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A Study on the Rate of Convergence of Chlodovsky-Durrmeyer Operator and Their Bézier Variant

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Abstract: *In this paper, we have studied the Bézier variant of Chlodovsky-Durrmeyer operators $D_{m,\vartheta}$ for function f measurable and locally bounded on the interval [0,∞). In this we improved the result given by Ibikl E. And Karsli H. [14]. We estimate the rate of pointwise convergence of $\left(D_{m,\vartheta} f\right)(x)$ at those $x > 0$ at which the one-sided limits $f(x+), f(x-)$ exist by using the Chanturia modulus of variation. In the special case $\vartheta = 1$ the recent result of Ibikl E. And Karsli H. [14] concerning the Chlodowsky-Durrmeyer operators D_m is essentially improved and extended to more general classes of functions.*
Keywords: *Rate of convergence, Chlodovsky-Durrmeyer operator, Bézier basis, Chanturia modulus of variation, p-th power variation.*

I. Introduction

For a function the classical Bernstein-Durrmeyer operators (see [7]) M_n applied to f are define as

$$(M_m f)(x) = (m+1)\sum_{i=0}^{m} p_{m,i}(x)\int_0^1 f(t)p_{m,i}(t)dt, \quad x \in [0,1] \tag{1}$$

where $p_{m,i}(x) = \binom{m}{i}x^i(1-x)^{m-i}$.

Several researchers have studied approximation properties of the operators M_n ([8], [10]) for function of bounded variation defined on the interval [0, 1]. After that Zeng and Chen [22] defined the Bézier variant of Durrmeyer operators as

$$\left(M_{m,\vartheta}\, f\right)(x) = (m+1)\sum_{i=0}^{m} Q_{m,i}^{(\vartheta)}(x)\int_0^1 f(t)p_{m,i}(t)dt, \tag{2}$$

where $Q_{m,i}^{(\vartheta)}(x) = J_{m,i}^{\vartheta}(x) - J_{m,i+1}^{\vartheta}(x)$ and $J_{m,i}(x) = \sum_{j=i}^{m} p_{m,j}(x)$ for $i = 0,1,2,\dots,m$, $J_{m,m+i}(x) = 0$ are the Bézier basis function which is introduced by P. Bézier [4] and estimated the rate of convergence of $M_{m,\vartheta}\, f$ for functions of bounded variation on the interval [0,1].

Let $X_{loc}[0,\infty)$ be the class of all complex valued function measurable and locally bounded on the interval [0,∞). For $f \in X_{loc}[0,\infty)$ the Chlodowsky–Durrmeyer operator D_m are defined as

$$(D_m f)(x) = \frac{m+1}{a_m}\sum_{i=0}^{m} P_{m,i}\left(\frac{x}{a_m}\right)\int_0^{a_m} f(t)P_{m,i}\left(\frac{t}{a_m}\right)dt, \quad 0 \le x \le a_m. \tag{3}$$

where (a_m) is a positive increasing sequence with the properties

$$\lim_{m\to\infty} a_m = \infty \quad and \quad \lim_{m\to\infty}\frac{a_m}{m} = 0 \tag{4}$$

For $f \in X_{loc}[0,\infty)$ and $\vartheta \ge 1$, we introduce the Bézier variant of Chlodowsky–Durrmeyer operators $D_{m,\vartheta}$ as follows

$$\left(D_{m,\vartheta} f\right)(x) = \frac{m+1}{a_m}\sum_{i=0}^{m} Q_{m,i}^{(\vartheta)}\left(\frac{x}{a_m}\right)\int_0^{a_m} f(t)P_{m,i}\left(\frac{t}{a_m}\right)dt, \quad 0 \le x \le a_m. \tag{5}$$

Obviously, $D_{m,\vartheta}$ is a positive linear operator and $\left(D_{m,\vartheta} 1\right)(x) = 1$. In particular, when $\vartheta = 1$ the operators (5) reduce to operators (3).

