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Some Fixed Point Theorems for pair mapping in Complex Valued b-Metric Space

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Abstract: *In this paper, we prove some common fixed point results for pair of rational type of contractive mappings in the setting of complex valued b-metric spaces. Our results extend, generalize and improve the corresponding results of A. K .Dubey' [48].*
Key word: *Complex valued b-metric space; common fixed point; Rational expression type of contractive mappings.*

I. Introduction.

Fixed point theorem general known as the Banach contraction mapping theorem. Banach Contraction principle [4] is a basic result in fixed point theory. Later, a larger number of articles have been devoted to the improvement and generalization of the Banach Contraction Principle by using different form of contraction condition in various spacesIn this theory contraction is one of main tools to prove the existence and uniqueness of a fixed point. Banach Contraction Principle which give an answer on existence and uniqueness of a solution of an operator equation

$$Tx = x.$$

This equation is the most widely used for fixed point theorem in all analysis. This principle is contractive in nature and is one of the most useful tools in the study a non-linear equation. There are a lot of generalizations of the Banach Contraction Mapping Principle in the iterative.

The Banach Contractive Principle was used to establish the existence of a unique solution for a nonlinear integral equation [4]. There are many generalizations of the Banach Contraction Principle particularly in the metric space see for instance([1][2][3],[7][12][16],[19][20][21],[22][23][24]). There are a lot of generalization of this principle has been obtained in several directions.

In 1989, Bakhtin [25] introduce the concept of b-metric space. In 1993, Czerwik [26] online the results of b-metric space which is a generalized the famous Banach contractive Principle in metric space. Using this idea researcher presented generalization of the renowned Banach Fixed Point Theorem in the b-metric space for ([27],[28][29]). Many authers have studied the extension of Fixed Point Theorem in b-metric space, see for instance ([30],[31,32,33],[34],[35],[36],[37][38],[39],[40,41][42][43],[44],[45]&[46]&[47]).

In 1978, Feisher and khan [9] generalized the Banach Contraction Principle with rational expression and proved some fixed and common fixed point theorems.

Recently Azam et al. [2] introduced the notation of complex valued metric space and proved some common fixed point theorems for mapping satisfying rational inequality which are not meaningful in cone metric space.

In the same way, various authers have studied and prove the fixed point result for mapping satisfying different type contractive conditions in the framework of complex valued metric space (see [15] ,[8] , [5] ,[18],[17]). In 2013 Rao et al [13] introduce the concept of complex valued b- metric space which was more general then the well known complex valued metric space. In sequal A.A. Mukheimer [11] obtained common fixed point result satisfying certain rational expression in complex valued b-metric space. In sequel A.K. Dubey, etal. (2015) obtain fixed point results for the mapping satisfied rational expression in complex valued b-metric space. The main purpose of this paper is the present common fixed point results of two self mapping satisfying rational expression in complex valued b-metric spaces our results in this paper are generalization of work done. A.K. Dubey, et al in [48].

II. Preliminaries

Definition2.1 (see [30]) Let X be a non empty set and let $s \geq 1$ be a given real number .A function $d: X \times X \rightarrow [0, \infty)$ is called b-metric if for all $x, y, z \in X$ .The following condition are satisfied.
$(i) d(x, y) = 0 \, iff \, x = y$
$(ii) d(x, y) = d(y, x)$

$(iii) d(x,y) \leq S[d(x,z) + d(z,y)]$.
The pair (x,d) is called a b-metric space. The numbers ≥ 1 is called the coefficient of (X,d).
Example 2.1.1(see [10]) Let $X = \{-1,0,1\}$.Define mapping $d: X \times X \to R^+$ by

$$d(x,y) = d(y,x) \ \forall x,y \in X .$$

\ $$d(x,x) = 0, \ d(-1,0) = 4, \ d(-1,1) = 1, \quad d(0,1) = 2$$

Then $d: X \times X \to R$ or (X,d) is b-metric space, but not metric space, since the triangle inequality is not satisfied indeed.

$$d(-1,0) + d(0,1) = 4 + 2 = 6 > 1 = d(-1,1)$$

and $$d(-1,1) + d(-1,0) = 1 + 4 = 5 > 2 = d(0,1)$$

are both true but

$$d(-1,1) + d(0,1) = 1 + 2 = 3 < 4 = d(-1,1)$$

is not true. So (X,d) is b-metric space with S=4/3.
Example 2.1.2 (see [10]) Let $X = [0,1]$ and $d: X \times X \to [0,\infty]$ be defined by

$d(x,y) = (x-y)^2 \ \forall x,y \in X$. Clearly (X,d) is b-metric space.

