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A note on nilpotency in a Left Goldie near-ring

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Abstract: *In this paper we present an important result that a nil subnear-ring of a semiprime strictly left Goldie near-ring is nilpotent. It is to be noted that the essentiality of left near-ring subgroup here, arises as crucial from it's feeble nature. In contrast to such a result in ring theory, the crucial role played by substructures already mentioned appears here with very fascinating distinctiveness.*
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I. Introduction

Chowdhury et al [1] introduced the notions of a Goldie near-ring as well as that of a Goldie module [2,3] as two way generalizations of so-called Goldie ring - an exposition of A.W. Goldie through his classics,- a part of his thorough study of the structure of prime rings under ascending chain conditions [10] and semiprime rings with maximum conditions [11]. We discussed various aspects of a Goldie near-ring and of a Goldie module including the near-ring of quotients and its possible descending chain condition and decomposition of the zero of a Goldie module [2,3], an analogous of Artin-Rees theorem [2]. Also we delve into Some Aspects of Artinian (Noetherian) Part of a Goldie Ring and its Topological Relevance (8) as well as Wreath Sum of Near-rings and Near-ring Groups with Goldie structures (9)). It is easy to see that a nilpotent subring of a ring is necessarily nil. But converse is not true, however, we see that [12] Goldie character in a ring draw attention in its favor!.We here prove this interesting standard problem in a near-ring with Goldie characteristics taking into consideration various aspects of large or essential characters of its subalgebraic structures with proper justification. Moreover, in this connection, it would not be irrelevant to mention author's another new notion, what may be called the notion of a *nilpotent module-element* or a *nilpotent N-group element* [7] together with *a nil or a nilpotent submodule, or an N subgroup* etc.

II. Preliminaries

For the sake of completeness we would like to begin our discussion with the definition of a right near-ring (N,+,.) - an algebraic structure consisting of a non-empty set N equipped with two binary operations viz., addition (+) and multiplication (.),where the first one makes N- a group (not necessarily abelian) and the second one a semigroup with the one-way distributive law, viz. (a+b)c=ac+bc , for a,b,c $\in N$
For other relevant information regarding near-ring preliminaries we would like to refer Pilz [13].
Throughout this paper N will mean a right near-ring with unity (zero symmetric) unless otherwise specified.

2.1 Definitions:

2.1.1 An element $a \in N$ is *nilpotent* if there is a positive integer t such that $a^t=0$, $a^{t-1} \neq 0$.
2.1.2. A subnear-ring is *nil* if each element of the corresponding set is nilpotent.
2.1.3. A subnear-ring *I is nilpotent* if there is a positive integer t such that $I^t=0$, $I^{t-1} \neq 0$, (in the sense $i_1.i_2....i_t = 0$, for $i_j \in I$ and $i_1.i_2....i_{t-1} \neq 0$, for some $i_j \in I$)
Clearly, a nilpotent subnear-ring is nil but the converse is not true. For the converse, that is a nilpotent subring is nilpotent, we'll deal with so called sequentially nilpotent (or s-nilpotent) notion.
We note the following: the above situation is dealt with the following definition that would lead us to our expected goal. 2.1.4.
An element a($\in I$) is *sequentially nilpotent (s-nilpotent)* if for some positive integer k, we have($a_i \in I$)
$a_1.a_2...a_k = 0$, (a_1=a) . So if an element a (a_1=a) is s-nilpotent, then for some
$(a=)a_1, a_2,.., a_k \in I$, $a_1.a_2...a_k = 0 \Rightarrow (xa_1).a_2...a_k = 0$ and so xa_1 is s-nilpotent , i.e. any left multiple of a is also s-nilpotent.
And hence we

Note: a(∈*I*) would *be not s-nilpotent* if for any sequence of the type $< a_i >, a_i \in I$, with (a=a_1) we have $a_1.a_2...a_k \neq 0(\neq \prod_{i=1}^{k} a_i)$ whatever be the positive integer *k*]

2.1. 5. A sub near-ring *I* of *N* is *sequentially nilpotent*(*s-nilpotent*) if for each sequence $< a_i >, a_i \in I$ there is a positive integer *k* such that $a_i.a_2...a_k = 0(= \prod_{i=1}^{k} a_i)$.

Note:

(i) for an s-nilpotent sub near-ring *I* of *N*, each element of *I* is s-nilpotent.

(ii) if *I* is not *s*-nilpotent, then there is a sequence $< a_i >, a_i \in I$, for each *k*, $a_1.a_2...a_k \neq 0(\neq \prod_{i=1}^{k} a_i)$,and

2.1.6. a_1(∈*I*) *has an infinite sequence* if there is a sequence $< a_i >, a_i \in I$ such that for each *k*,

$$a_1.a_2...a_k \neq 0(\neq \prod_{i=1}^{k} a_i).$$

Note:

I is not *s*-nilpotent, then there is an a_1 (∈*I*) such that a_1 *has an infinite sequence*.

2.1.7 For *x* ∈*N* the set $l(x) = \{n \in N \mid nx = 0\}$ is the *left annihilator* of *x* in *N*.

And this a left ideal of *N*.

