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The Existence of (s, t)-Monochromatic-rectangles in a 2-colored Checkerboard

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Abstract: *In this paper, we study the two edge-coloring of $K_{m,n}$ such that $K_{m,n}$ contains a monochromatic subgraph $K_{s,2}$, $K_{s,3}$ or $K_{s,t}$. We find the relation between n, s by investigating a two coloring of a checkerboard.*
Keywords: *Complete bipartite graph, monochromatic subgraph*

I. Introduction

We often encounter problems related the Ramsey numbers [1] in many Mathematical Competitions of High School Students. In this paper, we use the ideas of the Ramsey numbers on the checkerboard problems. We follow [2] for the notations in graph theory and the definitions of the complete bipartite graph $K_{m,n}$, and follow [3] to construct the correspondence between the checkerboards and complete bipartite graphs.

Jiong-Sheng Li provides the minimal sizes of k-colored square checkerboards which have monochromatic-rectangles in [3], we define the generalized monochromatic-rectangles and discuss the existence of such rectangles in an m x n checkerboard. In this paper, we only consider the checkerboards which are arbitrarily colored by two colors and we called it two-colored checkerboard.

In the second chapter, we discuss the minimal columns of the 2-colored checkerboard which has (2, 2)-monochromatic-rectangles by fixing the rows. At the end, we convert the results to graphic problems. In the third chapter, we extend the second chapter to discuss the minimal columns of the 2-colored checkerboard which has (2, t)-monochromatic-rectangles by fixing the rows. At the end, we convert the results to graphic problems. In the forth chapter, we discuss the minimal columns of the 2-colored checkerboard which has (3, 2)-monochromatic-rectangles by fixing the rows. At the end, we convert the results to graphic problems.

In the fifth chapter, we extend the forth chapter to discuss the minimal columns of the 2-colored checkerboard which has (3, t)-monochromatic-rectangles by fixing the rows. At the end, we convert the results to graphic problems. In the second chapter to the fifth chapter, all results have been proved in [4], but we improve the proof such that be more general and we also propose some amendments in the third chapter.

In the last two chapter, we propose the generalized conclusions. We discuss the minimal columns of the 2-colored checkerboard which has (s, t)-monochromatic-rectangles by fixing the rows. At the end, we convert the results to graphic problems. By [3] we convert the grids of a checkerboard into the edges of a complete bipartite graph, the number of rows and columns correspond to the number of vertices in complete partite sets, x and Y, respectively.

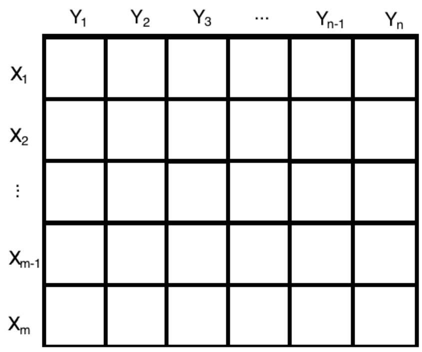

Figure 1: m x n checkerboard

If the grid in the *i*-th row and the *j*-th column of the checkerboard is black, then the correspond edge xiyj in the correspond complete bipartite graph is solid. And the white grid is correspond the dashed edge. The following is a 2-colored m x n checkerboard correspond to a 2-coloring $K_{m,n}$.

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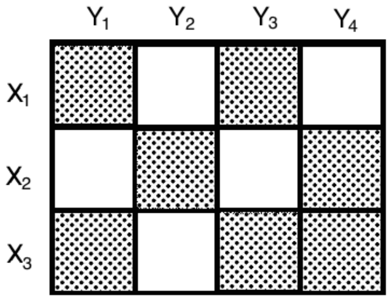

Figure 2: A 3 x 4 checkerboard

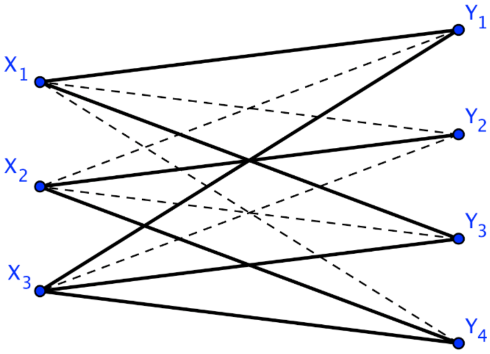

Figure 3: correspond complete bipartite graph of 3 x 4 checkerboard.

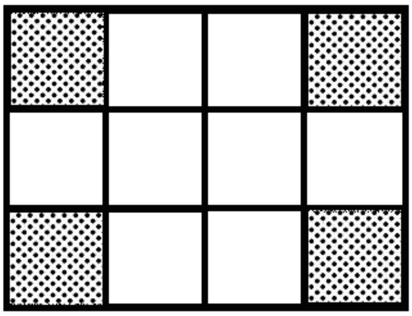

Figure 4: A (2, 2)-monochromatic-rectangle

1. (2, 2)-Monochromatic-rectangles in a Checkerboard

Definition 1

An m x n rectangle is called a ***(s, t)-monochromatic-rectangle***, if in first column there are s grids including the first one and the last one that have the same color, and there are other t-1 columns including the last column that are copies of the first column.

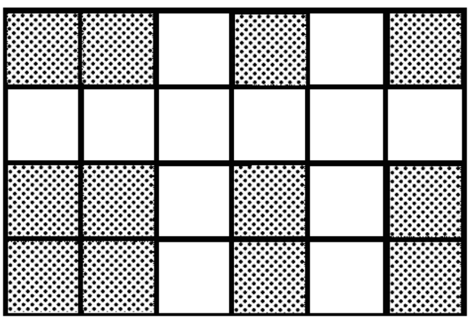

Figure 5: A (3, 4)-monochromatic-rectangle

1.1 The Case of 2 x n Checkerboard

If there are two (2, 1)-monochromatic-rectangles of the same color, then the checkerboard has a (2, 2)-monochromatic-rectangle. Otherwise, there is no (2, 2)-monochromatic-rectangle. Therefore, in a 2-colored 2 x n checkerboard a (2, 2)-monochromatic-rectangle may not exist.

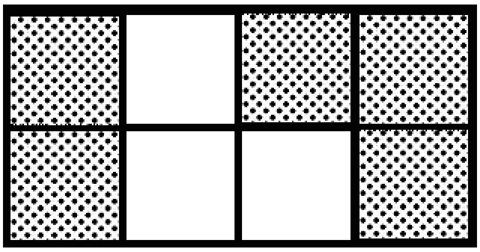

Figure 6: There are two (2, 1)-monochromatic-rectangles of the same color in the checkerboard.