Recently Agratini [1], Aniol and Pych–Taberska [3], Pych–Taberska [20], and Gupta [11, 12] have investigated the rate of pointwise convergence for Kantorovich and Durrmeyer Type Baskakov–Bézier and Bézier operators using a different approach. They have proved their theorems in terms of the Chanturia modulus of variation, which is a generalization of the classical Jordan variation. It is useful to point out that a deeper analysis of the Chanturia modulus of variation can be found in [6], but actually the modulus of variation was introduced for the first time by Langrange [18]. Although the Chanturia modulus of variation was defined as a

generalization of the classical variation nearly four decades years ago, it was not used to a sufficient extent to solve the problem mentioned above.
The paper is concerned with the rate of pointwise convergence of the operators (5) when *f* belong to $X_{loc}[0,\infty)$. Using the Chanturia modulus of variation defined in [6], we examine the rate of pointwise convergence of $(D_{m,\vartheta}f)(x)$ at the points of continuity and at the first kind discontinuity points of *f*.
For some important papers on different operators related to the present study we refer the readers to Gupta et. Al. [9, 21] and Zeng and Piriou [23]. It is necessary to point out that in the present paper we extend and improve the result of Ibikli E. and Karsli H.[14] for Chlodowsky-Durrmeyer operators.
We being by giving

Definition 1.1 Let f be a bounded function on a compact interval $I=[a,b]$. The modulus of variation $\mu_m(f;[a,b])$ of a function *f* is defined for nonnegative integers m as

$$\mu_0(f;[a,b])=0$$

and for $m\geq 1$ as

$$\mu_m(f;[a,b])=\sup_{\pi_m}\sum_{i=0}^{m-1}|f(x_{2i+1})-f(x_{2i})|,$$

where π_m is an arbitrary system of *m* disjoint intervals $(x_{2i},x_{2i+1}),\ i=0,1,\dots,m-1$, i.e.,
$a\leq x_0<x_1\leq x_2<x_3\leq\cdots\leq x_{2m-2}<x_{2m-1}\leq b$.
The modulus of variation of any function is a non-decreasing function of *m*. Some other properties of this modulus can be found in [6].
If $f\in BV_p(I),\ p\geq 1$, i.e., if *f* of *p*-th bounded power variation on *I*, then for every $i\in\mathbb{N}$,

$$\mu_i(f;I)\leq i^{1-1/p}V_p(f,I), \tag{6}$$

where $V_p(f,I)$ denotes the total *p*-th bounded power variation of *f* on *I*, defined as the upper bound of the set of numbers $\left(\sum_j\left|f(i_j)-f(l_j)\right|^p\right)^{1/p}$ over all finite systems of non-overlapping intervals $(i_j,l_j)\subset I$.
We also consider the class $BV_{loc}^p[0,\infty),\ p\geq 1$, consisting of all function of bounded *p*-th power variation on every compact interval $I\subset[0,\infty)$.
In the sequel it will be always assumed that the sequence (a_m) satisfies the fundamental conditions (4). The symbol $[a]$ will be denote the greatest integer not greater than a.

Remark. Now, let us consider the special case $\vartheta=1,\ p=1$, and let us suppose that function *f* is of bounded variation in the Jorden sense on the whole interval [0, ∞)
($f\in BV[0,\infty)$). Then, for all integers *m* such that $a_m>2x$ and $4a_m\leq m$, we have the following estimation for the rate of convergence of the Chlodowsky-Durrmeyer operators (3):

$$\left|(D_{m,\vartheta}f)(x)-\frac{f(x+)+f(x-)}{2}\right|\leq 2V\left(g_x;H_x(x\sqrt{a_m/m}\right)$$
$$+\frac{2^{10}a_m}{mx^2}\left(x\left(1-\frac{x}{a_m}\right)+\frac{a_m}{m}\right)\sum_{i=1}^{2[m/a_m]}V\left(g_x;H_x\left(\frac{x}{\sqrt{i}}\right)\right)$$
$$+\frac{4Ma_m}{mx^2}\left(x\left(1-\frac{x}{a_m}\right)+\frac{a_m}{m}\right)+\frac{2|f(x+)-f(x-)|}{\sqrt{\frac{mx}{a_m}\left(1-\frac{x}{a_m}\right)}},$$

where $M=\mathrm{Sup}_{0\leq x<\infty}|f(x)|$and $V(g_x;H)$ denotes the Jordan variation of g_x on the interval H.
The above estimation is essentially better than the estimation presented in [14]. Namely, it is easy to see that the right-hand side of the main inequality given in Theorem 1.1 in [14] is not convergent to zero for all function $f\in BV[0,\infty)$ and for all sequences (a_m) satisfying (4).