2.1.7(a) A near-ring is *left Goldie* if it satisfies the a.c.c. (ascending chain condition)on its left annihilators and it has no infinite direct sum of left ideals .

2.1.7(b) N is *strictly left Goldie* if it satisfies the a.c.c. on its left annihilators and it has no infinite independent family of left N-subgroups .

Example 1 : N = { o, a, b, c} is a near-ring under the operations defined by the following tables.

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

(i)

.	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
c	0	a	c	c

(ii)

Here we note that A = {0, a}, B = {0, a, b} and C = {0, a, c} are subsets of *N* and *BN* ⊆ *B*, *CN* ⊆ *C* whereas *NA*⊆ *A* and *AN* ⊆ *A*. Thus, we define the following

2.1.8 Definitions **:** A non-empty subset *S* of a near-ring *N* is

(i) a *right N-subset* of *N* if *SN* ⊆ *S*

(ii) a *left N-subset* of *N* if *NS* ⊆ *S* and

(iii) an *invariant subset* of *N* if *NS* ⊆ *S*, *SN*⊆ *S*.

It is clear that an invariant subset of a near-ring *N* is a left as well as right *N*-subset of *N*. Moreover, every left (right) N-subset contains the zero element of *N*.

2.1.9 (i) An ideal *I* of N is *strongly prime* if for two non zero invariant subsets *A* and *B* , *AB* ⊆*I* ⇒*A*⊆*I,* or *B*⊆*I.*

(ii) A near-ring is strongly prime if (0) is strongly prime.

2.1.10. Definition : If *N* is a near-ring then the group (*E*, +) is an *N-group* (near-ring group) NE when there exists a map *N* × *E* → *E*, (n, e) → ne such that

(i) (n1 + n2)e = n1e + n2e

(ii) (n1n2)e = n1(n2e)

(iii) 1. e = e, for all n1, n2 ∈ *N*, e ∈ *E*.
In what follows, *E* will stand for the near-ring group *NE*.
Clearly near-ring *N* can always be considered as an *N*- group. We shall write *NN* to denote *N* as an N-group.
Example 2 (Ex.1.18(c) [11]) : Let *G* be an additive group and *M(G)* be a (right) near-ring(of all maps from *G* to *G*) then *G* is an *M*(*G*) – group when

M(*G*) × *G* → *G* such that

(f, x) → f(x), for x∈ *G*, f ∈ *M*(*G*).
Example3 : Every left module *M* over a ring R is an *R*-group over the near-ring *R*.
2.1.11. Properties : *If E is an N-group then*
(i) 0. e = 0 (the first 0 is the zero element of N and the second 0 is the zero element of E).
(ii) (– n)e = –ne and

(iii) (n–n1) e = ne – n1e, for all e ∈ E; n, n1 ∈ N
2.1.12. Definitions: An *N*-group *E* is said to be an *N-group with acc on annihilators* if any ascending chain Ann(*M*1)⊂Ann(*M*2) ⊂Ann(*M*3)⊂ ... of annihilators of *subsets M1*, *M*2, *M*3, ... of *E* stops after a finite steps. Similarly, we define an *N-group E with dcc on annihilators* for any descending chain of the type Ann(*M*1)⊃Ann(*M*2) ⊃Ann(*M*3) ⊃ ... .

2.2. **Essential ideals and essential *N*-subgroups.**

2.2.1. Definitions: Let *A* and *B* be two N-subgroups of *E* such that *A*⊆*B* then *A* is an *essential N-subgroup of B* (denoted *A*⊆e*B*) if any *N*-subgroup *C*(≠0) of *B* has non-zero intersection with *A*. when *A*⊆ e *B*, we say *B* is an *essential extension of A in E*. Here an *essential left N-subgroup A of N* will mean an essential *N*-subgroup of *NN*.

An ideal *M* of *E* is an *essential ideal of E* (denoted *M*⊆ e *E*) if for any ideal *C* (≠0) of *E*,

M ∩ *C* ≠(0). If a left ideal *A* of *N* is an essential ideal of *NN* then *A* is *an essential left ideal of N*.

A left N-subgroup of *N* is *weakly essential* if for any non zero left ideal *I* of *N*, *A* ∩*I* ≠0
An essential left ideal *I* is weakly essential as a left *N*-subgroup. It is to be noted that an essential left *N*-subgroup *A* of *N* is also weakly essential . That the converse is not true is shown in example below.
Example4. (H(37), Page 341-342 [11]) : Consider the near-ring S_3 = {0, a, b, c, x, y) with operation addition [defined in table 1.3 (i)] and multiplication defined by the following table.

N = {0, a, b, c, x, y} is a near–ring under the operations defined by the following tables.