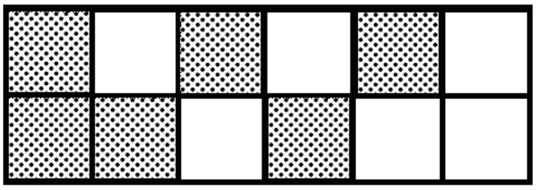

Figure 7: There is no two (2, 1)-monochromatic-rectangles of the same color in the checkerboard.

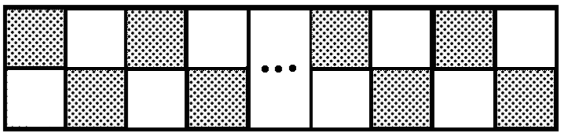

Figure 8: There is no two (2, 1)-monochromatic-rectangles of the same color in the checkerboard.

1.2 The Case of 3 x n Checkerboard

Definition 2

In a checkerboard, two (s, 1)-monochromatic-rectangles of the same color are **the same**, if one is a copy of the other one.

Definition 3

An n x 1 column **contains** a (s, 1)-monochromatic-rectangle means that there are s grids of the same color in the column.

Note: An n x 1 column contains at most $\binom{n}{s}$ distinct (s, 1)-monochromatic-rectangles.

Now, we consider

Lemma 1

In every 2-colored 3 x n checkerboard, n = 7 is the smallest number such that there exists a (2, 2)-monochromatic-rectangle.

Proof. To prove that we need to exhibit a 2-colored 3 x 6 checkerboard that has no (2, 2)-monochromatic-rectangles. In a column, there are at most $\binom{3}{2}$ distinct black (2, 1)-monochromatic-rectangles. So we can distribute the $\binom{3}{2}$ distinct black (2, 1)-monochromatic-rectangles and the $\binom{3}{2}$ distinct white (2, 1)-monochromatic-rectangles to the 6 columns, then the 2-colored 3 x 6 checkerboards have no (2, 2)-monochromatic-rectangles. By pigeonhole principle, there are at least $\left\lceil\frac{3\times 7}{2}\right\rceil = 11$ grids of the same color. Without loss of generality, let the color be black. Then we have $\sum_{i=1}^{7} d_i \geq 11$, where d_i is the number of black grids of the i^{th} column of the checkerboard. Assume 2-colored 3 x 7 checkerboard has a coloring such that there is no (2, 2)-monochromatic-rectangle, then any two columns don't contain the same black (2, 1)-

monochromatic-rectangles, each column contains $\binom{d_i}{2}$ distinct black (2, 1)-monochromatic-rectangles, and the total number of distinct black (2, 1)-monochromatic-rectangles is not more than $\binom{3}{2}$. So we have

$$\binom{d_1}{2} + \binom{d_2}{2} + \cdots + \binom{d_7}{2} \leq \binom{3}{2}.$$

Let $d_1 + d_2 + \ldots + d_7 = 11 + t$, where t is a positive integer. Then we can transform (\ref{E:1}) to

$$d_1^2 + d_2^2 + \ldots + d_7^2 \leqq 17 + t$$

By Cauchy-Schwarz inequality, we get

$$(d_1^2 + d_2^2 + \ldots + d_7^2)(1^2 + 1^2 + \ldots + 1^2) \geqq (d_1 + d_2 + \ldots + d_7)^2 \Longrightarrow (17+t) * 7 \geqq (11+t)^2$$

So, we have

$$t^2 + 15t + 2 \leqq 0$$

But t is positive, we reach a contradiction in the last inequality. Therefore, every 2-coloring of 3 x 7 checkerboard yields a (2, 2)-monochromatic-rectangle.

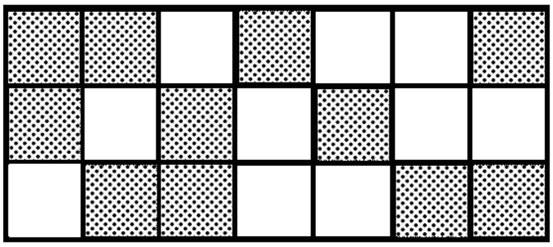

Figure 9: Every 2-colored 3 x 7 checkerboard contains a (2, 2)-monochromatic-rectangle.

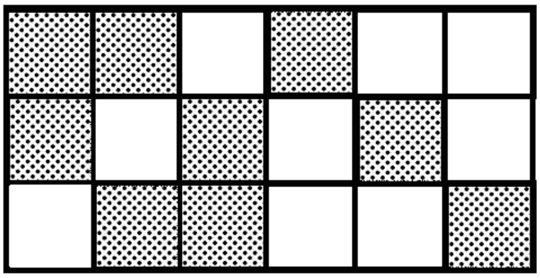

Figure 10: There is a 2-colored 3 x 6 checkerboard containing no (2, 2)-monochromatic-rectangles.

1.3 The Case of 4 x n Checkerboard

Lemma 2

In every 2-colored 4 x n checkerboard, n = 7 is the smallest number such that there exists a (2, 2)-monochromatic-rectangle.

Proof. By Lemma 1, in every 2-colored 3 x 7 checkerboard, there is a (2, 2)-monochromatic-rectangle. Therefore, in every 2-colored 4 x 7 checkerboard, there is a (2, 2)-monochromatic-rectangle.

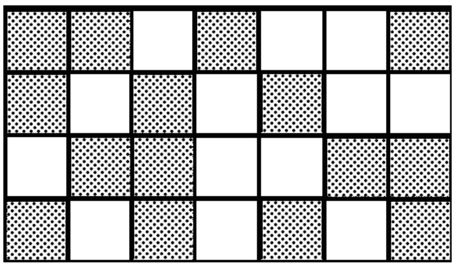

Figure 11: Every 2-colored 4 x 7 checkerboard contains a (2, 2)-monochromatic-rectangle.

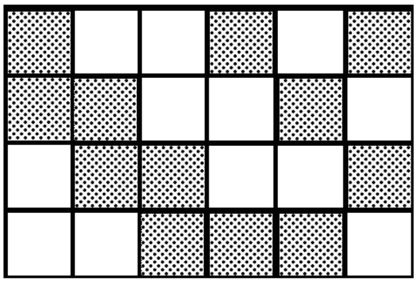

Figure 12: There is a 2-colored 4 x 6 checkerboard containing no (2, 2)-monochromatic-rectangles.

