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Weighted Sharing and Uniqueness of Entire Functions Concerning Differential Polynomials

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Abstract: *Using Nevanlinna value distribution theory, we study weighted sharing and uniqueness of entire functions concerning differential polynomials and obtain some results which improve the recent results due to Waghamore and Anand[1].*
Keywords: *Differential Polynomials, Entire Functions, Uniqueness, Value Distribution Theory, Weighted Sharing.*

I. Introduction and Definitions

In this paper we use the standard definitions and notations of the value distribution theory [2]. Let f and g be two nonconstant meromorphic functions and $a \in \mathbb{C} \cup \{\infty\}$. We say that f and g share the value a CM (counting multiplicities) if the zeros of $f - a$ and $g - a$ coincide in locations and multiplicities. If we do not consider the multiplicities, we say that f and g share the value aIM(ignoring multiplicities). We say that f and g share a function$h$ CM or IM if and only if $f - h$ and $g - h$ share 0 CM or IM. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $N(r, a;\ f)$ the counting function of all the a-points of f and by $\overline{N}(r, a;\ f)$ the corresponding one for which the multiplicity is not counted. For a positive integer k, we denote by $N_k(r, a;\ f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if and only if $m \le k$ and k times if and only if $m > k$.We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ possibly outside a set of finite linear measure. If $T(r, \alpha) = S(r, f)$ for a meromorphic function$\alpha = \alpha(z)$, then $\alpha$is called small function of f.

Definition 1.1. [3]Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a;\ f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \le k$ and $k + 1$ times if $m > k$. If $E_k(a;\ f) = E_k(a;\ g)$, we say that f and g share the value a with weight k.

The definition implies that if$f$ and g share a value a with weight k then z_0 is a zero of $f - a$ with multiplicity $m(\le k)$ if and only if it is a zero of $g - a$ with multiplicity$m(\le k)$ and z_0 is a zero of $f - a$ with multiplicity m(> k) if and only if it is a zero of $g - a$ with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f and g share (a, k) to mean that f and g share the value a with weight k. Clearly if f and g share (a, k) then f and g share (a, p) for all integers$p, 0 \le p < k$. Also we note that f and g share a value a IM or CM if and only if f and g share $(a, 0)$ or (a, ∞) respectively.

If α is a small function of f and g, then f and g share α with weight k means that $f - \alpha$ and $g - \alpha$ share the value 0 with weight k.

In 1996 Fang and Hua [4] proved the following theorem:

Theorem A. [4]Let f and g be two nonconstant entire functions. Also let $n \ge 6$ be a positive integer. If $f^n f'$ and $g^n g'$ share the value 1 CM, then one of the following holds

(i) $f(z) = c_1 e^{cz}$; $g(z) = c_2 e^{-cz}$; where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$.

(ii) $f = kg$for a constant k such that $k^{n+1} = 1$.

In 2001, M. L. Fang and W. Hong [5] obtained the following result.

Theorem B. [5]Let f and g be two transcendental entire functions, $n \ge 11$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM,then$f \equiv g$.

In 2013, S. S. Bhoosnurmath and V. L. Pujari [6] obtained the following result.

Theorem C. [6]Let$f$ and g be two nonconstant entire functions, $n \ge 7$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share z CM, then $f \equiv g$.

Recently H. P. Waghamore and S. Anand [1] proved the following theorem:

Theorem D. [1]Let f and g be two nonconstant entire functions and n, m be positive integers such that $n \ge m + 6$. If $f^n(f-1)^m f'$ and $g^n(g-1)^m g'$ share z CM, then $f \equiv g$.

Since

$$f^n(f-1)^m f' = \frac{1}{n+1}(f^{n+1})'(f-1)^m$$
$$= \left[f^{n+1}\left(\frac{{}^nC_m}{n+m+1}f^m - \frac{{}^nC_{m-1}}{n+m}f^{m-1} + \cdots + \frac{{}^nC_0}{n+1}(-1)^m\right)\right]'.$$

Therefore it is natural to consider the uniqueness of meromorphic functions concerning more general kind differential polynomial, such as $[f^n L(f)]^{(k)}$, where

$$L(z) = \alpha_m z^m + \alpha_{m-1} z^{m-1} + \cdots + \alpha_0 \; \ldots \ldots \ldots (1.1)$$

and $\alpha_m \neq 0, \alpha_{m-1}, \alpha_{m-2}, \ldots, \alpha_0 \neq 0$ are complex constants.

In this paper we prove the following result

Theorem 1.1. Let f and g be two transcendental entire functions, $n \geq 1$, $m \geq 1$, $k \geq 0$, be three integers such that $n > 2k + m + 4$. If $[f^n L(f)]^{(k)}$ and $[g^n L(g)]^{(k)}$ share $(z, 2)$, where $L(z)$ is defined as in (1.1), then one of the following cases holds:

(i) $f = e^{\beta_1}$ and $g = e^{\beta_2}$; where β_1 and β_2 are nonconstant entire functions.

