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Typical Properties of Maximal Sperner Families of Type (K,K+1 And Upper Estimate

Kochkarev B.S.
Kazan (Volga region) federal University

Abstract: *The properties of the families F of finite subsets of n element set S are considered in the situation, where subsets of F are incomparable on the binary relation of inclusion and a)for any $A \notin F$ there exists some set $A' \in F$ such that either $A \subset A'$ or $A' \subset A$; b)for any $A \in F$ is place $|A| \in \{k, k+1\}, k \neq 0, k < \left\lceil \frac{n}{2} \right\rceil$. For these families F we introduce some parameter $r(F)$ and show: 1). $r(F) = \binom{n-1}{k}$, if $n \leq 5$; 2).if $n > 5$, then $r(F)$ can be less than $\binom{n-1}{k}$; 3) family F with minimum value of parameter $r(F)$ have some structure; 4)we prove that the proportion of families F with $r(F) < \binom{n-1}{k}$ tends to zero with growth of n. Finally, we receive the upper estimate for number $g(n,k)$ of considered families and make an assumption (hypothesis) that $g(n,k) \sim (k+1)2^{\binom{n-1}{k}}$.*
Keywords: *maximal Sperner family, typical property*

I. Introduction

Interest in the study of the properties of families of subsets of finite sets which are pair-wise incomparable binary relation of inclusion appeared in connection with the problem of Dedekind on the number of elements of free distributive structure with n generators [1]. One of the first works in this direction was E. Sperner's paper [2]. The class of maximal Sperner families considered in the present paper was first introduced by the author in the paper [3]. The main results of this work were previously published in arxiv: 1304.4363v1 [cs.DM] 16Apr 2013.

Definition 1 [2]. A family F of subsets of finite set $S = \{a_1, a_2, ..., a_n\}$ is said to be that of Sperner if none of elements $A \in F$ is a subset of any other element $A' \in F$.

Definition 2 [3]. A Sperner family F is said to be maximal if for any $A \subset S, A \notin F$ there exists a $A' \in F$ such that either $A \subset A'$ or $A' \subset A$.

Remark [4]. The property of being Sperner of a family F is invariant with respect to the following transforms:

a) $F \to \overline{F}$, where $\overline{F}$ is the family obtained from F by replacement of every A by its complement A';

b) $F \to F_{S'}$, where $F_{S'}$ is a family obtained from F by applying the substitution

$$S' = \begin{pmatrix} a_1 a_2 ... a_n \\ a_{i_1} a_{i_2} ... a_{i_n} \end{pmatrix}.$$

For the number $f(n)$ of maximal Sperner families of subsets of an n-element set S direct calculation gives us $f(1) = 2, f(2) = 3, f(3) = 7, f(4) = 29, f(5) = 376$ [5].

Definition 3[4]. We will say that a Sperner family F is of type $(k, k+1)$ if $|A| \in \{k, k+1\}$ for any $A \in F$.

If F is a Sperner family of type $(k, k+1)$, then we denote by $F^{(k)}, F^{(k+1)}$, respectively, the family of subsets $A \in F, |A| = k; A' \in F, |A'| = k+1$.

By virtue of item a) in the remark, if one studies maximal Sperner families of type $(k, k+1)$ of an n-element set, then it suffices to restrict oneself by the case $k < \left\lceil \frac{n}{2} \right\rceil$. Denote by $p_i(F)$ the number of elements $A \in F, |A| = k$, into which element $a_i \in S$ does not enter, while $q_i(F)$ stands for the number of elements $A \in F, |A| = k+1$, which the element a_i does enter in. Further, let

$r_i(F) = p_i(F) + q_i(F), r(F) = \max\{r_i(F)\}, i = \overline{1,n}$. Obviously, with any $n \geq 3$ the inequality is valid $r_i(F) \leq \binom{n-1}{k}$.

Definition 4 [4]. A number $0 \leq s \leq \binom{n-1}{k}$ is called admissible if there exists an maximal Sperner family F such that $r_i(F) = s$ for a certain $i, 1 \leq i \leq n$.

By virtue of item b) in the remark we have that if s is an admissible number, then for any $i = \overline{1,n}$ there can be found maximal Sperner family F with $r_i(F) = s$. Therefore, in what follows, we will consider, as a rule, a fixed element $a_n \in S$ in the capacity of element $a_i \in S$.