1.4 The Case of 5 x n Checkerboard

Lemma 3

In every 2-colored 5 x n checkerboard, n = 5 is the smallest number such that there exists a (2, 2)-monochromatic-rectangle.

Proof. To prove that we need to exhibit a 2-colored 5 x 4 checkerboard that has no (2, 2)-monochromatic-rectangles. By Lemma, we have a 2-colored 4 x 5 checkerboard which doesn't have (2, 2)-monochromatic-rectangles. We rotate the checkerboard, so we have the 2-colored 5 x 4 checkerboard that has no (2, 2)-monochromatic-rectangles. By pigeonhole principle, there are at least $\left\lceil \frac{5\times5}{2} \right\rceil = 13$ grids of the same color. Without loss of generality, let the color be black. Then we have $\sum_{i=1}^{5} d_i \geq 13$, where d_i is the number of black grids of the i^{th} column of the checkerboard. Assume two colored 5 x 5 checkerboard has a coloring such that there is no (2, 2)-monochromatic-rectangles, then any two columns don't contain the same black (2, 1)-monochromatic-rectangles, each column contains $\binom{d_i}{2}$ distinct black (2, 1)-monochromatic-rectangles, and the total number of distinct black (2, 1)-monochromatic-rectangles is not more than $\binom{5}{2}$. So, we have

$$\binom{d_1}{2} + \binom{d_2}{2} + \cdots + \binom{d_5}{2} \leq \binom{5}{2}$$

Let $d_1 + d_2 + \ldots + d_5 = 13 + t$, where t is a positive integer. Then we can transform (\ref{E:2}) to

$$d_1^2 + d_2^2 + \ldots + d_5^2 \leqq 23 + t$$

By Cauchy-Schwarz inequality, we get

$$(d_1^2 + d_2^2 + \ldots + d_5^2)(1^2 + 1^2 + \ldots + 1^2) \geqq (d_1 + d_2 + \ldots + d_5)^2 \Longrightarrow (23 + t) * 5 \geqq (13 + t)^2.$$

So we have

$$t^2 + 21\,t + 54 \leqq 0$$

But t is positive, we reach a contradiction in the last inequality. Therefore, every 2-colored 5 x 5 checkerboard yields a (2, 2)-monochromatic-rectangle.

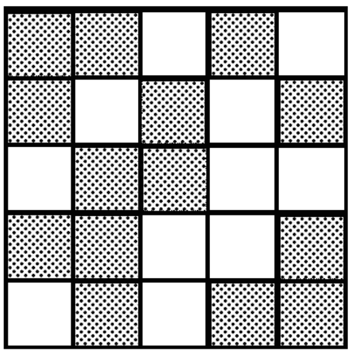

Figure 13: Every 2-colored 5 x 5 checkerboard contains a (2, 2)-monochromatic-rectangle.

1.5 Summary

The case of m x n checkerboard, where m ≧ 6, can be obtained by rotation of the rectangles. For example, 6 x 5 checkerboard can be considered to 5 x 6 checkerboard, so it has a (2, 2)-monochromatic-rectangle.
We can convert the above theorems to graphic problems. We have the following proposition.

- If n > 6, every 2-coloring of $K_{3,n}$ contains a monochromatic $K_{2,\,2}$
- If n > 6, every 2-coloring of $K_{4,n}$ contains a monochromatic $K_{2,\,2}$.
- If n > 4, every 2-coloring of $K_{5,n}$ contains a monochromatic $K_{2,\,2}$.

Example
Every 2-coloring of $K_{3,7}$ exists a monochromatic $K_{2,\,2}$ subgraph.

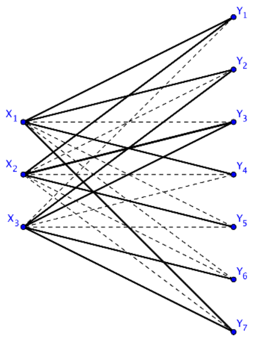

Figure 14: The subgraph induced by $\{X_1, X_3, Y_2, Y_7\}$ is a monochromatic copy of $K_{2,\,2}$.

Example
Every 2-coloring of $K_{4,7}$ exists a monochromatic $K_{2,\,2}$ subgraph.

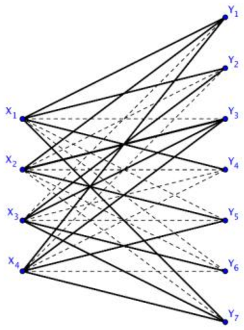

Figure 15: The subgraph induced by $\{X_1, X_3, Y_2, Y_7\}$ is a monochromatic copy of $K_{2,\,2}$.

Example
Every 2-coloring of $K_{5,5}$ exists a monochromatic $K_{2,\,2}$ subgraph.

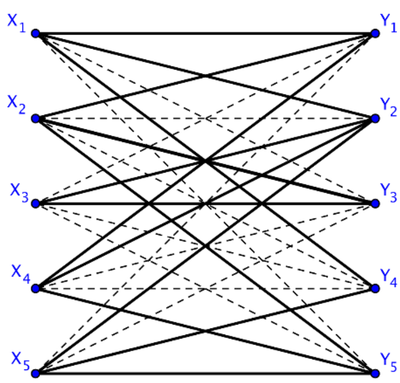

Figure 16: The subgraph induced by $\{X_4, X_5, Y_2, Y_5\}$ is a monochromatic copy of $K_{2,\,2}$.

II. (2, T)-Monochromatic-Rectangles In A Checkerboard

2.1 The Case of 2 x n Checkerboard
If there are t (2, 1)-monochromatic-rectangles of the same color, then the checkerboard has a (2, t)-monochromatic-rectangle. Otherwise, there is no (2, t)-monochromatic-rectangle.

