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Discreteness in Product and Weak Topological Systems

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Abstract: *We established (in our theorem 3.2) a very simple but absolutely very strong and su_cient condition for a general weak topology to transmit discreteness property to its range spaces. An immediate consequence of this when applied to product topology (in _nite- or in_nite-dimensions) is that all its range spaces are discrete if a product topology is discrete. Elsewhere we also obtained the conditions for the extension of the coverse: namely, how a discrete range space may induce discreteness on a weak topology.*

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I. Introduction

It is known [1] that if all the factor spaces are discrete then the product topology is discrete for a finite dimensional topological product space $X = \prod_{i=1}^{n} X_i$. What happens to the factor spaces in terms of discreteness if the product topology is discrete? We prove that all the factor spaces are discrete if the product topology is discrete—in finite- or infinite-dimensional situations. Further, we established the conditions under which the range spaces are guaranteed to be discrete topological spaces when a weak topology, in general, is discrete.

II. Main Results|Finite Dimensional Case

Definition 2.1 *If τ is the weak topology on X generated by the family $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}$ of topological spaces, together with the family $\{f_\alpha\}_{\alpha\in\Delta}$ of functions, we shall call the triple $[(X,\tau), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ a weak topological system.*

Definition 2.2 *A product topological system is a triple $[(X,\tau), \{(X_\alpha, \tau_\alpha)\}, \{p_\alpha\}]$ of a topological product space (X,τ), a family of topological spaces $\{(X_\alpha, \tau_\alpha)\}$ which, together with the family $\{p_\alpha\}$ of projection maps, induce the product topology on X.*

Let

$$X = \prod_{i=1}^{n} X_i$$

be the Cartesian product of a finite number of topological spaces. Then sets of the form

$$A_k = \bigcap_{i=1, i\neq k}^{m} p_i^{-1}(G_i) \bigcap p_k^{-1}(\{x_k^*\}),$$

where G_i is open in X_i and the p_is are the projection maps, will be of use in the proof of our first theorem. We see that

$$A_k = \{(x_1, x_2, \cdots, x_k^*, \cdots, x_n) : x_i \in G_i, 1 \le i \le m \le n,\ i \ne k,\\ x_k = x_k^*, x_j \in X_j, j > m\} \subset X.$$

To see the fact of A more clearly, we note that for any subset (open or not) G_i of X_i, $p_i^{-1}(G_i) = \{\bar{x} \in X : p_i(\bar{x}) \in G_i\} = \{\bar{x} \in X : p_i(\bar{x}) = x_i \in G_i\} = \{\bar{x} \in X : x_i \in G_i$ and $x_j \in X_j, \forall j \ne i, 1 \le i, j \le n\} = \{\bar{x} \in X :$ the i-th coordinate of $\bar{x}$ is in the subset G_i of X_i and the other coordinates come from the other factor spaces without restriction $\}$. Hence, if there are m open sets, $G_i, 1 \le i \le m \le n$, from m factor spaces of X, the intersection of their inverse images $\bigcap\limits_{i=1}^{m} p_i^{-1}(G_i)$ is

$\bigcap\limits_{i=1}^{m} p_i^{-1}(G_i) = \{\bar{x} \in X :$ the i-th coordinate of $\bar{x}$ must come from the (open) subset G_i in $X_i, 1 \le i \le m \le n$, and the other coordinates come freely from the remaining *unaffected* factor spaces$\}$.

If we consider a singleton $\{x_k^*\}$ in X_k, then it is easy to see that

$$p_k^{-1}(\{x_k^*\}) = \{\bar{x} \in X : x_i \in X_i \text{ and } x_k = x_k^*, i \ne k\} =\\ \{\bar{x} = (x_1, x_2, \cdots, x_k^*, \cdots, x_n) : x_i \in X_i, i \ne k\}.$$

It follows from all these that

$$A_k = \bigcap_{i=1, i\ne k}^{m} p_i^{-1}(G_i) \bigcap p_k^{-1}(\{x_k^*\}) = \{\bar{x} \in X : x_i \in G_i \text{ and}\\ x_k = x_k^*, 1 \le i \le m \le n, x_j \in X_j, j > m\}, i, j \ne k,$$

where we mean by x_t the t-th coordinate of the tuple $\bar{x}$. We also observe that a fixed point $\bar{x}^*$ in X is one in which all the coordinates $x_1, x_2, \cdots, x_n$ are fixed to $x_1^*, x_2^*, \cdots, x_n^*$, respectively. So, such a fixed point may be denoted as $\bar{x}^* = (x_1^*, x_2^*, \cdots, x_n^*)$. As we have seen, to fix only one coordinate x_k^* of a tuple in X, we can take the inverse image