Theorem 1 [4]. The number $s = \binom{n-1}{k} - 1$ is not admissible.

Thus it seems to be intrinsic to state the question on admissible values s in the limits $0 \leq s \leq \binom{n-1}{k}$ [3-5].

Theorem 2 [6]. For an maximal Sperner family of type $(k, k+1), n \geq 3,$ of the set S, all numbers $0,1,\ldots,\binom{n-1}{k} - 2, \binom{n-1}{k}$ are admissible.

Let $\Sigma(n)$ be a class of finite sets, where n runs over a certain set of indexes (for example, the set of non-negative integers N [7], whose power increases monotonically with n growing.

Definition 5. A certain property α with respect of elements of the set $\Sigma(n)$ is called typical [8] if

$$\lim_{n\to\infty} \frac{|\Sigma_\alpha(n)|}{|\Sigma(n)|} = 1 \text{ or } \lim_{n\to\infty} \frac{|\Sigma_{\alpha'}(n)|}{|\Sigma(n)|} = 0,$$

where $\Sigma_\alpha(n)(\Sigma_{\alpha'}(n))$ stands for the set of elements from $\Sigma(n)$, which possess (do not possess) the property α.

In this case it sometimes is said that α fulfills for almost all elements from $\Sigma(n)$.

In the capacity of $\Sigma(n)$ we will consider the class of maximal Sperner families subsets $F = F^{(k)} \cup F^{(k+1)}$ of type $(k, k+1)$ of finite set S and, in the capacity of the property α, the value of the parameter $r(F) = \binom{n-1}{k}$ of the maximal Sperner family F of type $(k, k+1)$. In what follows we will speak about type only in specific cases of k. For k we consider [4] the values $k < \left\lceil \frac{n}{2} \right\rceil, k \neq 0$.

In [4], for the necessary condition (Theorem 2) obtained in [3], a clarification was made, namely, it was shown that it holds only for $n \leq 5$. For all $n \geq 6$ we constructed in [4] the maximal Sperner families F with $r(F) < \binom{n-1}{k}$. Next, we will assume that $n \geq 6$.

The principal objective of the present paper is to prove the fact that for almost all maximal Sperner families F of type $(k,k+1)$ the value of the parameter $r(F)$ equals $\binom{n-1}{k}$ without restrictions with respect to k, we also are going to obtain the upper estimates for the respective combinatorial numbers.
The following assertions are given without proofs in vie of their evidence.

Proposition 1. If F is an maximal Sperner family of type $(k,k+1)$, then $r(F) < \binom{n-1}{k}$ if and only if for any $i = \overline{1,n}$ one can find a subset $A, |A| = k, a_i \notin A$ such that neither $A,$ nor $A \cup \{a_i\}$ belong to F.

Corollary 1. If F is an maximal Sperner family of type $(k,k+1)$, then $r(F) = \binom{n-1}{k}$ if and only if there exists $i \in \overline{1,n}$ such that for any $A, |A| = k, a_i \notin A$ one has either $A \in F$ or $A \cup \{a_i\} \in F$.

Proposition 2. If $F = F^{(k)} \cup F^{(k+1)}$ is an maximal Sperner family such that $F^{(k)} \neq Ø$, $F^{(k+1)} \neq Ø$, then for one to have $r(F) < \binom{n-1}{k}$ it suffices that $F^{(k+1)}$ consists of sets B_i such that $B_j \cap B_k = Ø$, $j \neq k$.

The condition given in Proposition 2 is not a necessary one. For example, for the set $S = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ Sperner family

$$F = \{\{a_1,a_2,a_3\},\{a_4,a_5,a_6\},\{a_3,a_5,a_6\},\{a_1,a_4\},\{a_1,a_5\},\{a_1,a_6\},\{a_2,a_4\},\{a_2,a_5\},\{a_2,a_6\},\{a_3,a_4\}\}$$

is maximal and such that $r(F) = 8 < \binom{5}{2}$.

II. Induction algorithm for construction of all maximal Sperner families of type (k,k+1) and its corollaries

The following induction algorithm for constructing all maximal Sperner families of type $(k,k+1)$ is suggested.