Example2.1.3 (see [11]) Let (X,d) be a metric space and $d^*(x,y) = (d(x,y))^p$ with $p \geq 1$ is a real number. Then (X,d^*) is a b-metric space with $S = 2^{p-1}$.

An ordinary metric d is a real valued function from a set $X \times X$ into R where X is non empty set that is $d: X \times X \to R$. A complex number $z \in \mathbb{C}$ is an ordered pair of real numbers where first co-ordinate is called $Re(z)$ and second co-ordinate is called $Im(z)$. Thus a complex valued metric space d is a function from a set $X \times X$ into $\mathbb{C}$, where X is the non-empty set and $\mathbb{C}$ is the set of complex number .
That is $d: X \times X \to \mathbb{C}$ .Let $z_1, z_2 \in \mathbb{C}$ difine a partial order $\leq$ on $\mathbb{C}$ as follows
$z_1 \leq z_2$ iff $Re(z_1) \leq Re(z_2)$, $Im(z_{1)} \leq Im(z_2)$.
It follow that $z_1 \leq z_2$ if one of the following condition are satisfied:
$(i) Re(z_1) = Re(z_2) \quad and \ Im(z_1) < Im(z_2)$
$(ii) Re(z_1) < Re(z_2) \quad and \ Im(z_1) = Im(z_2)$
$(iii) Re(z_1) < Re(z_2) \quad and \ Im(z_1) < Im(z_2)$
$(iv) Re(z_1) = Re(z_2) \quad and \ Im(z_1) = Im(z_2)$
In (i),(ii),(iii) we have $|z_1| \leq |z_2|$. In (iv), we have $|z_1| = |z_2|$. So $|z_1| \leq |z_2|$ In particular $|z_1| \lneqq |z_2|$ if $z_1 \neq z_2$ and one of (i), (ii), (iii) is satisfy. In this case $|z_1| < |z_2|$. We will write $z_1 < z_2$ iff (iii) satisfy. Further

$$0 \leq z_1 \lneqq z_2 \Rightarrow |z_1| \leq |z_2|.$$

$$z_1 \leq z_2 \ and \ z_2 < z_3 \Rightarrow z_1 < z_3 .$$

Definition 2.2 (see [2]) Let x be a nonempty set. Suppose that the mapping $d: X \times X \to \mathbb{C}$ satisfies the following conditions:
$(i) d(x,y) \geq 0 \ and \ d(x,y) = 0 \Leftrightarrow x = y.$
$(ii) d(x,y) = d(y,x)$ $(symmetic)$
$(iii) d(x,y) \leq d(x,z) + d(z,y)$ $(the\ trinagle\ inequalities)$.
Then d is called a complex valued metric on X and (X,d) is called a complex valued metric space.
Example 2.2.1(see [8]) Let $X = \mathbb{C}$ Define the mapping $d: X \times X \to C$ by

$d(x,y) = i|x-y|$, $\forall x,y \in X$. Then (X,d) is complex valued metric space.

Example 2.2.2 (see [19]) Let $X = \mathbb{C}$ Define the mapping $d: X \times X \to \mathbb{C}$ by

$$d(x,y) = e^{ik}|x-y| \ where \ k \in R \ and, \forall x,y \in X ,$$

Where $k \in \left[0, \frac{\pi}{2}\right]$, $\forall x,y \in X$ .Then (X,d) is called a complex valued metric space.
Definition 2.3 (see [4]) Let X be a nonempty set and mapping $d: X \times X \to \mathbb{C}$ satisfy the following conditions:
$(i) 0 \leq d(x,y) \ and \ d(x,y) = 0 \ iff \ x = y \quad \forall x,y \in X$.
$(ii) \ d(x,y) = d(y,x)$
$(iii) d(x,y) \leq S[d((x,z) + d(z,y)]$.
Where $s \geq 1$ is a real number. Then d is called complex valued metric space and (X,d) is called complex valued b-metric space.
Example2.3.1 (see [13]) Let $X = [0,1]$. Define a complex valued metric $d: X \times X \to \mathbb{C}$ by

$$d(x,y) = |x-y|^2 + i|x-y|^2 , \forall x,y \in X$$

Then (X,d) is a complex valued b-metric space with S=2.
Remark 2.4: If S=1, then the complex valued b-metric space always reduces to a complex valued metric space. Thus every complex valued metric space is a complex valued b-metric space, but not conversely. This generalizes the notation of a complex valued b-metric space over complex valued metric space.
Definition 2.5 (see [13]) Let (X,d) be a complex valued b-metric space consider the following:

(i) A point $x \in X$ is called interior point of a set $A \subset X$ whenever point. There exists $0 < r \in C$ such that

$$B(x,r) = \{y \in X: d(x,y) < r\} \subseteq A.$$

(ii) A point $x \in X$ is called a limit point of a set A whenever there exists for every $0 < r \in C$,

$$B(x,r) \cap (A - X) = \emptyset.$$

(iii) A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A.

(iv) A subset $A \subseteq X$ is called closed whenever each element of a A belong to A.

(v) A subbasis for a Housdroff topology τ on X is a family

$$F = \{B(x,r): x \in X \ and \ 0 < r\}.$$

Definition 2.6 (see [13]) Let (X,d) be a complex valued b-metric space and $\{x_n\}$ a sequence in X and $x \in X$ consider the following:

(i) If for every $c \in C$,with $0 < r$,there is $n \in N$ such that for all $d(x_n, x) < c$. Then $\{x_n\}$ is said to be convergent , $\{x_n\}$ coverges to x and x is the limit point of $\{x_n\}$ .We donote this by $\lim_{n\to\infty} x_n = x \ \ or \{x_n\} \to x \ as \ n \to \infty$

(ii)If for every $c \in \mathbb{C}$ with $0 < r$, there is $n > N \ d(x_n, x_{n+m}) < r$. where $m \in N$. Then $\{x_n\}$ is said to be Cauchy sequence.

(iii)If every Cuachy sequence in X is convergent, then (X,d) is said to be a complete complex valued b-metric space.

Lemma 2.7(see [13]): let (X,d) be a complex valued b-metric space and Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and if only $\left|d(x_{n,}x\right| \to 0$ as $n \to \infty$.

Lemma 2.8 (see [13]): let (X,d) be a complex valued b-metric space and Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and if only $\left|d(x_{n,}x_{n+m})\right| \to 0$ as $n \to \infty$,where $m \in N$.

III. Main Results

The following results are generalizations of theorem 7 and 9 of A. K. Dubey [48].

Theorem 3.1 Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and $T_{1,}T_2: X \to X$ be a mapping satisfying the condition:

$$d\left(T_{1,}x, T_2\, y\right) \leq \frac{\lambda\, d^2(x,y)}{1+d(x,y)} + \mu\, d(y, T_{2,}y) \qquad (1)$$

for all $\forall x, y \in X$,where λ , μ are nonnegative reals with $s\lambda + \mu < 1$.Then T_1 and T_2have a unique common fixed point in X.

Proof. For any arbitrary point, $x_n \in X$. Define sequence $\{x_n\}$ in Xsuch that

$$x_{2n+1} = T_1 x_{2n} \text{ for } n = \{0,1,2,3,\dots\} \qquad (2)$$

$$x_{2n+2} = T_2 x_{2n+1} \text{ for } n = \{0,1,2,3,\dots\} \qquad (3)$$

Now, we show that the sequence $\{x_n\}$ is Cauchy : Let $x = x_{2n}$ & $y = x_{2n+1}$ in (1) we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(T_1 x_{2n}, T_2 x_{2n+1}) \\ &\leq \frac{\lambda\, d^2(x_{2n}, x_{2n+1})}{1+d(x_{2n}, x_{2n+1})} + \mu\, d(x_{2n+1}, T_2 x_{2n+1}) \\ &= \frac{\lambda\, d^2(x_{2n}, x_{2n+1})}{1+d(x_{2n}, x_{2n+1})} + \mu\, d(x_{2n+1}, x_{2n+2}) \qquad (4) \end{aligned}$$

Which implies that$|d(x_{2n+1}, x_{2n+2})| \ \leq \lambda \frac{|d(x_{2n}, x_{2n+1})|}{|1+d(x_{2n}, x_{2n+1})|} |d(x_{2n}, x_{2n+1})| + \mu |d(x_{2n+1}, x_{2n+2})|$ (5)