II. Auxilary Result

In this section we give certain results, which are necessary to prove the main result.
For this, let us introduce the following notation. Given any $x\in[0,a_m]$ and any non-negative integer q, we write

$$\psi_x^q(t):=(t-x)^q\quad for\ t\in[0,\infty),$$
$$W_{m,q}(x):=(D_m\psi_x^q)(x)\equiv\frac{m+1}{a_m}\sum_{i=0}^{m}P_{m,i}\left(\frac{x}{a_m}\right)\int_0^{a_m}(t-x)^qP_{m,i}\left(\frac{t}{a_m}\right)dt. \tag{7}$$

Lemma 2.1 If $m\in\mathbb{N},x\in[0,a_m]$, then

$$W_{m,0}(x)=1,\quad W_{m,1}(x)=\frac{a_m-2x}{m+2},$$

$$W_{m,2}(x) = \frac{2(m-3)(a_m - x)x}{(m+2)(m+3)} + \frac{2{a_m}^2}{(m+2)(m+3)}$$

and, for q > 1,

$$W_{m,2q}(x) = \left(\frac{a_m}{m}\right)^q \sum_{j=0}^{q} \beta_{j,q} \left(x\left(1 - \frac{x}{a_m}\right)\right)^{q-j} \left(\frac{a_m}{m}\right)^j, \quad (8)$$

where $\beta_{j,q}$ are real numbers independent of *x* and bounded uniformly in *m*. Moreover, for $m \geq 2$

$$W_{m,2q}(x) \leq 2\frac{a_m}{m}\left(x\left(1 - \frac{x}{a_m}\right) + \frac{a_m}{m}\right) \quad (9)$$

and, for q > 1,

$$W_{m,2q}(x) \leq c_q \left(\frac{a_m}{m}\right)^q \left(x\left(1 - \frac{x}{a_m}\right) + \frac{a_m}{m}\right)^q, \quad (10)$$

where c_q is a positive constant depending only on q.

Proof. Formulas for $W_{m,0}, W_{m,1}, W_{m,2}$ and inequality (9) follow by simple calculation. Suppose *q >1* and put $y := x/a_m$. Then $y \in [0,1]$ and

$$W_{m,2q}(x) = \frac{m+1}{a_m}\sum_{i=0}^{m} P_{m,i}(y) \int_0^{a_m} (t - y a_m)^{2q} P_{m,i}\left(\frac{t}{a_m}\right) dt$$

$$= (m+1)a_m^{2q} \sum_{i=0}^{m} P_{m,i}(y) \int_0^1 (s-y)^{2q} P_{m,i}(s) ds = a_m^{2q}\left(M_m \psi_y^{2q}\right)(y), \quad (11)$$

where M_m is the classical Bernstein-Durrmeyer operator (1).
The representation formula (8) follows at once from the known identity

$$\left(M_m \psi_y^{2q}\right)(y) = \sum_{j=0}^{q} \beta_{j,q,m} \left(\frac{y(1-y)}{m}\right)^{q-1} m^{-2j},$$

where $\beta_{j,q,m}$ are real numbers independent of y and bounded uniformly in *m* (see [13], Lemma 4.8 with $c = -1$). Now, let us observe that for $y \in [0, {}^1\!/_m]$ $y \in \left[1 - \frac{1}{m}, 1\right], m \geq 2$, one has $y(1-y) \leq \frac{m-1}{m^2}$ and

$$\left(M_m \psi_y^{2q}\right)(y) = \sum_{j=0}^{q} \left|\beta_{j,q,m}\right| \left(\frac{m-1}{m^3}\right)^{q-1} m^{-2j} \leq \sum_{j=0}^{q} \eta_{j,q} m^{-2q},$$

where $\eta_{j,q}$ are non-negative numbers depending only on *j* and *q*. If $y \in \left[\frac{1}{m}, 1 - \frac{1}{m}\right]$ then $\frac{1}{my\,(1-y)} \leq \frac{m}{m-1} \leq 2$ and

$$\left(M_m \psi_y^{2q}\right)(y) = \left(\frac{y(1-y)}{m}\right)^q \sum_{j=0}^{q} \left|\beta_{j,q,m}\right| \frac{1}{(my(1-y))^j},$$

$$\leq \left(\frac{y(1-y)}{m}\right)^q \sum_{j=0}^{q} \eta_{j,q} 2^j.$$

Consequently,

$$\left(M_m \psi_y^{2q}\right)(y) \leq \frac{C_q}{m^q}\left(y(1-y) + \frac{1}{m}\right)^q \quad with \ C_q = \sum_{j=0}^{q} \eta_{j,q} 2^j.$$

Taking in (11) and in the above inequality $y = x/a_m$ we easily obtain estimation (10).