1. The base of the algorithm is the family of all maximal Sperner families of type $(k,k+1)$ $\Omega(1)$ with the value of parameter $r_n(F) = 0$. Obviously, $\Omega(1)$ represents one maximal Sperner family composed from all subsets $A, |A| = k, a_n \in A$ and all subsets $B, |B| = k+1, a_n \notin B$.
2. Further construction of maximal Sperner families is realized by induction. Let $\Omega(t)$ be the set of maximal Sperner families which has been constructed at the step t $(\Omega(1) = \{F, r_n(F) = 0\})$. The set $\Omega(t+1)$ is formed by means of transforms of maximal Sperner families from $\Omega(t)$. Suppose that $F = F^{(k)} \cup F^{(k+1)}$ is an arbitrary maximal Sperner family from $\Omega(t)$. If F turns to be maximal Sperner family with $r_n(F) = \binom{n-1}{k}$, then F remains unchanged. However, if F is an maximal Sperner family with $r_n(F) < \binom{n-1}{k}$, then denote

$$M = \{A : |A| = k, a_n \notin A, A \notin F^{(k)}\},$$
$$L = \{B : |B| = k+1, a_n \in B, B \notin F^{(k+1)}\},$$
$$\Psi(F) = M \cup L,$$

and realize the following transforms of F : add to F a certain set C from $\Psi(F)$ and exclude from the set thus formed F' all subsets comparable by binary relation of inclusion with the set C. After exclusion of sets comparable with C, it might happen that among the resting subsets in $\Psi(F)$ there are subsets noncomparable by inclusion with subsets remaining after exclusion. By adding the latter subsets we obtain obviously maximal Sperner family. By proceeding the mentioned above transform of F with respect to all subsets from $\Psi(F)$ and analogous transforms of all maximal Sperner families F from $\Omega(t)$ with $r_n(F) < \binom{n-1}{k}$, we receive the set of maximal Sperner families which form the set $\Omega(t+1)$. In accordance with the results in [3], $\min_{F\in\Omega(t)} r_n(F) = t-1$ if $t \le \binom{n-1}{k} - 1$ and $r_n(F) = \binom{n-1}{k}$ if $F \in \Omega(\binom{n-1}{k})$.

$\bigcup_{t=1}^{\binom{n-1}{k}} \Omega(t)$ represents the set of all maximal Sperner families of type $(k, k+1)$.

Every time, when one passes from $\Omega(t)$ to $\Omega(t+1)$, $2 \le t \le \binom{n-1}{k}$, some of maximal Sperner family might be repeated during the algorithm's work. Therefore, at each step, one should foresee the deletion of maximal Sperner families which already have been obtained earlier in order to leave only pair-wise distinct families.

Lemma 1. Let $\widehat{F} = \widehat{F}^{(k)} \cup \widehat{F}^{(k+1)}$ be the maximal Sperner family obtained by induction algorithm from maximal Sperner family $F = F^{(k)} \cup F^{(k+1)}$ with $r_n(F) < \binom{n-1}{k}$ on a certain step t. Then, as soon as $r_i(F) = \binom{n-1}{k}$ and $\widehat{F}$ has been obtained by addition of the set $A, |A| = k (B, |B| = k+1),$ moreover $a_i \notin A (a_i \in B)$, then $r_i(\widehat{F}) = \binom{n-1}{k}$.

Proof. Since $r_i(F) = \binom{n-1}{k}$, then by virtue of Corollary 1 for any $A : |A| = k, a_i \notin A, A \notin F$ there can be found $B : |B| = k+1, a_i \in B, B \supset A$ such that $B \in F$ (for any $B : |B| = k+1, a_i \in B, B \notin F$ there can be found $A : |A| = k, a_i \notin A, A \subset B$ such that $A \in F$). Therefore, by the induction algorithm, one has

$$r_i(\widehat{F}) = (p_i(F)+1) + (q_i(F)-1) = \binom{n-1}{k} \; (r_i(\widehat{F}) = (p_i(F)-1) + (q_i(F)+1) = \binom{n-1}{k}).$$

Lemma 2. If $F = F^{(k)} \cup F^{(k+1)}, F^{(\alpha)} \ne \varnothing,\ \alpha \in \{k, k+1\}$ is an maximal Sperner family such that $r(F) = \binom{n-1}{k} = r_i(F)$, where equality holds for a unique $i \in \overline{1,n}$, then the maximal Sperner family $\widetilde{F}$, obtained by the induction algorithm by adding of an arbitrary set $B : |B| = k+1, B \notin F, a_i \notin B$ is the maximal Sperner family with $r(F) < \binom{n-1}{k}$.