+	0	a	b	c	x	y
0	0	a	b	c	x	y
a	a	0	y	x	c	b
b	b	x	0	y	a	c
c	c	y	x	0	b	a
x	x	b	c	a	y	0
y	y	c	a	b	0	x

o	0	a	b	c	x	y
0	0	0	0	0	0	0
a	0	a	b	c	0	0
b	0	a	b	c	0	0
c	0	a	b	c	0	0
x	0	0	0	0	0	0
y	0	0	0	0	0	0

Here non-zero left S_3-subgroups are {0, a}, (0, b}, {0, c}, {0, x, y) and S_3. {0, x, y) and S_3 are the only non-zero left ideals. This shows that the S_3-subgroup {0, x, y) is weekly essential but not an essential left S_3-subgroup.
However, the following example is sufficient to show the existence of near-ring where every weakly essential left *N*-subgroup is also essential.
3.2.16. Example (J(91), Page 343[11]) :
N = {0, 1, 2, 3, 4, 5, 6, 7} is a near-ring under addition modulo 8 and multiplication defined by the following table

∘	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	0	3	4	3	0	1
2	0	2	0	6	0	6	0	2
3	0	3	0	1	4	1	0	3
4	0	4	0	4	0	4	0	4
5	0	5	0	7	4	7	0	5
6	0	6	0	2	0	2	0	6
7	0	7	0	5	4	5	0	7

Here {0, 4} and {0, 2, 4, 6} are the left *N*-subgroup of *N* whereas the second one is the only non-zero proper left ideal of *N*. Thus each of them is weakly essential and they are essential too.

Example5. (J (22), Page- 342 - 343 [11]) :

The group N = {0,1,2,3,4,5,6,7} under addition modulo 8 is an N-group w.r.t. the multiplication defined by the following table

•	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	2	0	4	4	2
2	0	0	0	4	0	0	0	4
3	0	0	0	6	0	4	4	6
4	0	0	0	0	0	0	0	0
5	0	0	0	2	0	4	4	2

N-group NN has non-trivial N-subgroups {0, 4} and {0,2,4,6}. Hence each of them has non-zero intersection with other N-subgroups of NN and so each of them is an essential N-subgroup of NN. Also{0, 4} ⊆ e {0, 2, 4, 6} which shows the validity of the following lemma 2.2.2.

2.2.2. Lemma : *If A, B, C are N-subgroup of E such that A⊆B⊆C then A ⊆e B⊆ e C if and only if A ⊆ e C.*

Proof : Let *P* be a non-zero *N*-subgroup of *E* such that *P*⊆*C*. Since *B* ⊆e *C*, *B* ∩ *P* ≠ (0).

Also, *B*∩P⊆*B* and *A* ⊆e *B*. So (*B*∩*P*) ∩ A≠ (0).

Therefore, *P*∩*A* ⊆(*B*∩*P*) ∩ *A* ≠(0).

Hence *A*⊆e*C*

Conversely, let *A*⊆e*C*. Then *A*∩*B* ≠(0), (for *B* ⊆*C*).

If *M* is a non-zero *N*-subgroup of *E* such that *M*⊆ *B* ⊆ *C* then, *M* is a non zero *N*-subgroup of *C*. Since *A*⊆ e *C*, it follows that *A* ∩ *M* ≠(0) which gives *A* ⊆e *B*.

Again, if *H* is any non-zero *N*-subgroup of *E* with *H* ⊆*C*⊆ *E* then *A* ∩ *H* ≠(0), (for *A* ⊆e *C*).

So, *A*⊆ *B* ⇒ (0) ≠ *A* ∩ *H* ⊆ *B* ∩ *H*.

Thus, *B*⊆ e *C*. //

2.2.3. Lemma : *Let A and B be two N-subgroups of E such that B ⊆e A. If a (≠0) ∈ A then there exists an essential N-subgroup L of NN such that La ≠ (0).*

Proof : Write $L = \{n \in N \mid na \in B\}$. Clearly, $La \subseteq B \subseteq A$ and $Na \subseteq A$ as A is an N-subgroup of E, $a \in A$.

Since $1 \in N$, $Na \neq (0)$. Again, $B \subseteq_e A$ gives $B \cap Na \neq (0)$.

Let $(0 \neq)\ b \in B \cap Na$. Then B = na (say) for $n \in N$. Thus $b = na \in B$ which gives $n \in L$. Hence $b = na \in La$.

Therefore, $La \neq (0)$ (for $b \neq 0$).
Now, let $x, y \in L$ then $xa, ya \in B$.
So, $(x - y)a = xa - ya \in B$.
$\Rightarrow x - y \in L$. .... (i)

Also, since B is an N-group of E, for $n \in N$, $(nx)a = n(xa) \in B$ (for $xa \in B$)
Therefore, $nx \in L$. .... (ii)
Thus L is an N-subgroup of NN.

Again, for an N-subgroup $I\ (\neq 0)$ of NN,

$Ia = (0)$

$\Rightarrow Ia \subseteq B$

$\Rightarrow I \subseteq L$

$\Rightarrow L \cap I = I \neq (0)$

and, $Ia \neq (0)$

$\Rightarrow B \cap Ia \neq (0)$, (for Ia is an N-subgroup of A and $B \subseteq_e A$).

Now, let $(\neq)\ x \in B \cap Ia$ then $x = b = \alpha a$ for $b \in B$, $\alpha \in I$.

Then $\alpha a \in B$

$\Rightarrow \alpha \in L$, (by choice of L)

$\Rightarrow \alpha \in L \cap I$.