Lemma 2.2 Let $x \in (0, \infty)$ and let

$$K_{m,\vartheta}\left(\frac{x}{a_m}, \frac{t}{a_m}\right) := \frac{m+1}{a_m}\sum_{i=0}^{m} Q_{m,i}^{(\vartheta)}\left(\frac{x}{a_m}\right) P_{m,i}\left(\frac{t}{a_m}\right).$$

Then

$$\int_t^{a_m} K_{m,\vartheta}\left(\frac{x}{a_m}, \frac{u}{a_m}\right) du \leq \frac{\vartheta}{(t-x)^2} W_{m,2}(x) \quad if \quad x < t \quad (12)$$

and

$$\int_0^t K_{m,\vartheta}\left(\frac{x}{a_m}, \frac{u}{a_m}\right) du \leq \frac{\vartheta}{(x-t)^2} W_{m,2}(x) \quad if \quad 0 < t < x, \quad (13)$$

where $W_{m,2}(x)$ is given by (7) (with $q = 2$).

Proof. As is known $Q_{m,i}^{(\vartheta)}(x) \le \vartheta P_{m,i}(x)$ for $\vartheta \ge 1$. Hence, if $x < t$, then

$$\int_t^{a_m} K_{m,\vartheta}\left(\frac{x}{a_m},\frac{u}{a_m}\right)du \le \frac{1}{(t-x)^2}\int_t^{a_m}(u-x)^2 K_{m,\vartheta}\left(\frac{x}{a_m},\frac{u}{a_m}\right)du$$

$$\le \frac{1}{(t-x)^2}\left(D_{m,\vartheta}\psi_x^2\right)(x) \le \frac{\vartheta}{(t-x)^2}\left(D_m\psi_x^2\right)(x) = \frac{\vartheta}{(t-x)^2}W_{m,2}(x).$$

The proof of (13) is similar.

In order to formulate the next lemma we introduce the following intervals. If $x > 0$, we write

$$I_x(u) := [x+u, x] \cap [0,\infty) \quad if\ u < 0$$

$$I_x(u) := [x, x+u] \quad if\ u > 0$$

Lemma 2.3 Let $f \in X_{loc}[0,\infty)$ and let the one-sided limits $f(x+), f(x-)$ exist at a fixed point $x \in (0,\infty)$. Consider the function g_x defined by (6) and write $d_m := \sqrt{a_m/m}$. If $h = -x$ or $h = x$, then for all integers m such that $d_m \le 1/2$ we have

$$\left|\int_{I_x(h)} g_x(t)K_{m,\vartheta}\left(\frac{x}{a_m},\frac{t}{a_m}\right)dt\right| \le v_1(g_x; I_x(hd_m))$$

$$+\frac{8\vartheta W_{m,2}(x)}{h^2 d_m^2}\left[\sum_{j=1}^{n-1}\frac{v_j(g_x; I_x(jhd_m))}{j^3} + \frac{v_n(g_x; I_x(h))}{n^2}\right],$$

where $n = [1/d_m]$and $W_{m,2}(x)$ is estimated in (9).

Proof. Restricting the proof to $h = -x$ we define the point $t_j = x + jhd_n$ for $j = 1,2,3,\ldots,n+1$and we denote $t_{n+1} = 0$. Put $T_j = [t_j, x]$ for $j = 1,2,3,\ldots,n+1$ and we have

$$\int_{I_x(h)} g_x(t)K_{m,\vartheta}\left(\frac{x}{a_m},\frac{t}{a_m}\right)dt \le \int_{t_1}^{x} g_x(t)K_{m,\vartheta}\left(\frac{x}{a_m},\frac{t}{a_m}\right)dt$$

$$+\sum_{j=1}^{n} g_x(t_j)\int_{t_{j+1}}^{t_j} K_{m,\vartheta}\left(\frac{x}{a_m},\frac{t}{a_m}\right)dt + \sum_{j=1}^{n}\int_{t_{j+1}}^{t_j}\left(g_x(t)-g_x(t_j)\right)K_{m,\vartheta}\left(\frac{x}{a_m},\frac{t}{a_m}\right)dt$$

$$= I_1(m,x) + I_2(m,x) + I_3(m,x), \quad say$$

Clearly,

$$|I_1(m,x)| \le \int_{t_1}^{x}|(g_x(t)-g_x(x))|K_{m,\vartheta}\left(\frac{x}{a_m},\frac{t}{a_m}\right)dt$$

$$\le v_1(g_x; T_1)\int_0^{a_m} K_{m,\vartheta}\left(\frac{x}{a_m},\frac{t}{a_m}\right)dt = v_1(g_x; T_1).$$