$$p_k^{-1}(\{x_k^*\})$$

of the singleton of x_k^* in X_k, under the k-th projection map. Therefore, a singleton $\bar{x}^*$ in X may be seen as the intersection

$$\{\bar{x}^*\} = \bigcap_{i=1}^{n} p_i^{-1}(\{x_i^*\})$$

of singletons from <u>all</u> the factor spaces, under the projection maps. This is true in finite- as well as infinite-dimensional Cartesian product sets. That is, if $X = \prod\limits_{i=1}^{\infty} X_i$ and $\bar{x}^* \in X$, then the singleton of $\bar{x}^*$ can be expressed (in this context) as

$$\{\bar{x}^*\} = \bigcap_{i=1}^{\infty} p_i^{-1}(\{x_i^*\}),$$

and this shows that the product topology on an infinite-dimensional Cartesian product set is not discrete if all the factor spaces are discrete topological spaces, because singletons in such a space can neither emerge as sub-basic sets nor basic sets.

The proof of the following theorem, which makes use of the set A described above, can best be seen as proof from first principles. It is both rigorous and cumbrous, even doubtful except that its prediction is impeccably butressed later by theorem 3.2. Theorem 2.1 represented the first idea that came to us about how to prove this lofty intuition while theorem 3.2 is the fortuitous second thought.

Theorem 2.1 *Let* $\{(X_i, \tau_i) : i = 1, \cdots, n\}$ *be a finite number of topological spaces,* $X = \prod_{i=1}^{n} X_i$ *the Cartesian product of these spaces and let* τ_p *be the product topology on* X*, induced by these spaces. If* τ_p *is the discrete topology on* X*, then each* (X_i, τ_i) *is a discrete topological space,* $i \in \{1, \cdots, n\}$.

Proof:

$$X = X_1 \times X_2 \times \cdots \times X_n = \prod_{i=1}^{n} X_i$$

be the Cartesian product of a finite family of nonempty sets, each being a topological space. Suppose *the* product topology τ_p on X is discrete. We show that each X_i, $1 \leq i \leq n$, is a discrete topological space. Suppose one of the factor spaces, say X_k $(1 \leq k \leq n)$ is not a discrete topological space. [Then $\exists x_k^* \in X_k \ni \{x_k^*\}$ is not an open set in X_k. We consider sets of the form

$$A_k = \bigcap_{i=1, i \neq k}^{m} p_i^{-1}(G_i) \bigcap p_k^{-1}(\{x_k^*\}),$$

where G_i is open in X_i and the p_is are the projection maps. We see that

$$A_k = \{(x_1, x_2, \cdots, x_k^*, \cdots, x_n) : x_i \in G_i, 1 \leq i \leq m \leq n \text{ and } i, j \neq k, x_k = x_k^*, x_j \in X_j, j > m\} \subset X.$$

Since $\{x_k^*\} \subset X_k$ is not open in the topology of X_k, $p_k^{-1}(\{x_k^*\})$ is not in the subbase for this product topology τ_p on X which, by hypothesis, is discrete. Hence, in particular, sets of the form A are not open in X which, by hypothesis, is a discrete topological space. This is a contradiction. Hence every factor space of X must be a discrete topological space if X is a discrete product topological space. (Another way to look at sets of type A is to see them as subsets of $p_k^{-1}(\{x_k^*\})$ in X. Since $p_k^{-1}(\{x_k^*\})$ is not in the sub base, not all nonempty subsets of it are in the base. Conversely, if all subsets of $p_k^{-1}(\{x_k^*\})$ are open in X, then $p_k^{-1}(\{x_k^*\})$ must also be open in X.) That is, each $X_i (1 \leq i \leq n)$ is discrete if (X, τ) is discrete.

III. Generalizations-Infinite-dimensional Case,Weak and Box Topologies

Lemma 3.1 *Any one-to-one function f_α in a weak topological system $[(X,\tau),\{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Delta},\{f_\alpha\}_{\alpha\in\Delta}]$ is an open map.*

Proof:

Let $U\in\tau$ and let $f_\alpha(U)=U_\alpha$, where f_α is a one-to-one function in the weak topological system. Then $U=f_\alpha^{-1}(U_\alpha)$. Since $U\in\tau$, it follows that $f_\alpha^{-1}(U_\alpha)\in\tau$. $\Rightarrow U_\alpha\in\tau_\alpha$. (Note: If $U_\alpha\notin\tau_\alpha$, then $f_\alpha^{-1}(U_\alpha)$ is not in τ. This is because τ is built up from, or constructed by first collecting *all* sets of the form $f_\alpha^{-1}(U_\alpha)$, where $U_\alpha\in\tau_\alpha$, and leaving out *all other subsets* of X_α which are not τ_α-open. Conversely, the only subsets of X of the form $f_\alpha^{-1}(U_\alpha)$ which are directly τ-open are those for which U_α are τ_α-open; being the subbasic sets of τ.) That is, $f_\alpha(U)=U_\alpha\in\tau_\alpha$, if f_α is one-to-one and $U\in\tau$. Hence f_α is an open map.