Since $|1 + d(x_{2n}, x_{2n+1})| > |d(x_{2n}, x_{2n+1})|$, we get

$$|d(x_{2n+1}, x_{2n+2})| \leq \lambda |d(x_{2n}, x_{2n+1})| + \mu |d(x_{2n+1}, x_{2n+2})| \qquad (6)$$

and hence

$$|d(x_{2n+1}, x_{2n+2})| \leq \frac{\lambda}{1-\mu} |d(x_{2n}, x_{2n+1})| \qquad (7)$$

Similarly, we obtain

$$|d(x_{2n+2}, x_{2n+3})| \leq \frac{\lambda}{1-\mu} |d(x_{2n+1}, x_{2n+2})|. \qquad (8)$$

Since $s\lambda + \mu < 1$ and $s \geq 1$, we get $\lambda + \mu < 1$.

Therefore, with $\delta = \frac{\lambda}{1-\mu} < 1$ and for all $n \geq 0$. and consequently, we have

$$|d(x_{2n+1}, x_{2n+2})| \leq \delta |d(x_{2n}, x_{2n+1})| \leq \delta^2 |d(x_{2n-1}, x_{2n})| \leq \cdots \leq \delta^{2n+1} |d(x_0, x_1)|$$

$$|d(x_{2n+1}, x_{2n+2})| \leq \delta |d(x_{2n}, x_{2n+1})| \leq \delta^2 |d(x_{2n-1}, x_{2n})| \leq \cdots \leq \delta^{2n+1} |d(x_0, x_1)| \qquad (9)$$

Thus for any $m > n, m, n \in \mathbb{N}$ and since $s\delta = \frac{s\lambda}{1-\mu} < 1$, we get

$$\begin{aligned} |d(x_{2n}, x_{2m})| \ &\leq s|d(x_{2n}, x_{2n+1})| + s|d(x_{2n+1}, x_{2m})| \\ &\leq s|d(x_{2n}, x_{2n+1})| + s^2|d(x_{2n+1}, x_{2n+2})| + s^2|d(x_{2n+2}, x_{2m})| \\ &\leq s|d(x_{2n}, x_{2n+1})| + s^2|d(x_{2n+1}, x_{2n+2})| + s^3|d(x_{2n+2}, x_{2n+3})| \\ &\quad + s^3|d(x_{2n+3}, x_{2m})| \end{aligned}$$

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$$|d(x_{2n}, x_{2m})| \leq s|d(x_{2n}, x_{2n+1})| + s^2|d(x_{2n+1}, x_{2n+2})| + s^3|d(x_{2n+2}, x_{2n+3})| + \cdots$$
$$\ldots . +s^{2m-2n-1}|d(x_{2m-2}, x_{2m-1})| + s^{2m-2n}|d(x_{2m-1}, x_{2m})| \quad (10)$$

By using (9), we get

$$|d(x_{2n}, x_{2m})| \leq s\delta^{2n}|d(x_0, x_1)| + s^2\delta^{2n+1}|d(x_0, x_1)| + s^3\delta^{2n+2}|d(x_0, x_1)| + \cdots$$
$$\ldots + s^{2m-2n-1}\delta^{2m-2}|d(x_0, x_1)| + s^{2m-2n}\delta^{2m-1}|d(x_0, x_1)|$$
$$= \sum_{i=1}^{2m-2n} s^i \delta^{i+2n-1} |d(x_0, x_1)|. \quad (11)$$

Therefore,

$$|d(x_{2n}, x_{2m})| = \sum_{i=1}^{2m-2n} s^{i+2n-1}\delta^{i+2n-1} |d(x_0, x_1)|$$
$$= \sum_{t=n}^{2m-1} s^t\delta^t |d(x_0, x_1)| \quad (12)$$
$$\leq \sum_{t=2n}^{\infty} (s\delta)^t |d(x_0, x_1)|$$
$$= \frac{(s\delta)^{2n}}{1-(s\delta)} |d(x_0, x_1)|$$

and hence

$$|d(x_{2n}, x_{2m})| \leq \frac{(s\delta)^{2n}}{1-(s\delta)} |d(x_0, x_1)| \to 0 \; as \; n, m \to \infty \quad (13)$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists some $u \in X$ such that $x_n \to u$ as $n \to \infty$. Suppose that is not possible; then there exists $z \in X$ such that