Now, $\alpha = 0 \Rightarrow x = 0$, a contradiction.

So, $L \cap I \neq (0)$.

Therefore, L is an essential N-subgroup of NN such that $La \subseteq B$ and $La \neq (0)$.//

In an N-group E, the *singular N-subset of E* viz., the subset $Z1(E) = \{u \in E \mid Lu = (0)$, for some essential N-subgroup L of $NN\}$ plays an important role in our discussion.
N-group E is *N-non-singular* if $Z1(E) = 0$ and N is *left non-singular* if $Z1(N) = 0$. it is to be noted that $Z1(E)$ is <u>an N-subset of E</u> and $Z1(N)$ is <u>an invariant subset of N</u>

2.2.4. Lemma : For an $x \in E$, Ann(x) is an essential N-subgroup of NN if and only if $x \in Z1(E)$.[easy]

2.2.5. Lemma : *If I is an N-subgroup of NN and for $B \subseteq E$, Ann(B) $\subseteq_e$ I and Z1 (E) = (0) then Ann(B) = I.*

Proof : Let $(0 \neq)\ x \in I$ then by 2.2.3, there exists an essential N-subgroup L of NN such that $Lx \neq (0)$, $Lx \subseteq \mathrm{Ann}(B)$.

So, $(Lx)\, r_E(\mathrm{Ann}(B)) \subseteq \mathrm{Ann}(B)\, r_E(\mathrm{Ann}(B)) = (0)$
$\Rightarrow L(x\, r_E(\mathrm{Ann}(B))) = (0) \Rightarrow (x\, r_E(\mathrm{Ann}(B))) = (0)$ [for $Z1(E) = (0)$]
$\Rightarrow x \in \mathrm{Ann}(r_E(\mathrm{Ann}(B))) = \mathrm{Ann}(B)$

$\Rightarrow I \subseteq \mathrm{Ann}(B)$
Now considering the hypothesis, we get $\mathrm{Ann}(B) = I$. //
.

2.2.6. Lemma : *Let E be with acc on annihilators such that E is N-non-singular (i.e.Z1(E) = (0)). If N has no infinite direct sum of left ideals and every essential left ideal of N is an essential N-subgroup of NN then N satisfies the dcc on annihilators of subsets of E.*

Proof : Let X and Y be subsets of E such that $B = \mathrm{Ann}(X)$ and $C = \mathrm{Ann}(Y)$. Thus, B, C are N-subgroups of NN.

Now, if $B \subset C$ and B is an essential N-subgroup of C then by 2.2.5, $B = C$ as $B = \mathrm{Ann}(X)$. Hence B is not an essential N-subgroup of C. So, there exists an N-subgroup $D(\neq 0)$ of NN such that $D \subseteq C$, $B \cap D = (0)$.

Let $A_1 \supset A_2 \supset A_3 \supset$ ... be a strictly descending chain of annihilators of subsets of E. Since $A_1 \supset A_{i+1}$, by the above argument, there exists an N-subgroup Pi ($\neq 0$) of NN such that $P_i \subseteq A_i$ and $A_{i+1} \cap P_i = (0)$ .... (i)

Consider $M = \{X_m\}$, the family of all left ideals of N such that $Ai+1 \cap Xm = (0)$. The union of each chain of M is again a left ideal in M and satisfies the condition $Ai+1 \cap Xm = (0)$. Thus, by Zorn's Lemma ,M has a maximal element Xi (say) such that $Ai+1 \cap Xi = (0)$

.... (ii)

Again, $Ai+1$ and Xi being left ideals of N, $Ai+1 + Xi$ is also a left ideal of N.

Now, let V be a left ideal of N such that $(Ai+1+ Xi) \cap V = (0)$.

Now, ai+1 = xi + v, for some $ai+1 \in Ai+1$, $xi \in Xi$, $v \in V$.

$\Rightarrow v = - xi + ai+1 \in Xi + Ai+1 \subseteq Ai+1 + xi$

$\Rightarrow v \in (Ai+1 + Xi) \cap V = (0)$

$\Rightarrow ai+1 = xi \in Ai+1 \cap Xi = (0)$

$\Rightarrow Ai+1 \cap (Xi + V) = (0)$

Since Xi is maximal with condition $Ai+1 \cap Xi = (0)$, it follows that $Xi + V = Xi$ as $Xi \subseteq Xi + V$. This gives $V \subseteq Xi$ and so $V = V \cap Xi \subseteq V \cap (Ai+1 + Xi) = (0)$.

Thus, $Ai+1 + Xi$ is an essential left ideal of N such that $Ai+1 \cap Xi = (0)$ and the assumed hypothesis gives that $Ai+1 + Xi$ is an essential N-subgroup of NN. And so for Pi, chosen above, $Pi \cap (Ai+1 + Xi) \neq (0)$.

Suppose, $\alpha \in (Pi) = ai+1 + xi$, for $\alpha i \in Pi$, $ai+1 \in Ai+1$, $xi \in Xi$.

Then, $xi = - ai+1 + Pi \subseteq\in Ai+1 + Pi \subseteq Ai + Pi$, for $Ai+1 \subseteq Ai$. So, $xi \in Ai$ (for $Pi \subseteq Ai$) which gives $xi \in Ai \cap Xi$.