2.2 The Case of 3 x n Checkerboard
Theorem 1
If n > (6t - 6), where t $\geqq$ 2, then in every 2-colored 3 x n checkerboard. There is a (2, t)-monochromatic-rectangle.
Proof. If n = (6t - 6) + 1 = 6t - 5 (We only prove that every coloring of two colors 3n x (6t - 5) checkerboard, there is a (2, t)-monochromatic-rectangle.) By pigeonhole principle, there are at least $\left\lceil\frac{6t-5}{2}\right\rceil = 3\text{t} - 2$ columns that have at least two of same color grids. Without loss of generality, let the color be black. Then $d_i \geqq 2$, i = 1, 2, …, (3t - 2) , where d_i is the number of black grids of the i^{th} column of the checkerboard. Assume two colored 3 x (6t - 5) checkerboard has a coloring such that there is no (2, t)-monochromatic-rectangles, then any t columns don't contain the same black (2, 1)-monochromatic-rectangles, each column contains $\binom{d_i}{2}$ distinct black (2, 1)-monochromatic-rectangles, and the total number of distinct black (2, 1)-monochromatic-rectangles is not more than $(\text{t} - 1)\binom{3}{2}$. So we have,

$$\binom{d_1}{2} + \binom{d_2}{2} + \cdots + \binom{d_{3t-2}}{2} \leq (\text{t} - 1)\binom{3}{2}.$$

Because $d_i \geqq 2$,

$$\binom{2}{2} + \binom{2}{2} + \cdots + \binom{2}{2} \leq \binom{d_1}{2} + \binom{d_2}{2} + \cdots + \binom{d_{3t-2}}{2}.$$

Combining these two results shows 3t - 2 $\leqq$ 3t - 3 $\Longrightarrow$ 1 $\leqq$ 0 leads a contradiction.
So, If n > (6t - 6), where t $\geqq$ 2, then in every 2-coloring of 3 x n checkerboard. There is a (2, t)-monochromatic-rectangle.

2.3 The Case of 4 x n Checkerboard
Theorem 2
If n > (6t - 6), where s $\geqq$ 2, then in every 2-colored 4 x n checkerboard. There is a (2, t)-monochromatic-rectangle.
Proof. By above theorem, in every 2-colored 3 x (6t - 6) checkerboard, there is a (2, t)-monochromatic-rectangle. Therefore, in every 2-colored 4 x (6t - 6) checkerboard, there is a (2, t)-monochromatic-rectangle.

2.4 The Case of 5 x n Checkerboard
Theorem 3
If n > (5t - 6), where t $\geqq$ 2, then in every 2-colored 5 x n checkerboard. There is a (2, t)-monochromatic-rectangle, where t is even. And $\binom{d_1}{2} + \binom{d_2}{2} + \cdots + \binom{d_n}{2} > (2t - 3)\binom{5}{2}$, where d_i is the number of black grids of the i^{th} column of the checkerboard.

Proof. Suppose t = 2k - 2, where k is integer greater than two, we use induction on k. If n = (10k - 16) + 1 = 10k - 15 (We only prove that every coloring of two colors 5 x (10k - 15) checkerboard, there is a (2k - 2, 2)-monochromatic-rectangle.
Basis step: When k = 2, by Lemma, we have in every 2-coloring of 5 x 5 checkerboard, there is a (2, 2)-monochromatic-rectangle.
Suppose k = s is true, s $\geqq$ 2 and s is a positive integer for all 2-coloring of 5 x (10s - 15) checkerboard, there is a (2, 2s - 2)-monochromatic-rectangle. By pigeonhole principle, there are at least $\left\lceil\frac{5\times(10s-15)}{2}\right\rceil = 25\text{s} - 37$ grids of the same color. Without loss of generality, let the color be black. So, we have $\sum_{\text{i}=1}^{10\text{s}-15} d_i \geq 25\text{s} - 37$, and $\binom{d_1}{2} + \binom{d_2}{2} + \cdots + \binom{d_{10s-15}}{2} > (2s - 3)\binom{5}{2}$.
Induction step: When k = s + 1, n = 10(s + 1) - 15=10s - 5, by pigeonhole principle, there are at least $\left\lceil\frac{5\times(10s-5)}{2}\right\rceil = 25\text{s} - 12$ grids of the same color in 5 x (10s - 5) checkerboard. Without loss of generality, let the color be black.
We have $\sum_{\text{i}=1}^{10\text{s}-5} d_i = 25\text{s} - 12$.By induction hypothesis, $\sum_{\text{i}=11}^{10\text{s}-15} d_i = 25\text{s} - 37$, so $\sum_{\text{i}=1}^{10} d_i = 25$ and $\binom{d_i}{2}$ is the number of black (2, 1)-monochromatic-rectangles in the i^{th} column, I = 1, 2, …, (10s - 5). Assume two colored 5 x (10s - 5) checkerboard has a coloring such that there is no (2, 2s)-monochromatic-rectangles, then

any 2s columns don't contain the same black (2, 1)-monochromatic-rectangles, each column contains $\binom{d_i}{2}$ distinct black (2, 1)-monochromatic-rectangles, and the total number of distinct black (2, 1)-monochromatic-rectangles is not more than $(2s-1)\binom{5}{2}$. So we have

$$\binom{d_1}{2}+\binom{d_2}{2}+\cdots+\binom{d_{10s-5}}{2}\le(2s-1)\binom{5}{2}.$$

By induction hypothesis

$$\binom{d_{11}}{2}+\binom{d_{12}}{2}+\cdots+\binom{d_{10s-5}}{2}>(2s-3)\binom{5}{2}.$$

So, we have

$$\binom{d_1}{2}+\binom{d_2}{2}+\cdots+\binom{d_{10}}{2}<(2s-1)\binom{5}{2}-(2s-3)\binom{5}{2}\Rightarrow\binom{d_1}{2}+\binom{d_2}{2}+\cdots+\binom{d_{10}}{2}<20.$$

Hence,

$$(d_1^2 + d_2^2 + \ldots + d_{10}^2) - (d_1 + d_2 + \ldots + d_{10}) < 40.$$

$$d_1^2 + d_2^2 + \ldots + d_{10}^2 < 65.$$

By Cauchy-Schwarz inequality

$(d_1^2 + d_2^2 + \ldots + d_{10}^2)*10 \geqq (d_1 + d_2 + \ldots + d_{10})^2$.

So, we have

$$62.5 \leqq (d_1^2 + d_2^2 + \ldots + d_{10}^2) \leqq 64.$$

Because d_i are integer, therefore, $(d_1^2 + d_2^2 + \ldots + d_{10}^2)$ must be 63 or 64.

Case1: $(d_1^2 + d_2^2 + \ldots + d_{10}^2) = 63$
Since

$$d_2^2 + d_3^2 + \ldots + d_{10}^2 = 63 - d_1^2$$

and

$$d_2 + d_3 + \ldots + d_{10} = 25 - d_1,$$

, by Cauchy-Schwarz inequality, we have

$$(d_2^2 + d_3^2 + \ldots + d_{10}^2) *9 \geqq (d_2 + d_3 + \ldots + d_{10})^2$$

That is,

$$(63-d_1^2) *9 \geqq (25 - d_1)^2.$$

Hence,

$$10 d_1^2 - 50 d_1 + 58 \leqq 0.$$

Therefore,

$$1.83 \leqq d_1 \leqq 3.17.$$

By the integrity of d_1, we get that d_1 must be 2 or 3. Similarly, $d_2, d_3, \ldots, d_{10}$ must be 2 or 3.
But $d_1 + d_2 + \ldots + d_{10} = 25$, so $d_1, d_2, \ldots, d_{10}$ consists of five 2's and five 3's.
Therefore, $d_1^2 + d_2^2 + \ldots + d_{10}^2 = 65 \neq 63$ which leads to a contradiction.

