Omega} \times R,\ R)$; $f(x,\ t)t \geq 0$, for all $x \in \Omega$ and $t \in R$;

(f_2) For $a.e.x \in \Omega,\ \frac{f(x,\ t)}{t}$ is nondecreasing with respect to $t > 0$.

(f_3) $\lim\limits_{|t|\to 0} \frac{f(x,\ t)}{t} = p(x),\ \lim\limits_{|t|\to\infty} \frac{f(x,\ t)}{t} = \beta$ uniformly in $a.e.x \in \Omega$, where $0 \leq p(x) \leq L^\infty(\Omega), |p(x)|_\infty < \lambda_1$, and $\beta > \lambda_k$ $\beta \in (0,\ +\infty)$,for some integer $k \geq 2$, and $\beta \neq \lambda_n (n = 1,\ 2,\ \cdots)$.

$\lambda_n (n = 1,\ 2,\ \cdots)$ is the eigenvalue of

$$\begin{cases} \triangle^2 u - \frac{u}{|x|^4(lnR/|x|)^2} = \lambda u & ,\ x \in \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0 & ,\ x \in \partial\Omega. \end{cases} \tag{2}$$

Define

$$\lambda_1 = \inf_{u\in H}\{\int_\Omega (|\triangle u|^2 - \frac{u^2}{|x|^4(lnR/|x|)^2})dx :\ \int_\Omega u^2 dx = 1\}, \tag{3}$$

Define

$$\lambda_1 = \inf_{u\in H}\{\int_\Omega (|\triangle u|^2 - \frac{u^2}{|x|^4(lnR/|x|)^2})dx :\ \int_\Omega u^2 dx = 1\}, \tag{3}$$

and

$$\lambda_n = \inf_{u\in H}\{\int_\Omega (|\triangle u|^2 - \frac{u^2}{|x|^4(lnR/|x|)^2})dx : \int_\Omega u^2 dx = 1,\ \int_\Omega u\varphi_i dx = 0,\ i = 1,\ 2,\ \cdots,\ n-1\},$$

where φ_n is the eigenfunction corresponding to λ_n.

II. Preliminaries and statements

We used a new Hilbert space H, which is the completion of $H_0^2(\Omega)$, with respect to the norm

$$\|u\|_H^2 = \int_\Omega (|\triangle u|^2 - \frac{u^2}{|x|^4(lnR/|x|)^2})dx,$$

whose corresponding inner product is

$$\langle u,\ v\rangle = \int_\Omega \left(\triangle u \triangle v - \frac{uv}{|x|^4(lnR/|x|)^2}\right) \mathrm{d}x.$$

in [9], assume $1 \le p < 2$, $H_0^2(\Omega) \subset H(\Omega) \subset W_0^{1,p}(\Omega)$. We first give some notation. The functional $I : H \to R$ corresponding to Problem (1) is defined by

$$I(u) = \frac{1}{2}\|u\|_H^2 - \int_\Omega F(x,\ u)dx$$

where $F(x,\ u) = \int_0^u f(x,\ t)dt$. It is easy to see that I is a C^1 function and its gradient at u is given by

$$I'(u) = u - \mathcal{K}(u),\quad \mathcal{K} : H \to H,\quad \mathcal{K}(u) = (\triangle^2 - \frac{1}{|x|^4(\ln R/|x|)^2})^{-1} f(x,\ u).$$

Then $\langle \mathcal{K}(u),\ v\rangle = \int_\Omega f(x,\ u)vdx$ for all $v \in H$. We consider the convex cones $P = \{u \in H : u \ge 0\}$ and $-P = \{u \in H : u \le 0\}$, moreover, for $\epsilon > 0$, define

$$P_0 = \{u \in H : \mathrm{dist}(u,\ P) < \epsilon\}, \qquad -P_0 = \{u \in H : \mathrm{dist}(u,\ -P) < \epsilon\},$$

$$\overline{P} := P_0 \cup (-P_0), \qquad S = H \backslash \overline{P},$$

$$P_1 = \{u \in H : \mathrm{dist}(u,\ P) < \frac{\epsilon}{2}\}, \qquad -P_1 = \{u \in H : \mathrm{dist}(u,\ -P) < \frac{\epsilon}{2}\}.$$

Then P_0 is open convex, $\pm P \subset \pm D_0$, S is closed.