$$|d(u, T_1u)| = |z| > 0 \quad (14)$$

Now,

$$z = d(u, Tu) \leq sd(u, x_{2n+2}) + sd(x_{2n+2}, T_1u)$$
$$= sd(u, x_{2n+2}) + sd(T_2x_{2n+2}, T_1u) \quad (15)$$
$$\leq sd(u, x_{2n+2}) + \frac{s\lambda d^2(x_{2n+1}, u)}{1+d(x_{2n+1}, u)} + s\mu d(u, T_1u)$$

which implies that

$$|z| = |d(u, T_1u)| \leq s|d(u, x_{2n+2})| + \frac{s\lambda|d^2(x_{2n+1}, u)|}{1+|d(x_{2n+1}, u)|} + s\mu|d(u, T_1u)|. \quad (16)$$

Taking the limit of (16) as $n \to \infty$, we obtain that $|z| = |d(u, T_1u)| \leq 0$, a contradiction with (14).
So $|z| = 0$. Hence $T_1u = u$.
Similarly, we can show that $T_1u = u$.
Now show that T_1 and T_2 have unique common fixed point of T_1 and T_2. To show this, assume that u^* is another fixed point of T_1 and T_2 .Then,

$$d(u, u^*) = d(T_1 u, T_2 u^*) \leq \frac{\lambda d^2(u, u^*)}{1+d(u, u^*)} + \mu d(u^*, T_2u^*) \quad (17)$$

So

$$|d(u, u^*)| \leq \lambda \frac{|d^2(u, u^*)|}{|1+d(u, u^*)|} |d(u, u^*)| + \mu|d(u^*, T_2u^*)| \quad (18)$$

Since

$$|1 + d(u, u^*)| > |d(u, u^*)| \quad (19)$$

Therefore

$$|d(u, u^*)| < \lambda|d(u, u^*)| + \mu|d(u^*, u^*)| \quad (20)$$
$$= \lambda|d(u, u^*)| , \text{ a contradiction.}$$

So, $u = u^*$, which proves the uniqueness of fixed point in X. This completes the proof.

Theorem 3.2 Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and $T_1, T_2: X \to X$ be a mapping satisfying

$$d(T_1x, T_2y) \leq \lambda d(x, y) + \frac{\mu d(x, T_1x) d(y, T_2y)}{d(x, T_2y) + d(y, T_1x) + d(x, y)}, \quad (21)$$

for all $x, y \in X$ such that $x \neq y$, $d(x, T_2y) + d(y, T_1x) + d(x, y) \neq 0$, where λ, μ are nonnegative reals with $s\lambda + \mu < 1$ or $d(T_1x, T_2y) = 0$ if $d(x, T_2y) + d(y, T_1x) + d(x, y) = 0$. Then T_1& T_2 have a unique common fixed point in X.

Proof. For any arbitrary point, $x_0 \in X$.Define sequence $\{x_n\}$ in X such that

$$x_{2n+1} = T_1x_{2n} \quad \text{for } n = (0,1,2,3 \ldots .) \quad (22)$$
$$x_{2n+2} = T_2x_{2n+1} \quad \text{for } n = (0,1,2,3 \ldots .) \quad (23)$$

Now, we show that the sequence $\{x_n\}$ is Cauchy: Let $x = x_{2n}$ & $y = x_{2n+1}$ in (21) we have

$$d(x_{2n+1}, x_{2n+2}) = d(T_1x_{2n}, T_2x_{2n+1})$$
$$\leq \lambda(x_{2n}, x_{2n+1}) + \frac{\mu d(x_{2n}, T_1x_{2n}) d(x_{2n+1}, T_2x_{2n+1})}{d(x_{2n}, T_2x_{2n+1}) + d(x_{2n+1}, T_1x_{2n}) + d(x_{2n}, x_{2n+1})}$$
$$= \lambda(x_{2n}, x_{2n+1}) + \frac{\mu d(x_{2n}, x_{2n+1}) d(x_{2n+1}, x_{2n+2})}{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1}) + d(x_{2n}, x_{2n+1})} \quad (24)$$

which implies that

$$|d(x_{2n+1}, x_{2n+2})| \leq \lambda|d(x_{2n}, x_{2n+1})| + \frac{\mu|d(x_{2n+1},x_{2n+2})|}{|d(x_{2n},x_{2n+2})|+|d(x_{2n},x_{2n+1})|}|d(x_{2n}, x_{2n+1})|, \quad (25)$$

since

$$|d(x_{2n+1}, x_{2n+2})| \leq |d(x_{2n+1}, x_{2n})| + |d(x_{2n}, x_{2n+2})|. \quad (26)$$