By the Abel lemma on summation by parts and by (13) we have

$$|I_2(m,x)| \le |g_x(t_1)|\int_0^{t_1} K_{m,\vartheta}\left(\frac{x}{a_m},\frac{t}{a_m}\right)dt + \sum_{j=1}^{n-1}\left|g_x(t_{j+1}) - g_x(t_j)\right|\int_0^{t_{j+1}} K_{m,\vartheta}\left(\frac{x}{a_m},\frac{t}{a_m}\right)dt$$

$$\le \frac{\vartheta W_{m,2}(x)}{h^2 d_m^2}\left(|g_x(t_1) - g_x(x)| + \sum_{j=1}^{n-1}\left|g_x(t_{j+1}) - g_x(t_j)\right|\frac{1}{(j+1)^2}\right)$$

$$= \frac{\vartheta W_{m,2}(x)}{h^2 d_m^2}\left(|g_x(t_1) - g_x(x)| + \sum_{j=1}^{n-2}\sum_{i=1}^{j}|g_x(t_{i+1}) - g_x(t_i)|\left(\frac{1}{(j+1)^2} - \frac{1}{(j+2)^2}\right)\right.$$

$$\left. + \frac{1}{n^2}\sum_{i=1}^{n}|g_x(t_{i+1}) - g_x(t_i)|\right)$$

$$\le \frac{\vartheta W_{m,2}(x)}{h^2 d_m^2}\left(v_1(g_x; T_1) + 2\sum_{j=1}^{n-2}\frac{v_{j+1}\left(g_x; T_{j+1}\right)}{(j+1)^3} + \frac{v_n(g_x; T_n)}{n^2}\right)$$

$$\le \frac{\vartheta W_{m,2}(x)}{h^2 d_m^2}\left(2\sum_{j=1}^{n-1}\frac{v_j\left(g_x;T_j\right)}{j^3}+\frac{v_n(g_x;T_{n+1})}{n^2}\right).$$

Next, in view of (13) and the Abel transformation,

$$|I_3(m,x)| \le \sum_{j=1}^{n} v_1\left(g_x;[t_{j+1},t_j]\right)\int_{t_{j+1}}^{t_1} K_{m,\vartheta}\left(\frac{x}{a_m},\frac{t}{a_m}\right)dt$$

$$\le \frac{\vartheta W_{m,2}(x)}{h^2 d_m^2}\sum_{j=1}^{n}\frac{v_1\left(g_x;[t_{j+1},t_j]\right)}{j^2}$$

$$= \frac{\vartheta W_{m,2}(x)}{h^2 d_m^2}\left(\sum_{i=1}^{n}\frac{v_1(g_x;[t_{i+1},t_i])}{n^2}+\sum_{j=1}^{n-1}\sum_{i=1}^{j} v_1(g_x;[t_{i+1},t_i])\left(\frac{1}{j^2}-\frac{1}{(j+1)^2}\right)\right)$$

$$\le \frac{\vartheta W_{m,2}(x)}{h^2 d_m^2}\left(\frac{v_n(g_x;T_{n+1})}{n^2}+6\sum_{j=1}^{n-1}\frac{v_j\left(g_x;T_{j+1}\right)}{(j+1)^3}\right)$$

$$\le \frac{\vartheta W_{m,2}(x)}{h^2 d_m^2}\left(\frac{v_n(g_x;T_{n+1})}{n^2}+6\sum_{j=2}^{n}\frac{v_j\left(g_x;T_j\right)}{j^3}\right)$$

Combining the result and observing that $T_j = I_x(jhd_m)$ we get the desired estimation for $h = -x$. In the case of $h = x$ the proof runs analogously; we use inequality (12) instead of (13).

III. Main Results

In this section we prove our main theorems.