(ii) $f = tg$ for a constant t such that $t^p = 1$, where $p = n + m - i, \alpha_{m-i} \neq 0$ for some $i = 0, 1, \ldots,$ m.

(iii) f and g satisfy algebraic euation $Q(x_1, x_2) = 0$, where

$$Q(x_1, x_2) = x_1^n(\alpha_m x_1^m + \alpha_{m-1} x_1^{m-1} + \cdots + \alpha_0) - x_2^n(\alpha_m x_2^m + \alpha_{m-1} x_2^{m-1} + \cdots + \alpha_0)$$

II. Lemmas

In this section we present some lemmas which are required in the sequel.

Lemma 2.1. [7] Let f be a nonconstant meromorphic function and let $\alpha_l \neq 0, \alpha_{l-1}, \alpha_{l-2}, \ldots, \alpha_0$ be small functions with respect to f. Then

$$T(r, \alpha_l f^l + \alpha_{l-1} f^{l-1} + \cdots + \alpha_0) = lT(r, f) + S(r; f).$$

Lemma 2.2. [3] Let f and g be two nonconstant meromorphic functions sharing $(1, 2)$. Then one of the following cases holds:

(i) $f \equiv g$.

(ii) $T(r) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + S(r)$

(iii) $fg \equiv 1$,

where $T(r) = \max\{T(r, f), T(r, g)\}$ and $S(r) = o\{T(r)\}$.

Lemma 2.3. [8] Let f be a nonconstant meromorphic function and k be a positive integer. Also let c be a nonzero finite complex number. Then

$$T(r, f) \leq N_{k+1}(r, 0; f) + \overline{N}\left(r, 0; f^{(k)} - c\right) + \overline{N}(r, \infty; f) - N_0(r, 0; f^{k+1}) + S(r, f),$$

where $N_0(r, 0; f^{k+1})$ denotes the counting function of the zeros of $f^{(k+1)}$ which are not zeros of $f(f^{(k)} - c)$.

Lemma 2.4. [9] Let f be a nonconstant meromorphic function and p, k be two positive integers. Then

$$N_p\left(r, 0; f^{(k)}\right) \leq T\left(r, f^{(k)}\right) - T(r, f) + N_{p+k}(r, 0; f) + S(r; f),$$

and

$$N_p\left(r, 0; f^{(k)}\right) \leq N_{p+k}(r, 0; f) + k\overline{N}(r, \infty; f) + S(r; f).$$

Lemma 2.5. Let f and g be two nonconstant entire functions. Also let $F = [f^n L(f)]^{(k)}$ and $G = [g^n L(g)]^{(k)}$, where $L(z)$ is defined as in (1.1). If there exists three nonzero constants $\lambda_1, \lambda_2, \lambda_3$, such that $\lambda_1 F + \lambda_2 G = \lambda_3$ then $n \leq 2k + m + 2$.

Proof. Since f and g are entire functions therefore by Lemma 2.1, Lemma 2.3 and Lemma 2.4 we have

$$(n + m)T(r, f) \leq N_{k+1}\left(r, 0; f^n L(f)\right) + \overline{N}(r, 0; F - \frac{\lambda_3}{\lambda_1}) + S(r, f)$$

$$\leq N_{k+1}\left(r, 0; f^n L(f)\right) + \overline{N}(r, 0; G) + S(r, f)$$

$$\leq N_{k+1}\left(r, 0; f^n L(f)\right) + N_{k+1}\left(r, 0; g^n L(g)\right) + k\overline{N}\left(r, 0; g^n L(g)\right) + S(r, f) + S(r, g)$$

$$\leq N_{k+1}\left(r, 0; f^n L(f)\right) + N_{k+1}\left(r, 0; g^n L(g)\right) + S(r, f) + S(r, g)$$

$$\leq (k + m + 1)T(r, f) + (k + m + 1)T(r, g) + S(r, f) + S(r, g) \ldots \ldots \ldots \quad (2.1)$$

Similarly we have

$$(n + m)T(r, f) \leq (k + m + 1)T(r, g) + (k + m + 1)T(r, f) + S(r, f) + S(r, g) \ldots \ldots \ldots \quad (2.2)$$

From (2.1) and (2.2) we have

$$(n - 2k - 2 - m)\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g) \ldots \ldots \ldots \quad (2.3)$$

From (2.3) we get $n \leq 2k + m + 2$.

III. Proof of the Main Result

Proof of Theorem 1.1:

Let $F = f^n L(f)$, $G = g^n L(g)$, $F_1 = [f^n L(f)]^{(k)}$, $G_1 = [g^n L(g)]^{(k)}$, $F^* = \frac{F}{z}$ and $G^* = \frac{G}{z}$. Clearly F^* and G^* share $(1, 2)$ and ∞ IM. Hence by Lemma 2.2 one of the following holds:

(i) $F^* \equiv G^*$.