Case2: $(d_1^2 + d_2^2 + \ldots + d_{10}^2) = 64$
Since

$$d_2^2 + d_3^2 + \ldots + d_{10}^2 = 64 - d_1^2$$

and

$$d_2 + d_3 + \ldots + d_{10} = 25 - d_1.$$

by Cauchy-Schwarz inequality, we have

$$(d_2^2 + d_3^2 + \ldots + d_{10}^2)*9 \geqq (d_2 + d_3 + \ldots + d_{10})^2$$

$$d_2 + d_3 + \ldots + d_{10} \geqq 25 - d_1.$$
$$(64 - d1^2)*9 \geqq (25 - d_1)^2$$

$$10\, d_1^2 - 50\, d_1 + 49 \leqq 0$$

$$1.34 \leqq d_1 \leqq 3.66$$

By the integrity of d_1, we get that d_1 must be 2 or 3. Similarly, $d_2, d_3, \ldots, d_{10}$ must be 2 or 3.
But $d_1 + d_2 + \ldots + d_{10} = 25$, so $d_1, d_2, \ldots, d_{10}$ consists of five 2's and five 3's.
Therefore, $d_1^2 + d_2^2 + \ldots + d_{10}^2 = 65 \neq 64$ which leads to a contradiction.

So for all 2-colored 5 x (10s - 5) checkerboard, there is a (2, 2s)-monochromatic-rectangle.

By induction, all k $\geqq$ 2 and k is positive integer, in every 2-colored 5 x (5t - 6) checkerboard, there is a (2, t)-monochromatic-rectangle, where t=2k - 2.
Lemma 4
In every 2-colored 5 x 11 checkerboard, there is a (2, 3)-monochromatic-rectangle.
Proof. By pigeonhole principle, there are at least $\left\lceil \frac{5\times 11}{2} \right\rceil = 28$ grids of the same color. Without loss of generality, let the color be black. Then we have $\sum_{i=1}^{11} d_i \geq 28$, where d_i is the number of black grids of the i^{th} column of the checkerboard. Assume two colored 5 x 11 checkerboard has a coloring such that there is no monochromatic-rectangles, then any three columns don't contain the same black (2, 1)-monochromatic-rectangles, each column contains $\binom{d_i}{2}$ distinct black (2, 1)-monochromatic-rectangles, and the total number of distinct black (2, 1)-monochromatic-rectangles is not more than $2\binom{5}{2}$. So we have

$$\binom{d_1}{2} + \binom{d_2}{2} + \cdots + \binom{d_{11}}{2} \leq 2\binom{5}{2}.$$

Let $d_1 + d_2 + \ldots + d_{11} = 28 + t$, where t is a positive integer. Then we have

$$d_1^2 + d_2^2 + \ldots + d_{11}^2 \leqq 68 + t.$$

By Cauchy-Schwarz inequality, we get

$$(d_1^2 + d_2^2 + \ldots + d_{11}^2)(1^2 + 1^2 + \ldots + 1^2) \geqq (d_1 + d_2 + \ldots + d_{11})^2 \Longrightarrow (68 + t)*11 \geqq (28 + t)^2.$$

So, we have

$$t^2 + 45t + 36 \leqq 0.$$

But t is positive, the last inequality is contradiction. Therefore, every 2-colored 5 x 11 checkerboard yields a (2, 3)-monochromatic-rectangle.

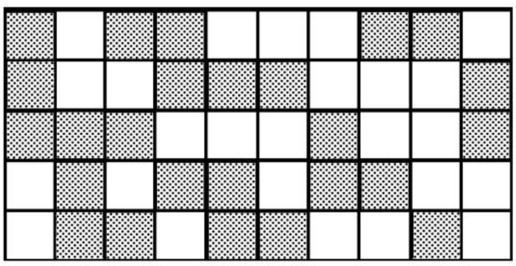

Figure 17: There is a 2-colored 5 x 10 checkerboard containing no a (2, 3)-monochromatic-rectangle

Theorem 4
If n > (5t - 5), where t $\geqq$ 2, then in every 2-colored 5 x n checkerboard. There is a (2, t)-monochromatic-rectangle, where s is odd. And $\binom{d_1}{2} + \binom{d_2}{2} + \cdots + \binom{d_n}{2} > (2t-2)\binom{5}{2}$, where d_i is the number of black grids of the i^{th} column of the checkerboard.

Proof. Suppose t = 2k - 1, where k is integer greater than two, we use induction on k, If n = (10k - 10) + 1 = 10k - 9 (we only prove that every 2-colored 5 x (10k - 9) checkerboard, there is a (2, 2k - 1)-monochromatic-rectangle.

Basis step: When k = 2, by Lemma, we have in every 2-colored 5 x 11 checkerboard, there is a (2, 3)-monochromatic-rectangle.

Suppose k = s is true, s ≧ 2 and s is a positive integer. Then for all 2-colored 5 x (10s - 9) checkerboard, there is a (2, 2s - 1)-monochromatic-rectangle. By pigeonhole principle, there are at least $\left\lceil\frac{5\times(10s-9)}{2}\right\rceil = 25\text{s} - 22$ grids of the same color. Without loss of generality, let the color be black. So, we have $\sum_{\text{i}=1}^{10\text{t}-9} d_i = 25\text{s} - 22$ and $\binom{d_1}{2} + \binom{d_2}{2} + \cdots + \binom{d_{10s-9}}{2} > (2s-2)\binom{5}{2}$.

Induction step: When k = s + 1, n = 10(s + 1) – 9 = 10s + 1, by pigeonhole principle, there are at least $\left\lceil\frac{5\times(10s+1)}{2}\right\rceil = 25\text{s} + 3$ grids of the same color in 5 x (10s + 1) checkerboard. Without loss of generality, let the color be black.

We have $\sum_{\text{i}=1}^{10\text{s}+1} d_i = 25\text{s} + 3$.