It is easy to prove that the weak solution of (1) are the critical points of the function

$$I(u) = \frac{1}{2}(\int_\Omega |\triangle u|^2 \,\mathrm{d}x - \int_\Omega \frac{u^2}{|x|^4(lnR/|x|)^2}\,\mathrm{d}x) - \int_\Omega F(x,\ u)\,\mathrm{d}x. \tag{4}$$

where $F(x,\ u) = \int_0^u f(x,\ t)dt$.

For any $\varphi \in H$,

$$\langle I'(u),\ \varphi\rangle = \int_\Omega \triangle u \cdot \triangle\varphi \,\mathrm{d}x - \int_\Omega \frac{u\varphi}{|x|^4(lnR/|x|)^2}\,\mathrm{d}x - \int_\Omega f(x,\ u)\varphi\,\mathrm{d}x. \tag{5}$$

Theorem Assume that f satisfies $(f_1),(f_2)$ and (f_3), problem (1) has a sign-changing solution.

Proposition 1. (see [9]) Assume $G \in C^1(E,\ R)$ and $\mathcal{K}(\pm P_0) \subset \pm P_1$, a compact subset A of X links to a closed subset B of $E \backslash D$ with respect to Φ^* ,

$$a_0 := \sup_A G \le b_0 := \inf_B G.$$

If G satisfies (w-PS) $_c$ condition for any $c \in [b_0, \ \sup_{(t,\ u)\in[0,\ 1]\times A} G((1-t)u)]$, then $\mathcal{K}[a^* - \varepsilon, \ a^* + \varepsilon] \cap (H\backslash(-P \cup P)) \neq \emptyset$ for all ε small, where

$$a^* = \inf_{\Gamma\in\Phi^*} \sup_{\Gamma([0,\ 1],\ A)\bigcap S} G(u) \in [b_0, \sup_{\Gamma([0,\ 1],\ A)\bigcap S} G((1-t)u)].$$

Moreover, $K_{a^*} \subset B$, if $a^* = b_0$.

Remark. Proposition 1 is still true if G satisfies (PS) condition, since the (PS) condition implies the $(w-PS)$ condition.

III. Existence of sign-changing solutions

Definition 1. Any sequence $\{u_n\}$ satisfying

$$\sup_n |J(u_n)| < \infty, \qquad (1 + \|u_n\|)J'(u_n) \to 0,$$

is called a weak Palais-Smale sequence (in short, $(w-PS)$ sequence). If any weak (PS) sequence of J possesses a convergent subsequence, we say that J satisfies the $(w-PS)$ condition. If the supremum in (6) is replaced by: $J(u_n) \to c$ as $n \to \infty$, we say that J satisfies the $(w-PS)$ at level c, written as $(w-PS)_c$.

Define a class of contractions of E as follows:

$\Phi = \{\Gamma(\cdot, \ \cdot) \in C([0, \ 1]\times E, \ E)|\Gamma(0, \ \cdot)$; for each $t \in [0, \ 1)$, $\Gamma(t, \ \cdot)$ is a homeomorphism of E onto itself and $\Gamma^{-1}(\cdot, \ \cdot)$ is continuous on $[0, \ 1]\times E$; there exists an $x_0 \in E$ such that $\Gamma(1, \ x) = x_0$ for each $x \in E$ and that $\Gamma(t, \ x) \to x_0$ as $t \to 1$ uniformly on bounded subsets of $E\}$.