Now, if xi = 0 then $Pi \in Ai+1$ which gives $Pi \in Ai+1 . Pi = (0)$. So, $Pi = 0$.

Therefore, $Pi \cap (Ai+1 + Xi) = (0)$ and this is a contradiction. Hence $xi \neq 0$ and therefore

$Ai \cap Xi \neq (0)$.

Let $Ci = Ai \cap Xi$, a non-zero left ideal of N.

Then, $Ci \cap Ai+1 = (Ai \cap Xi) \cap Ai+1$

$= (Ai+1 \cap Ai) \cap Xi$

$= Ai+1 \cap Xi$, (as $Ai \supset Ai+1$)

$= (0)$, [by (ii)]

Therefore, when $Ai \supset Ai+1$, we get a non-zero ideal $Ci = Ai \cap Xi$ such that $Ci \cap Ai+1 = (0)$

.......(iii)

Now, for different values of i, we get an infinite family $\{C1, C2, C3, ...\}$ of non-zero left ideals of N such that (iii) holds.

Also, $Ci = Ai \cap Xi \subseteq Ai$ ....(iv)

Therefore, $C1 \cap C2 \subseteq C1 \cap A2 = (0)$, [by (iii) and (iv)]

Again, $C1 \cap (C2 + C3) \subseteq C1 \cap (A2 + A3)$, [by (iv)]

$\subseteq C1 \cap A2$, as $A2 \supset A3$

$= (0)$, [by (iii)]

$\Rightarrow C1 \cap (C2 + C3) = (0)$ ....(v)

And if $x \in C2 \cap (C1 + C3)$ then

$x = c2 = c1 + c3$, for $ci \in Ci$, i = 1,2,3.

$\Rightarrow c1 = c2 - c3 \in C2 + C3$

So, $c1 \in C2 \cap (C2 + C3) = (0)$, [by (v)]

$\Rightarrow c1 = 0$ and $c2 = c3 \in C3$.

$\Rightarrow C2 \in C2 \cap C3 \subseteq C2 \cap A3 = (0)$,

[by (iii) and (iv)]

$\Rightarrow c2 = 0$ and hence $C2 \cap (C1 + C3) = (0)$.

Similarly, $C3 \cap (C1 + C2) = (0)$. Thus $C1 \oplus C2 \oplus C3$ is a direct sum of non-zero left ideals of N.

Proceeding in this way, we find an infinite direct sum $C1 \oplus C2 \oplus C3 \oplus$...... of nonzero left ideals of *N*. This goes against our hypotheses and hence there exists a t $\in$ Z^+ such that *A*t = *A*t+1 = *A*t+2 = ... Therefore, *N* satisfies the dcc on annihilators of subset of *E*. //

2.2.7. Lemma : *$Z_1(N)$ (= {x ∈N | Ax = (0), for some essential left N-subgroup A of N}) is an invariant subset of N.*

Proof : Let x∈ $Z_1(N)$. Then *A*x = (0), for some essential left *N*-subgroup *A* of *N*. So, by 2.2.3, for any n ($\neq$0)∈*N* there exists an essential left N-subgroup *L* of *N* such that

*L*n $\subseteq$*A*, *L*n $\neq$(0).

This gives, *L*(nx) = (*L*n) x $\subseteq$*A*x = (0)

nx∈ $Z_1(N)$.

And, *A* (xn) = (*A*x)n = (0)

xn∈ $Z_1(N)$ //

2.2.8. Lemma : *A strongly semiprime near-ring N with acc on left annihilators has no non-zero nil left N-subset of N.*

Proof : Let *A* be any non-zero left *N*-subset of *N*. Since *N* satisfies the acc on left annihilators, we can choose a ($\neq$0) ∈*A* with 1(a) as large as possible.

Now, a*N*a = (0)

$\Rightarrow$ $(Na)^2$ = (*N*a)(*N*a) = *N*(a*N*a) = (0)

And *N*a being a non-zero left *N*-subset of *N*(1∈*N*,a$\neq$ 0), we meet a contradiction to 3.2.5.[*N* being strongly semi prime has no non-zero nilpotent left or right *N*-subset**]**

So, a*N*a $\neq$(0).

Let x∈ *N* be such that axa$\neq$ 0

Now, xa $\neq$0 (otherwise axa = 0)

x $\neq$1(a)

Again, z∈1(a)$\Rightarrow$ za = 0

$\Rightarrow$ z(axa) = (za)xa = 0

$\Rightarrow$ z∈ 1(axa)

1(a)∈ 1 (axa)

But 1(a) being maximal, 1(axa) = 1(a)

So, x^2 $\notin$l(axa)

$\Rightarrow$ x (axa) = 0

$\Rightarrow$ $(xa)^2$ =0

$\Rightarrow$ (xax)a= 0

$\Rightarrow$xax1(a) = 1(axa)

$\Rightarrow$(xax)(xax)= 0

$\Rightarrow$ $(xa)^3$ $\neq$0 and so on.