By induction hypothesis, $\sum_{\text{i}=11}^{10s-15} d_i = 25s - 22$, so $\sum_{\text{i}=1}^{10} d_i = 25$ and $\binom{d_i}{2}$ is the number of black-bars in the i^{th} column, I = 1, 2, …, (10s + 1).

Assume two colored 5 x (10s + 1) checkerboard has a coloring such that there is no (2, 2s + 1)-monochromatic-rectangles, then any 2s + 1 columns don't contain the same black (2, 1)-monochromatic-rectangles, each column contains $\binom{d_i}{2}$ distinct black (2, 1)-monochromatic-rectangles, and the total number of distinct black (2, 1)-monochromatic-rectangles is not more than 2s $\binom{5}{2}$. So we have

$$\binom{d_1}{2} + \binom{d_2}{2} + \cdots + \binom{d_{10s+1}}{2} \leq 2\text{s}\binom{5}{2}.$$

By induction hypothesis

$$\binom{d_1}{2} + \binom{d_2}{2} + \cdots + \binom{d_{10s+1}}{2} > (2s-2)\binom{5}{2}.$$

So, we have

$$\binom{d_1}{2} + \binom{d_2}{2} + \cdots + \binom{d_{10s}}{2} < 2s\binom{5}{2} - (2s-2)\binom{5}{2} = 20$$

Hence,

$$(d_1^2 + d_2^2 + \ldots + d_{10}^2) - (d_1 + d_2 + \ldots + d_{10}) < 40,$$

and

$$d_1^2 + d_2^2 + \ldots + d_{10}^2 < 65.$$

By Cauchy-Schwarz inequality

$$(d_1^2 + d_2^2 + \ldots + d_{10}^2)*10 \geqq (d_1 + d_2 + \ldots + d_{10})^2$$

$$(d_1^2 + d_2^2 + \ldots + d_{10}^2) \geqq 62.5$$

Hence, $62.5 \leqq (d_1^2 + d_2^2 + \ldots + d_{10}^2) \leqq 64$.

Because d_i is integer, $d_1^2 + d_2^2 + \ldots + d_{10}^2$ must be 63 or 64

Case1: $d_1^2 + d_2^2 + \ldots + d_{10}^2 = 63$

$$d_2^2 + d_3^2 + \ldots + d_{10}^2 = 63 - d_1^2$$

and

$$d_2 + d_3 + \ldots + d_{10} = 25 - d_1,$$

, by Cauchy-Schwarz inequality, we have

$$(d_2^2 + d_3^2 + \ldots + d_{10}^2)*9 \geqq (d_2 + d_3 + \ldots + d_{10})^2$$

That is,

$$(63-d_1^2)*9 \geqq (25 - d_1)^2.$$

Hence,

$$10 d_1^2 - 50 d_1 + 58 \leqq 0.$$

Therefore,

$$1.83 \leqq d_1 \leqq 3.17.$$

By the integrity of d_1, we get that d_1 must be 2 or 3. Similarly, $d_2, d_3, \ldots, d_{10}$ must be 2 or 3.
But $d_1 + d_2 + \ldots + d_{10} = 25$, so $d_1, d_2, \ldots, d_{10}$ consists of five 2's and five 3's.
Therefore, $d_1^2 + d_2^2 + \ldots + d_{10}^2 = 65 \neq 63$ which leads to a contradiction.
Case2: $(d_1^2 + d_2^2 + \ldots + d_{10}^2) = 64$

Since

$$d_2^2 + d_3^2 + \ldots + d_{10}^2 = 64 - d_1^2$$

and

$$d_2 + d_3 + \ldots + d_{10} = 25 - d_1.$$

by Cauchy-Schwarz inequality, we have

$$(d_2^2 + d_3^2 + \ldots + d_{10}^2)*9 \geqq (d_2 + d_3 + \ldots + d_{10})^2$$
$$d_2 + d_3 + \ldots + d_{10} \geqq 25 - d_1.$$
$$(64 - d1^2)*9 \geqq (25 - d_1)^2$$

$$10 d_1^2 – 50 d_1 + 49 \leqq 0$$

$$1.34 \leqq d_1 \leqq 3.66$$

By the integrity of d_1, we get that d_1 must be 2 or 3. Similarly, $d_2, d_3, \ldots, d_{10}$ must be 2 or 3.
But $d_1 + d_2 + \ldots + d_{10} = 25$, so $d_1, d_2, \ldots, d_{10}$ consists of five 2's and five 3's.
Therefore, $d_1^2 + d_2^2 + \ldots + d_{10}^2 = 65 \neq 64$ which leads to a contradiction.

So for all 2-colored 5 x (10s - 5) checkerboard, there is a (2, 2s + 1)-monochromatic-rectangle.

By induction, for all $k \geqq 2$ and k be integer, in every 2-colored 5 x (5t - 5) checkerboard, there is a (2, t)-monochromatic-rectangle, where t = 2k - 1.

2.5 Summary
We can convert the above theorems to graphic problems. We have the following proposition.
Let $s \geqq 2$

- If n > 6(t - 1), every 2-coloring of $K_{3,n}$ exists a monochromatic $K_{2,t}$ subgraph.
- If n > 6(t - 1), every 2-coloring of $K_{4,n}$ exists a monochromatic $K_{2,t}$ subgraph.
- If n > (10t - 16), every 2-coloring of $K_{5,n}$ exists a monochromatic $K_{2,(2t-2)}$ subgraph.
- If n > (10t - 10), every 2-coloring of $K_{5,n}$ exists a monochromatic $K_{2,(2t-1)}$ subgraph.

III. (3, 2)-Monochromatic-rectangles in a Checkerboard

3.1 The Case of 3 x n Checkerboard
If there are two columns of all grids are of the same color, then the checkerboard has a (3, 2)-monochromatic-rectangle. Otherwise, there is no (3, 2)-monochromatic-rectangle.
3.2 The Case of 4 x n Checkerboard
If every column of a 4 x n checkerboard has two black grids and two white grids, then it doesn't have s a (3, 2)-monochromatic-rectangle. Therefore, for every two color 4 x n checkerboard, there exists a coloring such that has no (3, 2)-monochromatic-rectangles in the 4 x n checkerboard.
3.3 The Case of 5 × n Checkerboard
Lemma 5

In every 2-colored 5 x n checkerboard, n = 21 is the smallest number such that there exists a (3, 2)-monochromatic-rectangle.