Obviously, $\Gamma(t, \ u) = (1-t)u \in \Phi$. Let $\Phi^* = \{\Gamma \in \Phi|\Gamma(t, \ D) \subset D\}$. Then $\Gamma(t, \ u) = (1-t)u \in \Phi^*$.

The following concept of linking can be found in[8, 10]

Definition 2. A subset A of E is linked (with respect to Φ) to B of E if $A \bigcap B = \emptyset$, for every $\Gamma \in \Phi$ there is a $t \in [0, \ 1]$ such that $\Gamma(t, \ A) \cap B \neq \emptyset$.

It is easy to see that if A links B with respect to Φ, then A also links B with Φ^* .

Lemma 1. The Hilbert space H is embedded into $L^2(\Omega)$ and the embedding is compact.

Proof. From Theorem A.2 of [11] , there exist $R_0 > 0, \ C_1 > 0$ such that $\forall R \geq R_0, \quad \forall u \in H_0^2$, we have that

$$\int_\Omega |\triangle u|^2 \, dx - \int_\Omega \frac{u^2}{|x|^4(lnR/|x|)^2} \, dx \geq C_1 \|u\|^2_{W_0^{1,p}(\Omega)}, \tag{6}$$

where $1 \leq p < 2$. Since $H_0^2(\Omega)$ is dense in $H(\Omega)$, then the above inequalities are hold on for any $u \in H(\Omega)$. It's easy to check that, $H(\Omega) \subset W_0^{1,\ p}(\Omega)$, so $H(\Omega) \hookrightarrow W_0^{1,\ p}(\Omega)$. Furthermore, if $p > \frac{3}{2}$, by Sobolev embedding theorem, the embedding $W_0^{1,\ p}(\Omega) \hookrightarrow L^2(\Omega)$ is compact. By [11] $H(\Omega) \hookrightarrow L^2(\Omega)$ and the embedding is compact, i.e. $H(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$.

Lemma 2. The minimizing problem (4) has a solution φ_1.

Proof. Let $\{u_n\}$ be a sequence, satisfies

$$\|u_n\|_H^2 \to \lambda_1,\ with \int_\Omega u_n^2 dx = 1.$$

Then $\{u_n\}$ is bounded in H . By $H \hookrightarrow\hookrightarrow L^2(\Omega)$, passing to a subsequence, still denoted by $\{u_n\}$, such that

$$u_n \to u,\ with \int_\Omega u^2 dx = 1.$$

Note that

$$\|\frac{u_n - u_m}{2}\|_H^2 + \|\frac{u_n + u_m}{2}\|_H^2 = \frac{1}{2}(\|u_n\|_H^2 + \|u_m\|_H^2)$$

for all $n,\ m \geq 1$, then

$$\|\frac{u_n - u_m}{2}\|_H^2 \leq \frac{1}{2}(\|u_n\|_H^2 + \|u_m\|_H^2) - \lambda_1 \int_\Omega (\frac{u_n + u_m}{2})^2 dx \to 0,$$

as $n,\ m \to \infty$. Hence, $\{u_n\}$ is a Cauchy sequence in H , which means u_n strongly converges to some φ_1 in H , and $\|\varphi_1\|_H^2 = \lambda_1$.

Lemma 3. $\lambda_n \to \infty$ as $n \to \infty$.

Proof. We may suppose that λ_n is bounded, then there exist $K > 0$ such that

$$0 < \lambda_n < K.$$

Then $\{u_n\}$ is bounded in H . By Lemma 1, passing to a subsequence, still denoted by $\{u_n\}$. But by definition of λ_n , we know that for $n \neq k$

$$\|u_k - u_n\|_{L^2}^2 = \int_\Omega |u_k - u_n|^2 dx = \int_\Omega u_k^2 dx - 2\int_\Omega u_k u_n dx + \int_\Omega u_n^2 dx = 2.$$

This is a contradiction.

Lemma 4. I satisfies the (PS) condition.