Proof. Indeed, the added set B is comparable with some $1 \le m \le k$ sets $A : |A| = k, A \in F$. Therefore, by the induction algorithm, these sets are excluded from F and $r_i(\widetilde{F}) = \binom{n-1}{k} - m$. For $r_j(\widetilde{F}), j \ne i$, we obviously will have $r_j(\widetilde{F}) < \binom{n-1}{k}$, quod erat demonstradum.

Theorem 3 [9]. If $F = F^{(k)} \cup F^{(k+1)}$ is the maximal Sperner family with $r_n(F) = 0$, then

$r_i(F)=\binom{n-1}{k}, i\neq n$.

Theorem 4 [9]. Let $F=F^{(k)}\cup F^{(k+1)}, F^{(i)}\neq\varnothing, i\in\{k,k+1)$ is the maximal Sperner family such that for any $B\in F^{(k+1)}$ $a_i\in B$ (for any $A\in F^{(k)}$ $a_i\notin A$). Then $r(F)=r_i(F)=\binom{n-1}{k}$.

Corollary 2 [9]. For any the maximal Sperner family F such that $\left|F^{(k+1)}\right|=1(\left|F^{(k)}\right|=1)$ $r(F)=\binom{n-1}{k}$.

Corollary 3 [9]. If n is odd, then for any maximal Sperner family F of type $(\left\lfloor\frac{n}{2}\right\rfloor,\left\lceil\frac{n}{2}\right\rceil)$ such that $\left|F^{(\left\lceil\frac{n}{2}\right\rceil)}\right|=2$, $r(F)=\binom{n-1}{\left\lfloor\frac{n}{2}\right\rfloor}$.

Theorem 5. Let $F=F^{(k)}\cup F^{(k+1)}, F^{(j)}\neq\varnothing,\ \ j\in\{k,k+1\}$ is a maximal Sperner family. Then $r_i(F)=\binom{n-1}{k}-\left|F_i^{(k)}\right|+\left|F_i^{(k+1)}\right|$, where $F_i^{(k+1)}$ is the family of all subsets B from $F^{(k+1)}$ such that $a_i\in B$ and $F_i^{(k)}$ is the family of all subsets $A,|A|=k,a_i\notin A$ comparable with the subsets $F^{(k+1)}$.

Proof. Really since $\binom{n-1}{k}-\left|F_i^{(k)}\right|=p_i(F)$ and $\left|F_i^{(k+1)}\right|=q_i(F)$, then

$$r_i(F)=p_i(F)+q_i(F)=\binom{n-1}{k}-\left|F_i^{(k)}\right|+\left|F_i^{(k+1)}\right|.$$

Corollary 4. If $F=F^{(k)}\cup F^{(k+1)}, F^{(j)}\neq\varnothing.\ j\in\{k,k+1\}$ is maximal Sperner family such that for any pair B_i,B_j of subsets from $F^{(k+1)}$ $B_i\cap B_j=\varnothing$ and $\Sigma_{B_i\in F^{(k+1)}}|B_i|=(k+1)\left|F^{(k+1)}\right|,(k\neq 1)$, then $r_i(F)=r(F)$, if $a_i\in B$ for a subset B from $F^{(k+1)}$.