Proof. To prove that we need to exhibit a two colored 5 x 20 checkerboard has no (3, 2)-monochromatic-rectangles. In a column, there are at most $\binom{5}{3}$ distinct black (3, 1)-monochromatic-rectangles. So we can distribute the $\binom{5}{3}$ distinct black (3, 1)-monochromatic-rectangles and the $\binom{5}{3}$ distinct white (3, 1)-monochromatic-rectangles to the 20 columns, then the two colored 5 x 20 checkerboards have no (3, 2)-monochromatic-rectangles. By pigeonhole principle, there are at least $\left\lceil\frac{21}{2}\right\rceil = 11$ columns with at least three grids of the same color. Without loss of generality, let the color be black. Then $d_i \geqq 3$, i = 1, 2, …, 11, where d_i is the number of black grids of the i^{th} column of the checkerboard. Assume 2-colored 5 x 21 checkerboard has a coloring such that there is no (3, 2)-monochromatic-rectangles, then any two columns don't contain the same black (3, 1)-monochromatic-rectangles, each column contains $\binom{d_i}{3}$ distinct black (3, 1)-monochromatic-rectangles, and the total number of distinct black (3, 1)-monochromatic-rectangles is not more than $\binom{5}{3}$. So we have

$$\binom{d_1}{3} + \binom{d_2}{3} + \cdots + \binom{d_{11}}{3} \leq \binom{5}{3}.$$

Because $d_i \geqq 3$, we have $11 \leq \binom{5}{3} = 10$ which leads to a contradiction from the last inequality.

So, if n > 21, then for every 2-coloring of 5 x n checkerboard, there is a (3, 2)-monochromatic-rectangle.

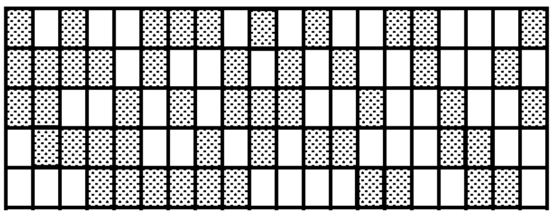

Figure 18: There is a 2-colored 5 x 20 checkerboard containing no a (3, 2)-monochromatic-rectangle.

3.4 The Case of 6 x n Checkerboard

Lemma 6

In every 2-colored 5 x n checkerboard, n = 21 is the smallest number such that there exists a (3, 2)-monochromatic-rectangle.

Proof. By Lemma, in every 2-colored 5 x 21 checkerboard, there is a (3, 2)-monochromatic-rectangle. Therefore, in every 2-colored 6 x 21 checkerboard, there is a (3, 2)-monochromatic-rectangle.

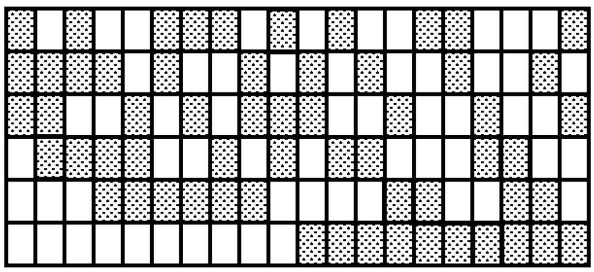

Figure 19: There is a 2-colored 6 x 20 checkerboard containing no a (3, 2)-monochromatic-rectangle.

3.5 Summary

We can convert the above theorems to graphic problems. We have the following proposition.

- If n > 20, every 2-coloring of $K_{5,n}$ contains a monochromatic $K_{3,2}$.
- If n > 20, every 2-coloring of $K_{6,n}$ contains a monochromatic $K_{3,2}$.

IV. (3, t)-Monochromatic-rectangles in a Checkerboard

4.1 The Case of 5 x n Checkerboard

Theorem 5

If n > 20(t - 1), where t $\geqq$ 2, then in every 2-colored 5 x n checkerboard, there is an (3, t)-Monochromatic-rectangle.

Proof. If n = (20t - 20) + 1 = 20t - 19 (We only prove that every 2-colored 5 x (20t - 19) checkerboard, there is a (3, t)-monochromatic-rectangle.) By pigeonhole principle, there are at least $\left\lceil\frac{20t-19}{2}\right\rceil = 10t - 9$ columns that have at least three of same color grids. Without loss of generality, let the color be black. Then $d_i \geqq 3$, i = 1, 2, …, (10t - 9), where d_i is the number of black grids of the i^{th} column of the checkerboard. Assume 2-colored 3 x (20t - 19) checkerboard has a coloring such that there is no (3, t)-Monochromatic-rectangles, then any s columns don't contain the same black (3, 1)-monochromatic-rectangles, each column contains $\binom{d_i}{3}$ distinct black (3, 1)-monochromatic-rectangles, and the total number of distinct black (3, 1)-monochromatic-rectangles is not more than $(t-1)\binom{5}{3}$. So we have,

$$\binom{d_1}{3} + \binom{d_2}{3} + \cdots + \binom{d_{10t-9}}{3} \leq (t-1)\binom{5}{3}.$$

Because $d_i \geqq 3$, we have $10t - 9 \leq (t-1)\binom{5}{3} = 10t - 10$ which leads to a contradiction.

So, if n > (20t - 20), where t ≧ 2, then in every 2-colored 5 x n checkerboard, there is an (3, t)-Monochromatic-rectangle.

4.2 The Case of 6 x n Checkerboard

Theorem 6

If n > 20(t - 1), where s ≧ 2, then in every 2-colored 6 x n checkerboard, there is a (3, t)-Monochromatic-rectangle.

Proof. By Theorem, in every 2-colored 5 x (20t - 20) checkerboard, there is a (3, t)-Monochromatic-rectangle . Therefore, in every 2-colored 6 x (20t - 20) checkerboard, there is a (3, t)-Monochromatic-rectangle.

4.3 Summary

We can convert the above theorems to graphic problems. We have the following proposition.

Let s ≧ 2

- If n > 20(t - 1), every 2-coloring of $K_{5,n}$ exists a monochromatic $K_{3,t}$ subgraph.
- If n > 20(t - 1), every 2-coloring of $K_{6,n}$ exists a monochromatic $K_{3,t}$ subgraph.

V. (s, 2)-Monochromatic-rectangles in a Checkerboard

5.1 The Case of (2s - 2) x n Checkerboard

If every column of a (2s - 2) x n checkerboard has s-1 black grids and s-1 white grids, then it doesn't have a (s, 2)-monochromatic-rectangle. Therefore, for every two color 2s - 2 x n checkerboard exists a coloring such that has no (s, 2)-monochromatic-rectangle in the (2s - 2) x n checkerboard.