Proof. Assume $\{u_n\} \subset H$, $I(u_n) \to C$, $I'(u_n) \to 0$ as $n \to \infty$. We first prove that $\{u_n\}$ is bounded in H . In fact, otherwise, we may suppose that $\|u_n\|_H \to \infty (n \to \infty)$. Set $\omega_n = \frac{u_n}{\|u_n\|_H}$. Obviously, ω_n is bounded in H . Passing to a subsequence, still denoted by ω_n , we may assume that, for some $\omega \in H$,

$\omega_n \rightharpoonup \omega$, in H , $\omega_n \to \omega$, a.e.in Ω , $\omega_n \to \omega$, in $L^2(\Omega)$.

We claim that $\omega \not\equiv 0$. In fact, by the condition (f_1) and (f_2) , we see that for all $x \in \Omega,\ t \in R$, $\exists b > 0$ such that $|f(x,\ t)| \leq b|t|$. so we have $|\frac{F(x,\ t)}{t^2}| \leq b$ for all $x \in \Omega$. By $I(u_n) = \frac{1}{2}\|u_n\|_H^2 - \int_\Omega F(x,\ u_n)\,\mathrm{d}x$ and $|I(u_n)| \to C(n \to \infty)$, we have

$$o(1) = \frac{1}{2} - \int_\Omega \frac{F(x,\ u_n)}{\|u_n\|_H^2} dx, \tag{7}$$

where $o(1)$ denotes any quantity which tends to zero as $n \to \infty$.

Supposing $\omega \equiv 0$, we know that $\omega_n \to 0$ in $L^2(\Omega)$ and it follows from (8) that

$$\frac{1}{2} = \int_\Omega \frac{F(x,\ u_n)}{u_n^2}\omega_n^2 dx + o(1) \le b\int_\Omega \omega_n^2 dx + o(1) \to 0, \qquad n \to \infty.$$

which is impossible, so $\omega \not\equiv 0$. By $I'(u_n) \to 0$ as $n \to \infty$ and

$$\langle I'(u_n),\ \varphi\rangle = \int_\Omega \triangle u_n \triangle \varphi\, \mathrm{d}x - \int_\Omega \frac{u_n\varphi}{|x|^4(lnR/|x|)^2}\,\mathrm{d}x - \int_\Omega f(x,\ u_n)\varphi\,\mathrm{d}x,$$

we have

$$\int_\Omega \triangle\omega_n \triangle\varphi\,\mathrm{d}x - \int_\Omega \frac{\omega_n\varphi}{|x|^4(lnR/|x|)^2}\,\mathrm{d}x - \int_\Omega \frac{f(x,\ u_n)}{u_n}\omega_n\varphi\,\mathrm{d}x = o(1), \quad \forall\varphi \in H. \quad (8)$$

By there exists $b > 0$ such that $|f(x,\ t)| \le b|t|$ for all $x \in \Omega,\ t \in R$. If $\omega(x) = 0$, then

$$\frac{f(x,\ u_n)}{u_n}\omega_n \to 0 = \beta\omega(x), \qquad n \to \infty.$$

If $\omega(x) \ne 0$, then we have $|u_n| = \|u_n\|_H|\omega_n| \to \infty$ as $n \to \infty$. Thus, by the condition (f_3), we have

$$\frac{f(x,\ u_n)}{u_n}\omega_n \to \beta\omega(x), \qquad n \to \infty.$$

Therefore,

$$\frac{f(x,\ u_n)}{u_n}\omega_n \to \beta\omega(x), \qquad a.e.\ x \in \Omega.$$

Since $|f(x,\ t)| \le b|t|$ for all $x \in \Omega,\ t \in R$, we see that $\{\frac{f(x,\ u_n)}{u_n}\omega_n\}$ is bounded in $L^2(\Omega)$, thus there exists a subsequence such that $\frac{f(x,\ u_n)}{u_n}\omega_n \rightharpoonup \beta\omega$ in $L^2(\Omega)$. Hence

$$\int_\Omega \frac{f(x,\ u_n)}{u_n}\omega_n\varphi dx \to \int_\Omega \beta\omega\varphi dx, \qquad n \to \infty. \qquad (9)$$