Proof. Really, according to theorem 5 we have $r_i(F)=\binom{n-1}{k}-\left|F_i^{(k)}\right|+1$, if a_i belong to a B from $F^{(k+1)}$ and $r_j(F)<r_i(F)$ if a_i do not belong to none B from $F^{(k+1)}$, i.e. $r(F)=r_i(F)$. Evidently, if $\frac{n}{k+1}$ is the whole and $\Sigma_{B_i\in F^{(k+1)}}|B_i|=n$ then for all $i=\overline{1,n}$

$$r(F)=r_i(F)=\binom{n-1}{k}-\left|F_i^{(k)}\right|+1=\binom{n-1}{k}-(\frac{n}{k+1}-1)(k+1).$$

Corollary 5. If $k<\left\lfloor\frac{n}{2}\right\rfloor, F=F^{(k)}\cup F^{(k+1)}, F^{(j)}\neq\varnothing,\ \ j\in\{k,k+1\}$ is maximal Sperner family such that for any pair B_i,B_j of subsets from $F^{(k+1)}$ $B_i\cap B_j=\varnothing$ and $\left|F^{(k+1)}\right|=\left\lfloor\frac{n}{k+1}\right\rfloor$, then $r(F)=\min_{F:r(F)<\binom{n-1}{k}} r(F)$ and the number of such maximal Sperner families is

$$\binom{n}{k+1}\binom{n-(k+1)}{k+1}\cdots\binom{n-(\left\lfloor\frac{n}{k+1}\right\rfloor-1)}{k+1}.$$

Proof. According to corollary 4 $r(F)=r_i(F)=\binom{n-1}{k}-\left|F_i^{(k)}\right|+1$, if a_i belong to a B from $F^{(k+1)}$. Evidently in the conditions of corollary 5 $\left|F_i^{(k)}\right|=\max_{F:r(F)<\binom{n-1}{k}}\left|F_i^{(k)}\right|$. The second affirmation evidently. Evidently also, that $\binom{n-1}{k}-(\left\lfloor\frac{n}{k+1}\right\rfloor-1)(k+1)\le r(F)\le\binom{n-1}{k}, r(F)\ne\binom{n-1}{k}-1.$

If n is even, then the maximal Sperner family $F=F^{(\frac{n}{2}-1)}\cup F^{(\frac{n}{2})}$, where $F^{(\frac{n}{2})}=\{\{a_{i_1},a_{i_2},...,a_{i_{\frac{n}{2}}}\},\{a_{j_1},a_{j_2},...a_{j_{\frac{n}{2}}}\},i_k\notin\{j_1,j_2,...,j_{\frac{n}{2}}\},k=\overline{1,\frac{n}{2}}\}$ is the maximal Sperner family with $r(F)=r_i(F)=\binom{n-1}{\frac{n}{2}-1}-\frac{n}{2}, i=\overline{1,n}$. Evidently the number of these maximal Sperner families is $\binom{n}{\frac{n}{2}}2^{-1}$.

Theorem 4 delivery to us the sufficient condition for $r(F)=r_i(F)=\binom{n-1}{k}$. We will prove that this condition these is also the necessary condition.

Corollary 6. If $r(F)=r_i(F)=\binom{n-1}{k}$, then for any $B\in F^{(k+1)}$ $a_i\in B$.

Proof. Really, according to theorem 5 $r_i(F)=\binom{n-1}{k}-\left|F_i^{(k)}\right|+\left|F_i^{(k+1)}\right|$, but since $r_i(F)=\binom{n-1}{k}$, then $\left|F_i^{(k+1)}\right|=\left|F_i^{(k)}\right|$, consequently for all $B\in F^{(k+1)}$ $a_i\in B$.

Theorem 6. If n is odd and $F=F^{(\lfloor\frac{n}{2}\rfloor)}\bigcup F^{(\lceil\frac{n}{2}\rceil)}$ is maximal Sperner family, then minimum $\left|F^{(\lceil\frac{n}{2}\rceil)}\right|$ such that $r(F)<\binom{n-1}{\lfloor\frac{n}{2}\rfloor}$ is 3.

Proof. According to corollary 3 , if n is odd, then for any maximal Sperner family F of type $(\left\lfloor\frac{n}{2}\right\rfloor,\left\lceil\frac{n}{2}\right\rceil)$ such, that $\left|F^{(\lceil\frac{n}{2}\rceil)}\right|=2$ $r(F)=\binom{n-1}{\lfloor\frac{n}{2}\rfloor}$. Let now $F=F^{(\lfloor\frac{n}{2}\rfloor)}\bigcup F^{(\lceil\frac{n}{2}\rceil)}$ be the maximal Sperner family, where $F^{(\lceil\frac{n}{2}\rceil)}=\{B_1,B_2,B_3:B_1\bigcap B_2=a_i\ne a_n,a_i\notin B_3\}$. Evidently the number of these maximal