5.2 The Case of (2s - 1) x n Checkerboard

Lemma 7

If n > 20(t - 1), where s ≧ 2, then in every 2-colored 6 x n checkerboard, there is a (3, t)-Monochromatic-rectangle.

Proof. To prove that we need to exhibit a 2-colored (2s - 1) x $2\binom{2s-1}{s}$ checkerboard that has no (2, 2)-monochromatic-rectangles. In a column, there are at most $\binom{2s-1}{s}$ distinct black (s, 1)-monochromatic-rectangles. So we can distribute the $\binom{2s-1}{s}$ distinct black (s, 1)-monochromatic-rectangles and the $\binom{2s-1}{s}$ distinct white (s, 1)-monochromatic-rectangles to the $2\binom{2s-1}{s}$ columns, then the 2-colored (2s - 1) x $2\binom{2s-1}{s}$ checkerboards have no (s, 2)-monochromatic-rectangles. By pigeonhole principle, there are at least $\left\lceil\frac{2\binom{2s-1}{s}+1}{2}\right\rceil = \binom{2s-1}{s} + 1$ columns with at least s grids are of the same color. Without loss of generality, let the color be black. Then $d_i \geqq s$, i = 1, 2, …, $\binom{2s-1}{s} + 1$, where d_i is the number of black grids of the i^{th} column of the checkerboard. Assume 2-colored $(2s-1)$ x $(2\binom{2s-1}{s} + 1)$ checkerboard has a coloring such that there is no (s, 2)-monochromatic-rectangles, then any two columns don't contain the same black (s, 1)-

monochromatic-rectangles, each column contains $\binom{d_i}{s}$ distinct black (s, 1)-monochromatic-rectangles, and the total number of distinct black (s, 1)-monochromatic-rectangles is not more than $\binom{2s-1}{s}$. So we have

$$\sum_{k=1}^{\binom{2s-1}{s}+1} \binom{d_k}{s} \leq \binom{2s-1}{s}$$

.

Because $d_i \geqq s$,

$$\sum_{k=1}^{\binom{2s-1}{s}+1} \binom{s}{s} \leq \binom{2s-1}{s} \Rightarrow \binom{2s-1}{s} + 1 \leq \binom{2s-1}{s}$$

which leads to a contradiction.

So, if n > 2 $\binom{2s-1}{s} + 1$, then in every 2-colored $(2s-1)$ x (2 $\binom{2s-1}{s} + 1$) checkerboard. There is a (s, 2)-monochromatic-rectangle.

5.3 The Case of 2s x n Checkerboard

Lemma 8

In every 2-colored$(2s)$ x (2 $\binom{2s-1}{s} + 1$) checkerboard, there is a (s, 2)-monochromatic-rectangle.

Proof. By Lemma, in every 2-colored $(2s-1)$ x (2 $\binom{2s-1}{s} + 1$) checkerboard, there is a (s, 2)-monochromatic-rectangle. Therefore, in every 2-colored $(2s)$ x (2 $\binom{2s-1}{s} + 1$) checkerboard, there is a (s, 2)-monochromatic-rectangle.

5.4 Summary

We can convert the above theorems to graphic problems to get following propositions.

- If n > 2 $\binom{2s-1}{s} + 1$), every 2-coloring of $K_{(2s-1),n}$ contains a monochromatic $K_{s,\,2}$.
- If n > 2 $\binom{2s-1}{s} + 1$), every 2-coloring of $K_{2s,n}$ contains a monochromatic $K_{s,\,2}$.

VI. (s, t)-Monochromatic-rectangles in a Checkerboard

6.1 The Case of (2s - 1) x n Checkerboard

Theorem 7

In every 2-colored $(2s-1)$ x $(2(\mathrm{t}-1)\binom{2s-1}{s} + 1)$ checkerboard, there is a (s, t)-monochromatic-rectangle.

Proof. By pigeonhole principle, there are at least $\left\lceil \frac{2(\mathrm{t}-1)\binom{2s-1}{s}+1}{2} \right\rceil = (\mathrm{t}-1)\binom{2s-1}{s} + 1$ columns with at least s grids are of the same color. Without loss of generality, let the color be black. Then $d_i \geqq$ t, i = 1, 2, …, $(\mathrm{t}-1)\binom{2s-1}{s} + 1$, where d_i is the number of black grids of the i^{th} column of the checkerboard. Assume 2-colored $(2s-1)$ x $(2(\mathrm{t}-1)\binom{2s-1}{s} + 1)$ checkerboard has a coloring such that there is no (s, t)-monochromatic-rectangle, then any s columns don't contain the same black (s, 1)-monochromatic-rectangles, each column contains $\binom{d_i}{s}$ distinct black (s, 1)-monochromatic-rectangles, and the total number of distinct black (s, 1)-monochromatic-rectangles is not more than $(\mathrm{t}-1)\binom{2s-1}{s}$. So, we have

$$\sum_{k=1}^{(t-1)\binom{2s-1}{s}+1} \binom{d_k}{s} \leq (\mathrm{t}-1)\binom{2s-1}{s}$$

Because $d_i \geqq s$,

$$\sum_{k=1}^{(t-1)\binom{2s-1}{s}+1} \binom{s}{s} \leq (t-1)\binom{2s-1}{s} \Rightarrow (t-1)\binom{2s-1}{s} + 1 \leq (t-1)\binom{2s-1}{s}$$

which leads to a contradiction.
So, if n > $2(t-1)\binom{2s-1}{s}+1$, then in every 2-colored (2s - 1) x ($2(t-1)\binom{2s-1}{s}+1$) checkerboard. There is a (s, t)-monochromatic-rectangle.

6.2 The Case of (2s) x n Checkerboard

Theorem 8

In every 2-colored (2s) x ($2(t-1)\binom{2s-1}{s}+1$) checkerboard, there is a (s, t)-monochromatic-rectangle.
Proof. By Theorem, in every 2-colored (2s - 1) x ($2(t-1)\binom{2s-1}{s}+1$) checkerboard, there is a (s, t)-monochromatic-rectangle. Therefore, in every 2-colored (2s) x ($2(t-1)\binom{2s-1}{s}+1$) checkerboard, there is a (s, t)-monochromatic-rectangle.

6.3 Summary

We can convert the above theorems to graphic problems. We have the following proposition.

- If n > ($2(t-1)\binom{2s-1}{s}+1$), every 2-coloring of $K_{(2s-1),n}$ contains a monochromatic $K_{s,t}$.
- If n > ($2(t-1)\binom{2s-1}{s}+1$), every 2-coloring of $K_{2s,n}$ contains a monochromatic $K_{s,t}$.

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