Using (9), (10) and $\omega_n \rightharpoonup \omega(n \to \infty)$ in H, we have

$$\int_\Omega (\triangle\omega\triangle\varphi - \frac{\omega\varphi}{|x|^4(\ln R/|x|)^2})dx = \beta\int_\Omega \omega\varphi dx.$$

This implies that ω is a nontrivial solution of the following problem

$$\begin{cases} \triangle^2 u - \frac{u}{|x|^4(lnR/|x|)^2} = \beta u &, \quad \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0 &, \quad \text{on } \partial\Omega. \end{cases}$$

.

which contradicts that $\frac{\beta}{\lambda_n}$. Therefore $\{u_n\}$ is bounded in H. Passing to a subsequence, we may assume that $u_n \rightharpoonup u$ in H. By $\langle I'(u_n),\ \varphi\rangle \to 0$ as $n \to \infty$, setting $\varphi = u_n - u$ yields

$$\int_\Omega (\triangle u_n \triangle(u_n - u) - \frac{u_n(u_n - u)}{|x|^4(\ln R/|x|)^2})dx = \int_\Omega f(x,\ u_n)(u_n - u)dx. \qquad (10)$$

By the condition (f_2) and (f_3) we know that for any $\epsilon > 0$, there exists $C_1 > 0$, such that

$$|f(x,\ t)| \le (|p(x)|_\infty + \epsilon)|t| + C_1|t|^{p-1} \quad (2 < p < +\infty) \tag{11}$$

Then by (12) , the H ö der inequality, we conclude that

$$\begin{aligned} |\int_\Omega f(x,\ u_n)(u_n - u)dx| &\le \int_\Omega |f(x,\ u_n)(u_n - u)|dx \\ &\le \int_\Omega ((|p(x)|_\infty + \epsilon)|u_n| + C_1|u_n|^{p-1})|u_n - u|dx \to 0, \end{aligned}$$

as $n \to \infty$. Since $u_n \rightharpoonup u(n \to \infty)$ in H and

$$\limsup_{n\to\infty} \|u_n\|_H \ge \liminf_{n\to\infty} \|u_n\|_H \ge \|u\|_H,$$

by (10) , we have

$$0 \le \limsup_{n\to\infty}(\|u_n\|_H - \|u\|_H) = \limsup_{n\to\infty}\langle u_n,\ u_n - u\rangle = \limsup_{n\to\infty}\int_\Omega f(x,\ u_n)(u_n - u)dx \to 0$$

So from $\|u_n\|_H \to \|u\|_H$ we derive that $u_n \to u$ in H .

Rewrite I as

$$I(u) = \frac{1}{2}\|u\|_H^2 - \frac{1}{2}\beta\|u^-\|_{L^2}^2 - \frac{1}{2}\beta\|u^+\|_{L^2}^2 - \int_\Omega H(x,\ u)dx, \qquad u \in H_0^2(\Omega),$$

where

$$H(x,\ u) :\ = \int_0^u h(x,\ t)dt; \quad h(x,\ t) = f(x,\ t) - (\beta t^+ - \beta t^-); \quad t^\pm = \max\{\pm t,\ 0\}.$$

Let E_k denote the eigenspace of $\lambda_k (k \ge 1)$ and $H_k = E_1 \cup \cdots \cup E_k$.

Lemma 5. $I(u) \to -\infty$ for $u \in H_k$ with $\|u\|_H \to \infty$.