Thus, $(xa)^t$ $\neq$0, for any t∈Z^+.

Therefore, *A* possesses a non-zero non nilpotent element xa. So *A* is not nill.

Hence *N* does not have any non-zero nil left *N*-subset of *N*. /

2.2.9. Lemma : *If N is a strongly semiprime near-ring with acc of left annihilators then N is left non-singular.*

Proof : Being *N* acc with acc on left annihilators, $Z_1(N)$ is a nil invariant subset of *N* and by above it follows that $Z_1(N)$ = 0. Thus the result follows**.**

Again N being strictly left Goldie, it is left Goldie. So it has no infinite direct sum of left ideals. And therefore as a special case of 2.2.6, we get the following ([5], Nat, Acad,Sci. Letters.)

2.2.10. Theorem : *If in a strongly semiprime strictly left Goldie near-ring N, every weakly essential left N-subgroup of N is also essential, then N satisfies the dcc on left annihilators*.

And now we get the following effective result for our purpose.

2.2.11 Corollary: *In A strongly semiprime strictly left Goldie near-ring N, if every weakly essential left N-subgroup of N is also essential, then N satisfies the a.c.c. on left as well as right annihilators.*

3. Main Result

3.1. Theorem

Suppose N satisfies the acc on left annihilators and I is a nil subring of N and I is not left s-nilpotent. Then there exists a sequence $< a_i >, a_i \in N$ *such that* $Na_i \neq 0$ *and the family* $\{Na_i\}$ *is an independent family, or the sum* $Na_1 + Na_2 + \ldots$ *is direct.*

Proof: It is assumed that *I* is not s-nilpotent. Then there is an element $y \in I$ such that *y* has an infinite chain $< y_i >, y_i \in I$ with $\left(yy_1y_2....y_{k-1}y_k \neq 0, \forall k\right)$. We now consider the following

We have $y_1 \in I$ such that $yy_1 \neq 0$,

$y_2 \in I$ such that $yy_1y_2 \neq 0$

$y_3 \in I$ such that $yy_1y_2y_3 \neq 0$

……. And so on.

So we clearly have the following possibilities

There exists $x \in I$ such that $yx \neq 0$ (for example y_1 is such an element, and we may have more than one such element!)

There exists $x \in I$ *N* such that $yy_1x \neq 0$ (for example y_2 is such an element, and we may have more than one such element!)

….. etc

Thus it is possible to define a sequence y_1, y_2, ..of *N* such that

$K_1 = \{x \in I \mid yx$ has an infinite chain}

$K_2 = \{x \in I \mid yy_1x$ has an infinite chain}

$K_3 = \{x \in I \mid yy_1y_2x$ has an infinite chain}

In general

$K_n = \{x \in I \mid y_1y_2..y_{n-2}y_{n-1}yx$ has an infinite chain}

As *N* satisfies the acc on left annihilators, now we consider the maximal element $l(y_n)$ with $y_n \in K_n$.

We now claim

For each i , $l(y_i) = l(y_i.y_{i+1..}y_{i+j})$ (for all $j \geq 1$)

In particular note that , $l(y_1) = l(y_1y_2y_3)$,(i=1, j=2)

As $x \in l(y_1) \Rightarrow xy_1 = 0$, clearly $xy_1y_2y_3 = 0$ which gives

easily, $x \in l(y_1y_2y_3)$ i.e. $\Rightarrow l(y_1) \subseteq l(y_1y_2y_3)$ ----(*)

Now $y_1 \in K_1$ with $l(y_1)$ maximum

$y_2 \in K_2$ [= {$x \in I$ |yy_1x has an infinite chain}] with $l(y_2)$ maximum

$y_3 \in K_3$ [= {$x \in I$ |yy_1y_2x has an infinite chain}] with $l(y_3)$ maximum

so, $yy_1y_2y_3$ has an infinite chain. And

i.e., $x(= y_1y_2y_3)y$ i.e., xy has an infinite chain (here, $x \in N$)

so, $x \in K_1$ i.e., $y_1y_2y_3 \in K_1$

And therefore, $l(y_1y_2y_3) \subseteq l(y_1)$ –(**)

[using the maximality of $l(y_1)$]

Now * gives and ** give

$l(y_1y_2y_3) = l(y_1)$ …..(***)

We now set

$a_1 = yy_1$, $a_2 = yy_1y_2$, $a_3 = yy_1y_2y_3$, etc.

In general, $a_n = yy_1 \ldots y_{n-1}y_n$

Suppose, $a_1y_1 \neq 0$, then $a_1y_1y_2 \neq 0$, for if $a_1y_1y_2 = 0$, then $a_1 \in l(y_1y_2) \Rightarrow y_1a_1 = 0$ [as $l(y_1) = l(y_1y_2)$], hence, $a_1 y_1 = 0$.

Similarly, we have, $a_1y_1y_2y_3 \neq 0$ for if $a_1y_1y_2y_3 = 0 \Rightarrow a_1 \in l(y_1y_2y_3) = l(y_1)$

Claim $a_n y_1 = 0$

Suppose $a_n y_1 \neq 0$, for some n.

we now consider the case for any k.