Therefore

$$|d(x_{2n+1}, x_{2n+2})| \leq \lambda|d(x_{2n}, x_{2n+1})| + \mu|d(x_{2n}, x_{2n+1})|$$
$$= (\lambda+\mu)|d(x_{2n}, x_{2n+1})| \quad (27)$$

Similarly, we obtain

$$|d(x_{2n+2}, x_{2n+3})| \leq (\lambda+\mu)|d(x_{2n+1}, x_{2n+2})| \quad (28)$$

Since $s\lambda + \mu < 1$ and $s \geq 1$, we get $\lambda + \mu < 1$.

Therefore, with $\delta = \lambda + \mu < 1$ and for all $n \geq 0$ and consequently, we have

$$|d(x_{2n+1}, x_{2n+2})| \leq \delta|d(x_{2n}, x_{2n+1})| \leq \delta^2|d(x_{2n-1}, x_{2n})| \leq \cdots \leq \delta^{2n+1}|d(x_0, x_1)| \quad (29)$$

$$|d(x_{2n+1}, x_{2n+2})| \leq \delta|d(x_{2n}, x_{2n+1})| \leq \delta^2|d(x_{2n-1}, x_{2n})| \leq \cdots \leq \delta^{2n+1}|d(x_0, x_1)|$$

Thus, for any $m > n$, $m, n \in \mathbb{N}$, we have

$$|d(x_{2n}, x_{2m})| \leq s|d(x_{2n}, x_{2n+1})| + s|d(x_{2n+1}, x_{2m})|$$
$$\leq s|d(x_{2n}, x_{2n+1})| + s^2|d(x_{2n+1}, x_{2n+2})| + s^2|d(x_{2n+2}, x_{2m})|$$
$$\leq s|d(x_{2n}, x_{2n+1})| + s^2|d(x_{2n+1}, x_{2n+2})| + s^3|d(x_{2n+2}, x_{2n+3})| + s^3|d(x_{2n+3}, x_{2m})| \quad (30)$$

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$$|d(x_{2n}, x_{2m})| \leq s|d(x_{2n}, x_{2n+1})| + s^2|d(x_{2n+1}, x_{2n+2})| + s^3|d(x_{2n+2}, x_{2n+3})| + \cdots$$
$$\ldots + s^{2m-2n-1}|d(x_{2m-2}, x_{2m-1})| + s^{2m-2n}|d(x_{2m-1}, x_{2m})|$$

By using (29), we get

$$|d(x_{2n}, x_{2m})| \leq s\delta^{2n}|d(x_0, x_1)| + s^2\delta^{2n+1}|d(x_0, x_1)| + s^3\delta^{2n+2}|d(x_0, x_1)| + \cdots$$
$$\ldots\ldots + s^{2m-2n-1}\delta^{2m-2}|d(x_0, x_1)| + s^{2m-2n}\delta^{2m-1}|d(x_0, x_1)|$$
$$= \sum_{i=1}^{2m-2n} s^i\delta^{i+2n-1}|d(x_0, x_1)|. \quad (31)$$

Therefore

$$|d(x_{2n}, x_{2m})| = \sum_{i=1}^{2m-2n} s^{i+2n-1}\delta^{i+2n-1}|d(x_0, x_1)|$$
$$= \sum_{t=2n}^{2m-1} s^t\delta^t|d(x_0, x_1)| \quad (32)$$
$$\leq \sum_{t=2n}^{\infty}(s\delta)^t|d(x_0, x_1)|$$
$$= \frac{(s\delta)^{2n}}{1-(s\delta)}|d(x_0, x_1)|$$

and hence

$$|d(x_{2n}, x_{2m})| \leq \frac{(s\delta)^{2n}}{1-(s\delta)}|d(x_0, x_1)| \to 0 \ as\ n, m \to \infty \quad (33)$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists some $u \in X$ such that $x_{2n} \to u$ as $n \to \infty$. Suppose that is not possible; then there exists $z \in X$ such that