Proof. By $(f_1), (f_2)$ and (f_3) , there exist $C, \epsilon > 0$, such that
$(x,\ s) \in \Omega \times R$, we have
For $u = u_- + u_0 \in H_k$ with $u_- \in H_{k-1},\ u_0 \in E_k$, and

$$I(u) = \frac{1}{2}\|u\|_H^2 - \frac{1}{2}\beta\|u^-\|_{L^2}^2 - \frac{1}{2}\beta\|u^+\|_{L^2}^2 - \int_\Omega H(x,\ u)dx.$$

We have that

$$\begin{aligned} I(u) &\le \frac{1}{2}\|u\|_H^2 - \frac{1}{2}\beta\|u\|_{L^2}^2 - \int_\Omega H(x,\ u)dx \\ &\le \frac{1}{2}(1 - \frac{\beta}{\lambda_{k-1}})\|u_-\|_H^2 + \frac{1}{2}(1 - \frac{\beta}{\lambda_k})\|u_0\|_H^2 - \int_\Omega H(x,\ u)dx \\ &= \frac{1}{2}(1 - \frac{\beta}{\lambda_k})\|u\|_H^2 + \frac{1}{2}(\frac{\beta}{\lambda_k} - \frac{\beta}{\lambda_{k-1}})\|u_-\|_H^2 - \int_\Omega H(x,\ u)dx \\ &\le \frac{1}{2}(1 - \frac{\beta}{\lambda_k})\|u\|_H^2 - \int_\Omega H(x,\ u)dx. \end{aligned}$$

Therefore, there exist an $\varepsilon > 0$ such that

$$I(u) \leq -\varepsilon \|u\|_H^2 - \int_\Omega H(x,\ u)dx,$$

for all $u \in H_k$. Recall that $\lim_{|t|\to\infty} \frac{h(x,\ t)}{t} = 0$, thus we have $\lim_{\|u\|\to\infty} \frac{I(u)}{\|u\|_H^2} \leq -\varepsilon$, which implies the conclusion of the lemma.

Lemma 6. There exists $\rho_0,\ c > 0$ such that $I(u) \geq c$ for $u \in H_{k-1}^\perp$ with $\|u\|_H = \rho_0$.

Proof. By $(f_1), (f_2)$ and (f_3) , we see that for any $\epsilon > 0$, there exist constant $C_2 > 0$, such that for all $(x,\ s) \in \Omega \times R$, we have

$$F(x,\ s) \leq \frac{1}{2}(|p(x)|_\infty + \epsilon)s^2 + C_2 s^p. \tag{12}$$

Choosing $\epsilon > 0$ small enough such that $|p(x)|_\infty + \epsilon < \lambda_1$, by (13) and the define of λ_1 , we have

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|_H^2 - \int_\Omega F(x,\ u)dx \\ &\geq \frac{1}{2}\|u\|_H^2 - \frac{1}{2}(|p(x)|_\infty + \epsilon)\int_\Omega u^2 dx - C_2\|u\|_{L^p}^p. \end{aligned} \tag{13}$$

Let $2 < p < m$, by Hölder inequality, we have

$$\begin{aligned} \|u\|_{L^p} &= \Big(\int_\Omega |u|^p dx\Big)^{\frac{1}{p}} = \Big(\int_\Omega |u|^{\lambda p + (p-\lambda p)} dx\Big)^{\frac{1}{p}} \\ &\leq \Big(\Big(\int_\Omega |u|^{\lambda p \frac{2}{\lambda p}} dx\Big)^{\frac{\lambda p}{2}} \Big(\int_\Omega |u|^{(p-\lambda p)\frac{m}{p-\lambda p}} dx\Big)^{\frac{p-\lambda p}{m}}\Big)^{\frac{1}{p}} \\ &= \Big(\int_\Omega |u|^2 dx\Big)^{\frac{\lambda}{2}} \Big(\int_\Omega |u|^m dx\Big)^{\frac{1-\lambda}{m}} \\ &\leq \|u\|_{L^2}^{\lambda} \|u\|_{L^m}^{1-\lambda}, \end{aligned}$$