Theorem 3.1 *Let $[(X,\tau),\{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Delta},\{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system. If there exists an $\alpha\in\Delta$ such that (X_α,τ_α) is discrete and (for this $\alpha\in\Delta$) f_α is one-to-one, then (X,τ) is discrete.*

Proof:

Let $x\in X$ be arbitrary. Then $\{x\}$ is a singleton. We need to show that $\{x\}\in\tau$. But for some α_0, $(X_{\alpha_0},\tau_{\alpha_0})$ is discrete and f_{α_0} is one-to-one. Now, $f_{\alpha_0}(\{x\})=\{x_{\alpha_0}\}$ so that $\{x\}=f_{\alpha_0}^{-1}(\{x_{\alpha_0}\})$. Hence $\{x\}$ is a subbasic open set in τ and hence a basic open set in τ. Since $x\in X$ is arbitrary, (X,τ) is discrete.

Theorem 3.1 extends the existing converse of theorem 2.1, to situations where all the range spaces are not necessarily discrete.

Theorem 3.2 *Let $[(X,\tau),\{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Delta},\{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system. If (X,τ) is discrete, then any range space (X_α,τ_α) for which f_α is onto and an open map is discrete.*

Proof:

Let (X_α,τ_α) be a range space for which f_α is onto and an open map; and let $\{x_\alpha\}\subset X_\alpha$ be a singleton in X_α. We need to show that $\{x_\alpha\}\in\tau_\alpha$. Since $x_\alpha\in X_\alpha$ and f_α is onto, there exists an $x\in X$ such that $f_\alpha(x)=x_\alpha$. For this $x\in X$, the singleton $\{x\}$ is τ-open, since (X,τ) is discrete. It follows that $\{x_\alpha\}=f_\alpha(\{x\})\in\tau_\alpha$, as f_α is an open map.

NOTE

Since the family of functions in theorem 2.1, the projection maps, are onto and open maps, theorem 3.2 aptly generalizes theorem 2.1, and it also serves as an alternative way of proving Theorem 2.1. It (theorem 3.2) does not only generalize theorem 2.1; it also extends it to situations in which the Cartesian product set (in 2.1) is infinite-dimensional. Hence this corollary.

Corollary 3.1 *If a product topology—on a finite- or infinite-dimensional Cartesian product set—is discrete, then each of the factor spaces is a discrete topological space.*

Corollary 3.2 *Let* $[(X,\tau),\{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Delta},\{f_\alpha\}_{\alpha\in\Delta}]$ *be a weak topological system. If for some* $\alpha_0\in\Delta$, f_{α_0} *is bijective (i.e. one-to-one and onto) then* $(X_{\alpha_0},\tau_{\alpha_0})$ *is discrete if and only if* (X,τ) *is discrete.*

Definition 3.1 *Let* $\{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Delta}$ *be any family of topological spaces and let* $X=\prod_{\alpha\in\Delta}X_\alpha$ *be the Cartesian product of these sets. Let* τ_b *be the box topology on* X*, induced by the projection maps* $p_\alpha: X\to X_\alpha$.

Corollary 3.3 *If a box topology* τ_b *is discrete, then* τ_α *is discrete,* $\forall\alpha\in\Delta$.

Proof:

Suppose τ_b is discrete and that τ_{α_0} is not discrete, for some $\alpha_0\in\Delta$. Then $\exists x^*_{\alpha_0}\in X_{\alpha_0}, \ni \{x^*_{\alpha_0}\}\notin\tau_{\alpha_0}$. So $p^{-1}_{\alpha_0}(\{x^*_{\alpha_0}\})$ is not a subbasic open set in τ_b. Hence in particular $p^{-1}_{\alpha_0}(\{x^*_{\alpha_0}\})$ is not open in τ_b. Contradiction! Hence each (X_α,τ_α) is discrete if (X,τ_b) is discrete.

An alternative way to prove this actually is to directly apply theorem 3.2 since (again) the projection maps which generate the box topology are onto and open maps.

IV. Summary

The single most important exposition of this paper is theorem 3.2. Theorem 3.2 (with its very short but succint proof) is an extension and generalization of theorem 2.1; and one immediate application of theorem 3.2 is corollary 3.1. As suggested in the abstract, theorem 3.2 is today the strongest condition for a weak topology in generak to transmit the property of discreteness to a range space.

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