$$|d(u, T_1u)| = |z| > 0 \quad (34)$$

So by using the triangular inequality and (21), we get

$$z = d(u, T_1u)$$
$$\leq sd(u, x_{2n+2}) + sd(x_{2n+2}, Tu)$$
$$= sd(u, x_{2n+2}) + sd(T_2x_{2n+1}, T_1u)$$
$$\leq sd(u, x_{2n+2}) + s\lambda d(T_2x_{2n+1}, u) + \frac{s\mu d(x_{2n+1}, T_1x_{2n+1})d(u, T_2u)}{d(x_{2n+1}, T_2u) + d(u, T_1x_{2n+1}) + d(x_{2n+1}, u)} \quad (35)$$
$$= sd(u, x_{2n+2}) + s\lambda d(x_{2n+1}, u) + \frac{s\mu d(x_{2n+1}, x_{2n+2})d(u, T_2u)}{d(x_{2n+1}, T_2u) + d(u, x_{2n+2}) + d(x_{2n+1}, u)}$$

which implies that

$$|z| = |d(u, T_1u)|$$
$$\leq s|d(u, x_{2n+2})| + s\lambda|d(x_{2n+1}, u)| + \frac{s\mu|d(x_{2n+1}, x_{2n+2})||d(u, T_1u)|}{|d(x_{2n+1}, T_1u)| + |d(u, x_{2n+2})| + |d(x_{2n+1}, u)|} \quad (36)$$

Taking the limit of (36) as $n \to \infty$, we obtain that $|z| = |d(u, T_1u)| \leq 0$, a contradiction with (34).

So $|z| = 0$. Hence $T_1u = u$.

Similarly $T_2u = u$.

Now show that T_1 and T_2 have unique common fixed point of T_1 and T_2.To show that u^* is another fixed point of T_1 and T_2 .Then,

$$d(u, u^*) = d(T_1u, T_2u^*)$$
$$\leq \lambda d(u, u^*) + \frac{\mu d(u, T_1u)d(u^*, T_2u^*)}{d(u, T_2u^*) + d(u^*, T_1u) + d(u, u^*)} \quad (37)$$

so that

$$|d(u, u^*)| \leq \lambda|\ d(u, u^*)| + \frac{\mu|d(u, T_1u)||d(u^*, T_2u^*)|}{|d(u, T_2u^*)| + |d(u^*, T_1u)| + |d(u, u^*)|} \quad (38)$$

$< \lambda |d(u,u^*)|$, a contradiction.

So $u = u^*$, which proves the uniqueness of fixed point in X. This complete the proof

Now, we consider the following case: $d(x_{2n}, T_2\, x_{2n+1}) + d(x_{2n+1}, T_1\, x_{2n}) + d(x_{2n}, x_{2n+1}) = 0$ (for any n) implies $d(T_1\, x_{2n}, T_2\, x_{2n+1}) = 0$ so that $x_{2n} = T_1\, x_{2n} = x_{2n+1} = T_2\, x_{2n+1} = x_{2n+2}$. Thus we have $x_{2n+1} = T_1\, x_{2n} = x_{2n}$, so there exists $K_1$and l_1 such that $K_1 = T_1 l_1 = l_1$. Using foregoing arguments, one can also show that there exists $K_2$and l_2 such that $K_2 = T_2 l_2 = l_2$. As $d(l_1, T_2 l_2) + d(l_2, T_1\, l_1) + d(l_1, l_2) = 0$ (due to definition) implies $d(T_1 l_1, T_2 l_2) = 0, K_1 = T_1\, l_1 = T_2 l_2 = K_2$, which in turn yields that $K_1 = T_1\, l_1 = T_1\, K_1$. Similarly, one can also have $K_2 = T_2\, K_2$. As $K_1 = K_2$ implies $T_1\, K_1 = K_1$, therefore $K_1 = K_2$ is fixed point of T.

We now prove that $T_1$and T_2 have unique common fixed point. For thus, assume that K_1^*in X is another fixed point of T. Then we have$T_2 K_1^* = K_1^*$. As $d(K_1, T_2 K_1^*) + d(K_1^*, T_1 K_1) + d(K_1, K_1^*) = 0$, therefore $d(K_1, K_1^*) = d(T_1 K_1, T_2 K_1^*) = 0$.

This implies that $K_1 = K_1^*$ which proves the uniqueness of common fixed point in$X$. This completes the proof of the theorem.

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