Theorem 3.1 Let $f \in X_{loc}[0,\infty)$ and let the one-sided limits $f(x+), f(x-)$ exist at a fixed point $x \in (0,\infty)$. Then, for all integers m such that $a_m > 2x$ and $4a_m \le m$ one has

$$\left|\left(D_{m,\vartheta} f\right)(x) - \frac{f(x+)+\vartheta f(x-)}{\vartheta+1}\right| \le 2\mu_1\left(g_x; H_x(x\sqrt{a_m/m}\right)$$
$$+\frac{32\vartheta}{x^2}\left(x\left(1-\frac{x}{a_m}\right)+\frac{a_m}{m}\right)\left[\sum_{j=1}^{n-1}\frac{\mu_j\left(g_x;H_x(jx\sqrt{a_m/m}\right)}{j^3}+\frac{\mu_n(g_x;H_x(x))}{n^2}\right]$$
$$+\frac{2\vartheta C_q}{x^{2q}}\varphi(a_m;f)\left(\frac{a_m}{m}\right)^q\left(x\left(1-\frac{x}{a_m}\right)+\frac{a_m}{m}\right)^q+\frac{2\vartheta|f(x+)-f(x-)|}{\sqrt{\frac{mx}{a_m}\left(1-\frac{x}{a_m}\right)}},$$

where $n = \left[\sqrt{m/a_m}\right], H_x(u) = [x-u, x+u]$ for $0 \le u \le x, \varphi(a;f) = \sup_{0\le t\le b}|f(t)|$

$$g_x(t) = \begin{cases} f(t) - f(x+) & if\ t > x, \\ 0 & if\ t = x, \\ f(t) - f(x-) & if\ 0 \le t < x, \end{cases} \tag{14}$$

q is an arbitrary positive integer and c_q is a positive constant depending only on q.

Proof. We decompose $f(t)$ into four parts as

$$f(t) = \frac{f(x+)+\vartheta f(x-)}{\vartheta+1}+\frac{f(x+)-f(x-)}{2}\left[sgn_x(t)+\frac{\vartheta-1}{\vartheta+1}\right]+g_x(t)$$
$$+\delta_x(t)\left[f(x)+\frac{f(x+)-f(x-)}{2}\right] \tag{15}$$

where $g_x(t)$ is defined as (14) and $sgn_x(t) := sgn(t-x)$,

$$\delta_x(t) = \begin{cases} 1, & x = t, \\ 0, & x \ne t, \end{cases} \tag{16}$$

From (15) we have

$$\left(D_{m,\vartheta} f\right)(x) = \frac{f(x+)+\vartheta f(x-)}{\vartheta+1}+\left(D_{m,\vartheta} g_x\right)(x)+\frac{f(x+)-f(x-)}{2}$$
$$\times\left[\left(D_{m,\vartheta} sgn_x\right)(x)+\frac{\vartheta-1}{\vartheta+1}\right]+\left[f(x)-\frac{f(x+)-f(x-)}{2}\right]\left(D_{m,\vartheta}\delta_x\right)(x).$$

For operators $D_{m,\vartheta}$ using (16) we can observe that the last term on the right hand side vanishes. In addition it is obvious that $\left(D_{m,\vartheta} 1\right)(x) = 1$. Hence we have

$$\left|\left(D_{m,\vartheta}f\right)(x)-\frac{f(x+)+\vartheta f(x-)}{\vartheta+1}\right|$$
$$\leq\left|\left(D_{m,\vartheta}g_x\right)(x)\right|+\left|\frac{f(x+)-f(x-)}{2}\right|\left|\left(D_{m,\vartheta}sgn_x\right)(x)+\frac{\vartheta-1}{\vartheta+1}\right|, \qquad (17)$$

In order to prove our theorem we need the estimates for $\left(D_{m,\vartheta}g_x\right)(x)$ and $\left(D_{m,\vartheta}sgn_x\right)(x)+\frac{\vartheta-1}{\vartheta+1}$.

To estimate $\left(D_{m,\vartheta}g_x\right)(x)$ with the help of the fixed points *x* and *2x*, we decompose it into three parts as follows:

$$\left|\int_0^{a_m} g_x(t)K_{m,\vartheta}\left(\frac{x}{a_m},\frac{t}{a_m}\right)dt\right|\leq\left|\int_0^{x} g_x(t)K_{m,\vartheta}\left(\frac{x}{a_m},\frac{t}{a_m}\right)dt\right|$$
$$+\left|\int_x^{2x} g_x(t)K_{m,\vartheta}\left(\frac{x}{a_m},\frac{t}{a_m}\right)dt\right|+\left|\int_{2x}^{a_m} g_x(t)K_{m,\vartheta}\left(\frac{x}{a_m},\frac{t}{a_m}\right)dt\right|$$
$$=\left|E_{1,\vartheta}(m,x)\right|+\left|E_{2,\vartheta}(m,x)\right|+\left|E_{3,\vartheta}(m,x)\right|, \qquad (18)$$

where $K_{m,\vartheta}\left(\frac{x}{a_m},\frac{t}{a_m}\right)$ is defined in Lemma 2.2.