Sperner families is $(n-1)\binom{n-2}{\lfloor \frac{n}{2} \rfloor}\binom{n-1}{\lceil \frac{n}{2} \rceil}$. If the subset B_3 not have of the common comparable subset $A, |A| = \lfloor \frac{n}{2} \rfloor$, with a subset from $\{B_1, B_2\}$, then $r(F) = \binom{n-1}{\lfloor \frac{n}{2} \rfloor} - \lceil \frac{n}{2} \rceil + 1$ and $\min_i r_i(F) = \binom{n-1}{\lfloor \frac{n}{2} \rfloor} - 2\lceil \frac{n}{2} \rceil + 1$.

Theorem 7. Almost for all maximal Sperner family F $r(F) = \binom{n-1}{k}$.

Proof. We will prove this only for the maximal Sperner family of the type $(\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)$ with n odd since for other cases the proof carry out analogously. We will estimate from above the number of maximal Sperner families F such that $r(F) < \binom{n-1}{\lfloor \frac{n}{2} \rfloor}$. Evidently, according to the inductive algorithm and to theorem 3 the number of these maximal Sperner families not exceed of the number of maximal Sperner families F with $r(F) < \binom{n-1}{\lfloor \frac{n}{2} \rfloor}$ received before maximal Sperner family with $\min_{F: r(F) < \binom{n-1}{\lfloor \frac{n}{2} \rfloor}} r(F)$ plus of the number of maximal Sperner families F' with $r_n(F') < \binom{n-1}{\lfloor \frac{n}{2} \rfloor}$ received later the maximal Sperner family F with $\min_{F: r(F) < \binom{n-1}{\lfloor \frac{n}{2} \rfloor}} r(F)$. According to lemma 2 and to the inductive algorithm the number $g'(n, \lfloor \frac{n}{2} \rfloor)$ maximal Sperner families F with $r(F) < \binom{n-1}{\lfloor \frac{n}{2} \rfloor}$ satisfy to inequality

$$g'(n, \lfloor \tfrac{n}{2} \rfloor) \le (n-1)^2 \binom{n-2}{\lfloor \frac{n}{2} \rfloor}\binom{n-1}{\lceil \frac{n}{2} \rceil}\lceil \tfrac{n}{2} \rceil + 2\lceil \tfrac{n}{2} \rceil (n-1)\binom{n-2}{\lfloor \frac{n}{2} \rfloor}\binom{n-1}{\lceil \frac{n}{2} \rceil}\left(\binom{2\lceil \frac{n}{2} \rceil}{0} + \binom{2\lceil \frac{n}{2} \rceil}{1} 2^1 + \ldots + \binom{2\lceil \frac{n}{2} \rceil}{\lceil \frac{n}{2} \rceil - 2} 2^{2\lceil \frac{n}{2} \rceil - 2}\right) < 2\lceil \tfrac{n}{2} \rceil (n-1)\binom{n-2}{\lfloor \frac{n}{2} \rfloor}\binom{n-1}{\lceil \frac{n}{2} \rceil} 2^{4\lceil \frac{n}{2} \rceil - 2}$$

. Since the number of maximal Sperner families with $r(F) = \binom{n-1}{\lfloor \frac{n}{2} \rfloor}$ more than $2^{\binom{n-1}{\lfloor \frac{n}{2} \rfloor}}$, then $\lim_{n \to \infty} \frac{g'(n, \lfloor \frac{n}{2} \rfloor)}{2^{\binom{n-1}{\lfloor \frac{n}{2} \rfloor}}} = 0$ From here directly follow

$g(n,k) < n 2^{\binom{n-1}{k}}$.

The above estimate received for all maximal families of type $(k, k+1)$ improve essentially the estimate $2^{3\binom{n-1}{k}}$ from [4].

Theorem 8. For $k = 1$ $g(n,1) \sim 2^n = 2 \cdot 2^{\binom{n-1}{1}}$

Proof. It is trivial to show that in this case $g(n,1) = 2^n - n$. Thus in this case we have

$g(n,1) \sim 2 \cdot 2^{\binom{n-1}{1}}$.

Hypothesis: we suppose that $g(n,k) \sim (k+1)2^{\binom{n-1}{k}}$.

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