where $\frac{1}{p} = \frac{\lambda}{2} + \frac{1-\lambda}{m}$. By the above inequality, Lemma 1, embedding theorem, and note that $\int_\Omega u^2 dx \leq \frac{1}{\lambda_k}\|u\|_H^2$, we have

$$\begin{aligned} \|u\|_{L^p}^p &\leq \|u\|_{L^2}^{\lambda p} \|u\|_{L^m}^{(1-\lambda)p} \\ &\leq (\frac{1}{\lambda_k}\|u\|_H^2)^{\frac{\lambda p}{2}} (C\|u\|_H)^{(1-\lambda)p} \\ &= C^{(1-\lambda)p} \lambda_k^{-\frac{\lambda p}{2}} \|u\|_H^p. \end{aligned}$$

By (14) and the above inequality, we have

$$\begin{aligned} I(u) &\geq \frac{1}{2}(1 - \frac{|p(x)|_\infty + \varepsilon}{\lambda_1})\|u\|_H^2 - C_2 C^{(1-\lambda)p} \lambda_k^{-\frac{\lambda p}{2}} \|u\|_H^p \\ &\geq c. \end{aligned}$$

for some $c > 0$ with

$$\rho_0 : \ = \|u\|_H = (\frac{\lambda_k^{\frac{\lambda p}{2}}}{4C_2 C^{(1-\lambda)p}}(1 - \frac{|p(x)|_\infty + \epsilon}{\lambda_1}))^{\frac{1}{p-2}}.$$

Define

$$B_m: \ = (E_k \cup E_{k+1} \cup \cdots E_m) \cap B_{\rho_0(0)}$$

where ρ_0 comes from Lemma 6. Let

$$A: \ = \{u = \nu + sy_0 : \nu \in H_{k-1},\ s \geq 0,\ \|u\|_H = R\} \cup (H_{k-1} \cap B_R(0)),\ y_0 \in H_k,\ \|y_0\|_H = 1,$$

Then A and B_m link each other for any $R > \rho_0 > 0$ [8], and each u of B_m is sign-changing, Let $P_m = P \cap H_m$, then it is easy to check that dist $(B_m, -P_m \cup P_m) = \delta_m > 0$ since B_m is compact. Define

$$\pm D_0(m,\ r): \ = \{u \in H_m:\ \text{dist}(u,\ \pm P_m) < \rho\}$$

$$\pm D_1(m,\ r): \ = \{u \in H_m:\ \text{dist}(u,\ \pm P_m) < \frac{\rho}{2}\}$$

Let $m > k+2$, consider $I_m = I|_{H_m}$, the gradient of I_m can be expressed as $I_m' = \text{id} - \text{Proj}\ _m\mathcal{K}$, where Proj $_m$ denotes the projection of H onto H_m , $\mathcal{K}$ is given by $\mathcal{K}(u) = (\triangle^2 - \frac{1}{|x|^4(\ln R/|x|)^2})^{-1} f(x,\ u)$.

Lemma 7. There exists $\rho \in (0,\ \delta_m)$ such that

$$\text{Proj}_m\mathcal{K}(\pm D_0(m,\ \rho)) \subset \pm D_1(m,\ \rho).$$

Proof. Write $u^{\pm} = \max\{\pm u,\ 0\}$. For any $u \in H_m$,

$$\|u^+\|_{L^2} = \min_{\omega \in (-P_m)} \|u - \omega\|_{L^2} \leq \frac{1}{\lambda_1^{1/2}} \min_{\omega \in (-P_m)} \|u - \omega\|_H = \frac{1}{\lambda_1^{1/2}} \text{dist}(u, -P_m), \qquad (14)$$

and, for each $s \in (2,\ +\infty)$, there exists a $C_s > 0$ such that

$$\|u^{\pm}\|_{L^s} = \min_{\omega \in \mp P_m} \|u - \omega\|_{L^s} \leq C_s \min_{\omega \in \mp P_m} \|u - \omega\|_H = C_s \text{dist}(u, \mp P_m), \qquad (15)$$