The estimations for $\left|E_{1,\vartheta}(m,x)\right|$ and $\left|E_{2,\vartheta}(m,x)\right|$are given in Lemma 2.3 in which we put $h=-x$ and $h=x$, respectively. Using the obvious inequality

$$V_j\left(g_x;I_x(-u)\right)+V_j\left(g_x;I_x(u)\right)\leq 2V_j\left(g_x;H_x(u)\right),$$

where $H_x(u)=[x-u,x+u],\quad 0<u\leq x$, we obtain

$$\left|E_{1,\vartheta}(m,x)\right|+\left|E_{2,\vartheta}(m,x)\right|\leq 2V_j\left(g_x;H_x\left(x\sqrt{a_m/m}\right)\right)$$
$$+\frac{16\vartheta W_{m,2}(x)m}{h^2d_m}\left[\sum_{j=1}^{n-1}\frac{V_j\left(g_x;H_x\left(jx\sqrt{a_m/m}\right)\right)}{j^3}+\frac{V_n\left(g_x;H_x(x)\right)}{n^2}\right]. \qquad (19)$$

Now, we estimate $\left|E_{3,\vartheta}(m,x)\right|$ Clearly, given any $q\in\mathbb{N}$, we have

$$\left|E_{3,\vartheta}(m,x)\right|\leq 2\varphi(a_m;f)\frac{m+1}{a_m}\sum_{i=0}^{m}Q_{m,i}^{(\vartheta)}\left(\frac{x}{a_m}\right)\int_{2x}^{a_m}P_{m,i}\left(\frac{t}{a_m}\right)dt$$
$$\leq 2\varphi(a_m;f)\frac{m+1}{x^{2q}a_m}\sum_{i=0}^{m}Q_{m,i}^{(\vartheta)}\left(\frac{x}{a_m}\right)\int_{2x}^{a_m}(t-x)^{2q}P_{m,i}\left(\frac{t}{a_m}\right)dt$$
$$\leq\frac{2\vartheta\varphi(a_m;f)}{x^{2q}}\frac{m+1}{a_m}\sum_{i=0}^{m}P_{m,i}\left(\frac{x}{a_m}\right)\int_{0}^{a_m}(t-x)^{2q}P_{m,i}\left(\frac{t}{a_m}\right)dt$$
$$=\frac{2\vartheta\varphi(a_m;f)}{x^{2q}}W_{m,2q}(x). \qquad (20)$$

Finally, replacing x by x/a_m in the result of X. M. Zeng and W. Chen [22] (sect. 3, pp. 9-11) we immediately get

$$\left|\left(D_{m,\vartheta}sgn_x\right)(x)+\frac{\vartheta-1}{\vartheta+1}\right|\leq\frac{4\vartheta}{\sqrt{m\frac{x}{a_m}\left(1-\frac{x}{a_m}\right)}}$$

Putting (18), (19), (20) and (21) into (17), we get the required result. Thus the proof of Theorem 1 is complete.

From Theorem 3.1 and inequality (6) we get

Theorem 3.2 Let $f\in BV_{loc}^{p}[0,\infty),p\geq 1$ and let $x\in(0,\infty)$. Then, for all integers *m* such that $a_m>2x$ and $4a_m\leq m$ we have

$$\left|\left(D_{m,\vartheta}f\right)(x)-\frac{f(x+)+\vartheta f(x-)}{\vartheta+1}\right|\leq 2V_p\left(g_x;H_x(x\sqrt{a_m/m}\right)$$
$$+\frac{2^{7+1/p}\vartheta}{x^2n^{1+1/p}}\left(x\left(1-\frac{x}{a_m}\right)+\frac{a_m}{m}\right)\sum_{i=1}^{(n+1)^2-1}\frac{V_p\left(g_x;H_x\left(\frac{x}{\sqrt{i}}\right)\right)}{\left(\sqrt{i}\right)^{2-1/p}}$$

$$+\frac{2\vartheta C_q}{x^{2q}}\varphi(a_m;f)\left(\frac{a_m}{m}\right)^q\left(x\left(1-\frac{x}{a_m}\right)+\frac{a_m}{m}\right)^q+\frac{2\vartheta|f(x+)-f(x-)|}{\sqrt{\frac{mx}{a_m}\left(1-\frac{x}{a_m}\right)}}.$$