$a_n\, y_1y_2 \ldots y_{k-1}y_k \neq 0$, for $l(y_1)= l(y_1y_2 \ldots y_{k-1}y_k)$

$a_n\, y_1y_2 \ldots y_{k-1}y_k = 0 \Rightarrow a_n \in l(y_1y_2 \ldots y_{k-1}y_k) = l(y_1)$

$\Rightarrow\ a_n\, y_1=0$, a contradiction

so, $a_n y_1 y_2 \ldots y_{k-1} y_k \neq 0$, i.e. $(yy_1y_2 \ldots y_n)\,(y_1y_2 \ldots y_{k-1}y_k) \neq 0$

$\Rightarrow (yy_1y_2 \ldots y_n y_1)\,(y_2y_3 \ldots y_k) \neq 0$

And this gives $y_2\, y_3 \ldots$ forms a chain for $yy_1y_2 \ldots y_ny_1 = y\,(y_1y_2 \ldots y_ny_1) = yx$, $x \in N$

$\therefore x = y_1y_2 \ldots y_n\, y_1 \in K_1 (\subseteq I)$ and since, in K_1, $l(y_1)$ is maximum,

$l(y_1 \ldots y_n \ldots y_1) \subseteq l(y_1)$ and if $\alpha \in l(y_1)$, $\alpha\, y_1=0 \Rightarrow \alpha\, y_1y_n \ldots y_1=0$

$\Rightarrow \alpha \in l(y_1y_n \ldots y_1) \Rightarrow l(y_1) \subseteq l(y_1y_n \ldots y_1) \Rightarrow l(y_1) = l(y_1y_n \ldots y_1)$ ….(α)

since, $y_1, y_2, \ldots, y_n \in I$, $y_n..y_2.y_1 \in I$,

$\because$ $y_n..y_2.y_1 \in I$ (nil), $y_1..y_2.y_n$ is nilpotent (since, $y_1..y_2.y_n \in I$-*nil*), say $(y_1..y_2.y_n)^2=0$
(note, here nilpotency of *I* is used!!)

and therefore, $(y_1y_2 \ldots y_n)\,(y_1y_2 \ldots y_n) = 0 \Rightarrow (y_1y_2 \ldots y_n)\,(y_1y_2 \ldots y_n)y_1 = 0$

$\Rightarrow (y_1y_2 \ldots y_n)\,(y_1y_2 \ldots y_ny_1) = 0 \Rightarrow (y_1y_2 \ldots y_n) \in l(y_1y_n \ldots y_1) = l(y_1)$ [by (α)]

$\Rightarrow y_1y_2 \ldots y_n\, y_1 = 0 \Rightarrow yy_1y_2 \ldots y_n\, y_1 = 0 \Rightarrow (yy_1y_2 \ldots y_n)y_1 = 0 \Rightarrow \mathbf{a_n\, y_1 = 0}$.

Similarly, for all i, $a_n\, y_i=0$, for all $n \geq i$.
Now we show that

(i) all $Na_i \neq 0$ for $1 \in N$
(ii) to show that the sum $Na_1+Na_2+\ldots$ is direct or
the family $Na_1, Na_2,$ is an independent family.

We first show that $Na_1 \cap Na_2 = 0$.

That is if $n_1a_1 = n_2a_2$ for some $n_1, n_2 \in N$, then $n_1a_1 = n_2a_2 = 0$

Now we note that $y \in I$ is such that y has an infinite sequence and choose l(y) to be maximum. And y_1y is such that y_1y has an infinite chain with $l(y_1)$ is maximum,
similarly, yy_1y_2 is such that yy_1y_2 has an infinite chain with $l(y_2)$ is maximum,

And therefore, $l(yy_1y_2) \subseteq l(y)$ …

But clearly we have, $l(y) \subseteq l(yy_1y_2)$……
Therefore, $l(y) = l(yy_1y_2)$. And hence,

$n_1a_1 = n_2\, a_2 \Rightarrow n_1a_1y_2 = n_2a_2\, y_2 = 0$ (as $a_2y_2=0$)

$\Rightarrow n_1yy_1y_2 = 0$(as $a_1=yy_1$) $\Rightarrow n_1 \in l(yy_1y_2) = l(y) \Rightarrow n_1y = 0$

$\Rightarrow n_1yy_1 = 0 \Rightarrow n_1a_1 = 0$

$\Rightarrow$ i.e. $n_1a_1 = n_2a_2 = 0$, thus $Na_1 \cap Na_2 = 0$, i.e. $Na_1 + Na_2$ is direct.