By assumption (f_2) and (f_3) , we have

$$|f(x,\ t)| \leq (|p(x)|_\infty + \epsilon)|t| + C_1|t|^{p-1}, \quad x \in \Omega,\ t \in R \qquad (16)$$

where $2 < p < +\infty$. Choosing $\epsilon = \frac{\lambda_1 - |p(x)|_\infty}{5}$, then $|p(x)|_\infty + \epsilon < \lambda_1$. Let $v = \text{Proj}_m\mathcal{K}(u)$, satisfies $\|v^{\pm}\|_H = \min_{\omega \in \mp P_m} \|v - \omega\|_H$ (Note here $v^{\pm}$ is not the positive or negative part of v). Then by $(15)-(17)$,

$$\begin{aligned}
&\text{dist}(v, -P_m)\|v^+\|_H \leq \|v^+\|_H^2 \\
&= \langle v, v^+ \rangle = \int_\Omega f(x,\ u^+)v^+ dx \\
&\leq \int_\Omega ((|p(x)|_\infty + \epsilon)|u^+| + C_1|u^+|^{p-1})|v^+| dx \\
&\leq (|p(x)|_\infty + \epsilon)\|u^+\|_{L^2}\|v^+\|_{L^2} + C_1\|u^+\|_{L^p}^{p-1}\|v^+\|_{L^p} \\
&\leq \frac{(\lambda_1 + 4|p(x)|_\infty)}{5\lambda_1} \text{dist}(u, -P_m)\|v^+\|_H + C_p \text{dist}(u, -P_m)^{p-1}\|v^+\|_H.
\end{aligned}$$

That is,

$$\text{dist}(v, -P_m) \leq \frac{(\lambda_1 + 4|p(x)|_\infty)}{5\lambda_1} \text{dist}(u, -P_m) + C_p \text{dist}(u, -P_m)^{p-1}.$$

So, there exists a $\rho < \delta_m$ such that $\text{dist}(v, -P_m) < \frac{1}{2}\rho$ for every $u \in -D_0(m,\ \rho)$. Similarly, $\text{dist}(v,\ P_m) < \frac{1}{2}\rho$ for every $u \in D_0(m,\ \rho)$. The conclusion follows.

Proof of Theorem Let $D_m = -D_0(m,\ \rho) \cup D_0(m,\ \rho)$, $S_m := H_m \backslash D_m$. By lemma 4-lemma 7, all conditions of Proposition 1 are satisfied. Therefore, there exists a $u_m \in S_m$ such that

$$G_m'(u_m) = 0, \qquad G_m(u_m) \in [b_0, \sup_{(t,\ u)\in[0,\ 1]\times A} G((1-t)u)]$$

To prove G has a sign-changing critical point, we just have to prove that u_m has a convergent subsequence whose limit is still sign-changing. The proof of the existence of a convergent subsequence of u_m is the same as the proof of (PS) condition of Lemma 4. We just proof the limits of the subsequence is sign-changing. It follows by conditions (f_3) ,

$$\begin{aligned}
\|u_m^{\pm}\|_H^2 &= \int_\Omega f(x,\ u_m^{\pm})u_m^{\pm}dx \\
&\le \int_\Omega ((|p(x)|_\infty + \epsilon)|u_m^{\pm}| + C_1|u_m^{\pm}|^{p-1})u_m^{\pm}dx \\
&\le \int_\Omega (|p(x)|_\infty + \epsilon)|u_m^{\pm}|^2 + C_1|u_m^{\pm}|^p dx \\
&\le \frac{(|p(x)|_\infty + \epsilon)}{\lambda_1}\|u_m^{\pm}\|_H^2 + C'\|u_m^{\pm}\|_H^p
\end{aligned}$$

for some constant $C' > 0$. Hence, $\|u_m^{\pm}\|_H \ge c_0 > 0$. This implies that the limit of the subsequence is also sign-changing.

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