In order to show this it is necessary to prove that the right-hand sides of the inequalities mentioned in the hypotheses of the theorems tend to zero as $m\to\infty$. In view of (4) we have $n=\left[\sqrt{m/a_m}\right]\to\infty$ as $m\to\infty$. So, in Theorem 1 it is enough to consider only the term

$$\wedge_n(x)=\sum_{j=1}^{n-1}\frac{\square_j(g_x;H_x(jxd_m))}{j^3},\quad where\ \ d_m=\sqrt{a_m/m}.$$

Clearly,

$$\wedge_n(x)=\sum_{j=1}^{n-1}\frac{\square_1(g_x;H_x(jxd_m))}{j^2}\le 4d_m\int_{d_m}^{nd_m}\frac{\square_1(g_x;H_x(xt))}{t^2}dt$$

$$\le 4d_m\int_1^{n+1}\square_1\left(g_x;H_x\left(\frac{x}{s}\right)\right)ds\le\frac{4}{n}\sum_{i=1}^{n}\square_1\left(g_x;H_x\left(\frac{x}{i}\right)\right).$$

Since the function g_x is continuous at *x* and $\square_1\left(g_x;H_x\left(\frac{x}{i}\right)\right)$ denotes the oscillation of g_x on the interval $H_x\left(\frac{x}{i}\right)$, we have

$$\lim_{i\to\infty}\square_1\left(g_x;H_x\left(\frac{x}{i}\right)\right)=0$$

and consequently,

$$\lim_{n\to\infty}\wedge_n(x)=0$$

As regards Theorem 3.2, it is easy to verify that in view of the continuity of g_x at *x*,

$$\lim_{n\to\infty}\frac{1}{n^{1+1/p}}\sum_{i=1}^{n^2-1}\frac{1}{\left(\sqrt{i}\right)^{1-1/p}}V_p\left(g_x;H_x\left(\frac{x}{\sqrt{i}}\right)\right)=0.$$

Thus we get the following approximation theorem.

Proof. Let $f\in BV_{loc}^p[0,\infty),p\ge 1$. In view of (6) and the notation $d_m=\sqrt{a_m/m}$, $n=\left[\sqrt{m/a_m}\right]$, we have

$$\sum_{j=1}^{n-1}\frac{\square_j\left(g_x;H_x(jxd_m)\right)}{j^3}\le\sum_{j=1}^{n-1}\frac{V_p\left(g_x;H_x(jxd_m)\right)}{j^{2+1/p}}\le(2d_m)^{2+1/p}\int_{d_m}^{nd_m}\frac{V_p\left(g_x;H_x(xt)\right)}{t^{2+1/p}}dt$$

$$\le\left(\frac{2}{n}\right)^{2+1/p}\int_1^{(n+1)^2}\frac{V_p\left(g_x;H_x\left(x/\sqrt{s}\right)\right)}{\left(\sqrt{s}\right)^{2+1/p}}ds\le\left(\frac{2}{n}\right)^{2+1/p}\sum_{i=1}^{(n+1)^2-1}\frac{V_p\left(g_x;H_x\left(x/\sqrt{i}\right)\right)}{\left(\sqrt{i}\right)^{2+1/p}}$$

and

$$\frac{\square_n\left(g_x;H_x(x)\right)}{n^2}\le\frac{V_p\left(g_x;H_x(x)\right)}{n^{2+1/p}}$$

moreover,

$$\square\left(g_x;H_x\left(x\sqrt{a_m/m}\right)\right)\le V_p\left(g_x;H_x\left(x\sqrt{a_m/m}\right)\right)$$

The estimation given in Theorem 3.2 now immediately follows from Theorem 3.1.

Corollary. Suppose that $f\in X_{loc}[0,\infty)$ (in particular, $f\in BV_{loc}^p[0,\infty),p\ge 1$) and that there exists a positive integer *q* such that

$$\lim_{m\to\infty}\left(\frac{a_m}{m}\right)^q\varphi(a_m;f)=0.$$

Then at every point $x\in[0,\infty)$ at which the limits $f(x+),f(x-)$ exist we have

$$\lim_{m\to\infty}\left(D_{m,\vartheta}f\right)(x)=\frac{f(x+)+\vartheta f(x-)}{\vartheta+1}$$

Obviously, the above relations hold true for every measurable function *f* bounded on [0, ∞), in particular for every function *f* of bounded *p*-th power variation ($p\ge 1$) on the whole interval [0, ∞).

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