Similarly, the sum $Na_1 + Na_2 + \ldots$.is direct.
Now we'll show that

3.2. Theorem

If $\{S_i = a_j \mid j \geq i\}$ *then* $r(S_i), i = 1,2,..$ *form a strictly ascending chain of right annihilators.*

Proof: Here, $S_1 = \{a_j \mid a_j \geq 1\} = \{a_1, a_2, a_3, \ldots\}$, $S_2 = \{a_j \mid a_j \geq 2\} = \{a_2, a_3, a_4, \ldots\}$

$S_3 = \{a_j \mid a_j \geq 3\} = \{a_3, a_4, a_5, \ldots\}$ …,

$S_i = \{a_j \mid a_j \geq i\} = \{a_i, a_{i+1}, a_{i+2}, \ldots\}$

Now, $S_1x=0 \Rightarrow a_1x = a_2x = a_3x = \ldots = 0$

and this $\Rightarrow a_2x = a_3x = .. = 0$

$\Rightarrow S_2x = 0 \Rightarrow r(S_1) \subseteq r(S_2)$,

similarly, $r(S_1) \subseteq r(S_2) \subseteq r(S_3) \subseteq r(S_4)$……

Again, $a_2\, y_2=0$ $a_3\, y_2 = a_4\, y_2 = \ldots = 0$ but, $a_1\, y_2 \neq 0$ (for $yy_1y_2 \neq 0$).

Hence, $y_2 \in r(S_2)$ and $y_2 \notin r(S_1)$. And therefore, $r(S_1) \subset r(S_2)$ similarly, $r(S_2) \subset r(S_3)$, i.e.
$r(S_1) \subset r(S_2) \subset r(S_3) \subset ....$
is a strictly ascending infinite chain of left annihilators.//

III. Main result

Now we prove the main results that we are aiming for.
3.3 Theorem: *If I is not nilpotent, then I is not s-nilpotent*.
[Note: so, if *I* is s-nilpotent the *I* is nilpotent, and if *I* is nil then it is s-nilpotent]
Proof: We consider $I, I^2, I^3, \ldots$ and clearly, we have
$I \supseteq I^2 \supseteq I^3 \supseteq \ldots$ and therefore,
$l(I) \subseteq l(I^2) \subseteq l(I^3) \subseteq ....$ and
by acc on left annihilators, we have an integer , say k, such that
$l(I^k) = l(I^s)$ for all
$s \geq k$.
So if we set $K = I^k$, then $l(K) = l(K^2)$. And $K^2 \neq 0$(for if $K^2 = 0$, then *I* appears as nilpotent, which is not true.
And thus $K^2 \neq 0$, that gives an $x_1 \in K$ such that $x_1 K \neq 0$.
And this gives $x_1 K^2 \neq 0$. For if $x_1 K^2 = 0$,
then $x_1 \in l(K^2) = l(K) \Rightarrow x_1 K = 0$, a contradiction.
Now, $x_1 K^2 \neq 0 \Rightarrow x_2 \in K$ such that $x_2 x_1 K \neq 0$. And so on.

Thus we get $x_3, x_4 \ldots$ are such that each of $x_1, x_1x_2, x_1x_2x_3, \ldots$ is non zero .
Therefore, the sequence $< x_n >$ is such that
each $x_i \in I$ and $\ldots . x_k .. x_2 x_1 \neq 0$.
Hence, *I* is not s-nilpotent.//
3.4 Theorem: *If N is a strongly semiprime strictly left Goldie near-ring where every weakly essential left N-subgroup of N is also essential, then the each nil-subring of N is nilpotent.*
Proof: Let *I* be a nil subnear-ring of *N*. (to prove that *I* is nilpotent!).
Suppose, *I* is not nilpotent. Then by above *I* is not s-nilpotent.
Then we have an infinite sequence $a_1, a_2, a_3, \ldots$ in *N* such that each Na_i is non zero and their sum is direct, and the chain
$r(S_1) \subset r(S_2) \subset r(S_3) \subset ....$ is a strictly infinite ascending chain of right annihilators. Now as *N* is with the acc on right annihilators (2.2.11 Corollary) such a sequence is not possible. Thus, I must be nilpotent.

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REFERENCES

[1]. Chowdhury, K.C. : Goldie near-rings, bull. Cal. Math.soc.80(1988), no. 4,261-269
[2]. Goldie modules, IJPAM 19(7), 641-652, july 1988
[3]. Goldie M-group, Australian math. Soc.
[4]. Goldie theorem analogue for Goldie near-rings,IJPAM,20(2),141- 149,Feb,1989
[5]. A note on regular left Goldie near-ring, nat.acad. sci. letters,vol.12, no.(1989),433-435
[6]. FSD N-subgroup with acc on annihilators IJPAM 24(2),747-755,Dec1993
[7]. N-groups with acc on annihilators –some topological properties, Mathematica Pannonica,15/1,(2004) , 65-84)
[8]. Some Aspects of Artinian (Noetherian) Part of a Goldie Ring and its Topological
[9]. Relevance, Southeast Asian Bulletin of Mathematics, 32:8(2008), 43-853
[10]. Wreath Sum of Near-rings and Near-ring Groups, Southeast Asian Bulletin of Mathematics, 36(2012), 169–18
[11]. Goldie, A.W. : The structure of prime rings under ascending chain conditions, proc. London math sc.(3) 8 (1958), 589-608
[12]. Semi-prime rings with maximum conditions proc.lond. math. Ssoc. 10(1960), 201-220
[13]. Lanski, C. :Nil sub rings of Goldie rings are nilpotent.,Canadian j. math. 21(1969) 904-907
[14]. Pilz, G : Near-rings(the theory and its applications) : North Holland Publ. comp.1977