(ii) $T(r) \leq N_2(r, 0; F^*) + N_2(r, 0; G^*) + N_2(r, \infty; F^*) + N_2(r, \infty; G^*) + S(r)$

(iii) $F^* G^* \equiv 1$,

where $T(r) = \max\{T(r, F^*), T(r, G^*)\}$ and $S(r) = o\{T(r)\}$.

So we have to consider the following cases.

Case I: $F^* \equiv G^*$.

Integrating we have

$$[f^n L(f)]^{(k-1)} \equiv [g^n L(g)]^{(k-1)} + c_{k-1},$$

where c_{k-1} is a constant. If $c_{k-1} \neq 0$, using Lemma 2.5 it follows that $n \leq 2k + m$, a contradiction. Hence $c_{k-1} = 0$. Repeating the same process for $k - 1$ times, we get

$$f^n L(f) \equiv g^n L(g) \quad \ldots\ldots\ldots \quad (3.1)$$

From (3.1) we have

$$f^n(\alpha_m f^m + \alpha_{m-1} f^{m-1} + \cdots + \alpha_0) \equiv g^n(\alpha_m g^m + \alpha_{m-1} g^{m-1} + \cdots + \alpha_0) \ldots\ldots\ldots \quad (3.2)$$

Let $t = \frac{f}{g}$.

If t is a constant then substituting $f = tg$ in (3.2) we get

$$\alpha_m g^{n+m}(t^{n+m} - 1) + \alpha_{m-1} g^{n+m-1}(t^{n+m-1} - 1) + \cdots + \alpha_0 g^n(t^n - 1) = 0, \quad \ldots\ldots\ldots \quad (3.3)$$

which implies that $t^p = 1$, where $p = n$+m-i, $\alpha_{m-i} \neq 0$ for some $i = 0,1,2, \ldots., m$.

Hence $f \equiv tg$, for a constant t, such that $t^p = 1$, where $p = n$+m-i, $\alpha_{m-i} \neq 0$ for some $i = 0,1,2, \ldots., m$.

If t is not a constant, then by (3.3) f and g satisfy the algebraic equation $Q(x_1, x_2) = 0$, where

$$Q(x_1, x_2) = x_1^n(\alpha_m x_1^m + \alpha_{m-1} x_1^{m-1} + \cdots + \alpha_0) - x_2^n(\alpha_m x_2^m + \alpha_{m-1} x_2^{m-1} + \cdots + \alpha_0)$$

Case II: In this case we have

$$T(r) \leq N_2(r, 0; F^*) + N_2(r, 0; G^*) + N_2(r, \infty; F^*) + N_2(r, \infty; G^*) + S(r), \quad \ldots\ldots\ldots \quad (3.4)$$

where $T(r) = \max\{T(r, F^*), T(r, G^*)\}$ and $S(r) = o\{T(r)\}$. Without loss of generality, we suppose that $T(r, f) \leq T(r, g), r \in I$, where I is a set of finite measure. By Lemma 2.1 and Lemma 2.4 we get

$$N_2(r, 0; F_1) \leq T(r, F_1) - (n + m)T(r, f) + N_{2+k}(r, 0; F) + S(r, F)$$

That is

$$N_2(r, 0; F_1) \leq T(r, F_1) - T(r, F) + N_{2+k}\big(r, 0; f^n L(f)\big) + S(r, f) \ldots\ldots\ldots \quad (3.5)$$

Since f and g are transcendental using Lemma 2.1 we have from (3.4)

$$T(r, F_1) \leq N_2(r, 0; F_1) + N_2(r, 0; G_1) + N_2(r, \infty; F_1) + N_2(r, 0; G_1) + S(r, F_1) + S(r, G_1)$$
$$\leq N_2(r, 0; F_1) + N_2(r, 0; G_1) + S(r, f) \ldots\ldots\ldots \quad (3.6)$$

Using Lemma 2.4 from (3.5) and (3.6) we have

$$(n + m)T(r, f) \leq N_2(r, 0; G_1) + N_{2+k}\big(r, 0; f^n L(f)\big) + S(r, f)$$
$$\leq N_{2+k}\big(r, 0; g^n L(g)\big) + N_{2+k}\big(r, 0; f^n L(f)\big) + S(r, f)$$
$$\leq (2m + 2k + 4)T(r, f) + S(r, f),$$

which contradicts with $n > m + 2k + 4$.

Case III: $F^* G^* \equiv 1$. Thatis $[f^n L(f)]^{(k)}[g^n L(g)]^{(k)} \equiv z^2$. Suppose, if possible, that z_0 is a zero of f of order p, then z_0 must be a zero of $[f^n L(f)]^{(k)}$ of order $np - k$. Since $n > k + 2$ therefore z_0 must be a zero of z^2 with the order at least 3. This is impossible. Therefore f has no zero. Hence $f = e^{\beta_1}$, where β_1 is a nonconstant entire function. Similarly we can prove that $g = e^{\beta_2}$,where β_2 is a nonconstant entire function.

This proves the theorem.

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