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On A Series the Complex Functions for Hardy - Sobolev Spaces with An applications

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Abstract: *We show the Concept of a Series on a Hardy-Sobolev space and give its atomic decomposition. As an application of a Seriesfunctions we shown a div-curl lemma.*

I. Introduction and Preliminaries

From [13], the Hardy space $H^1(\mathbb{C}^n)$is the space of locally integrable series functions f_r for which $H^1(\mathbb{C}^n)$

$$\sum_r M(f_r)(x) = \sup_{t>0} \sum_r |(\psi_r)_t * (f_r)(x)|$$

belongs to $L^1(\mathbb{C}^n)$, where $\psi_r \epsilon D(\mathbb{C}^n)$,
$(\psi_r)_t(x) = \frac{1}{t^n}\psi_r\left(\frac{x}{t}\right), t > 0, \int_{\mathbb{C}^n} \psi_r(x)dx = 1, supp\,\psi_r \subset B(0,1)$, a ball centered at the origin with radius 1. The norm of $H^1(\mathbb{C}^n)$is definedby

$$\sum_{r=0}^{\infty} H^1 \|f_r\|_{H^1(\mathbb{C}^n)} = \sum_{r=0}^{\infty} \|M(f_r)\|_{L^1(\mathbb{C}^n)}$$

Among many characterizations of Hardy spaces, the atomic decomposition is an important one. An $L^2(\mathbb{C}^n)$ a seriesfunctions a_r is an $L^1(\mathbb{C}^n)$ -atom if there exists a balln $B = B_{a_r}$ in $\mathbb{C}^n$ satisfying:
(1) $supp\, a_r \subset B$.
(2) $\sum_r \|a_r\|_{L^2(B)} \le |B|^{-1/2}$;
(3)$\sum_r \int_B a_r(x)\, dx = 0$
The basic result about atoms is the following atomic decomposition theorem (see [3] and [9,13]): A series function f_r on $\mathbb{C}^n$ belongs to $L^1(\mathbb{C}^n)$if and only if f_r has a decomposition

$$\sum_r^{\infty} f_r = \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \lambda_k\, (a_r)_k$$

where the $(a_r)_k$'s are $H^1(\mathbb{C}^n)$ -atoms and

$$\sum_{k=0}^{\infty} |\lambda_k| \le C \sum_{r=0}^{\infty} \|f_r\|_{H^1(\mathbb{C}^n)}$$

The tent space $\mathcal{N}^{\varepsilon-1}(\mathbb{C}^{n+1})(\varepsilon > 0)$ is the space of all measurable series functions Fon $\mathbb{C}_+^{n+1}$for which $S(F_s)\epsilon L^{\varepsilon-1}(\mathbb{C}^n)$, where $S(F_s)$is the square functions definedby

$$\sum_{s=0}^{\infty} S(F)(x) = \sum_{s=0}^{\infty} \left(\int_{\Gamma(x)} |F_s(y,t)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}$$

$\Gamma(x) = \{(y,t)\epsilon\ \mathbb{C}_+^{n+1}: |y-x| < t\}$is the cone whose vertex at $x\epsilon\ \mathbb{C}^n$. The norm of $F_s \epsilon \mathcal{N}^{\varepsilon-1}(\mathbb{C}_+^{n+1})$ is defined by

$$\sum_{s=0}^{\infty} \|F_s\|_{\mathcal{N}^{\varepsilon+1}(\mathbb{C}_+^{n+1})} = \sum_{s=0}^{\infty} \|S(F_s)\|_{L^{\varepsilon+1}(\mathbb{C}^n)}$$

An $\mathcal{N}^{\varepsilon-1}(\mathbb{C}_+^{n+1})$-atom is a series function $\alpha_r$supported in a tent $T(B) = \{(x,t)\epsilon\mathbb{C}_+^{n+1}: |x-x_0| \le \delta - t\} = (x,t)\} \in \mathbb{C}^n$, for which

$$\int_{T(B)} \sum_{r=0}^{\infty} |\alpha_r(x,t)|^2 \frac{dxdt}{t} \le |B|^{\frac{\varepsilon-3}{\varepsilon-1}}$$

In [5,13], Coifman, Meyer and Stein showed the following atomic decomposition theorem: any $F\epsilon\ \mathcal{N}^{\varepsilon-1}(\mathbb{C}_+^{n+1})$can be written as,

$$\sum_{s=0}^{\infty} F_s = \sum_{r=0}^{\infty}\sum_{k=0}^{\infty} \lambda_k\,(\alpha_r)_k$$

where the $(\alpha_r)_k$ are $\mathcal{N}^{\varepsilon-1}(\mathbb{C}_+^{n+1})$-atoms and

$$\sum_{k=0}^{\infty}|\lambda_k| \le C\sum_{s=0}^{\infty}\|F_s\|_{\mathcal{N}^{\varepsilon-1}(\mathbb{C}_+^{n+1})}$$

Let $D'(\mathbb{C}^n)$ denote the dual of $\mathrm{D}D(\mathbb{C}^n)$, often called the space of distributions.
For $f\epsilon\ D'(\mathbb{C}^n)$, its gradient is defined, in the sense of distributions, by

$$\sum_{r=0}^{\infty}\langle \nabla f_r,\varphi_r\rangle = -\int_{\mathbb{C}^n}\sum_{r=0}^{\infty} f_r\, div\ \varphi_r\, dx$$

for all $\varphi_r\epsilon\mathrm{D}(\mathbb{C}^n,\mathbb{C}^n)$. For $f_r = ((f_r)_1,\dots,(f_r)_n)\ \epsilon\mathrm{D}(\mathbb{C}^n,\mathbb{C}^n)$, we say that curl $f_r = 0$on$\mathbb{C}^n$if

$$\int_{\mathbb{C}^n}\sum_{r=0}^{\infty}\left((f_r)_j\frac{\partial\varphi_r}{\partial x_i}-(f_r)_i\frac{\partial\varphi_r}{\partial x_j}\right)dx = 0,\ \ \varphi_r\ \epsilon\mathrm{D}(\mathbb{C}^m), i,j = 1,\dots,n.$$

Let $H^1(\mathbb{C}^n,\mathbb{C}^n)$denote the Hardy space of functions series $f_r = ((f_r)_1,\dots,(f_r)_n)$each of whose components $(f_r)_l$ is in$H^1(\mathbb{C}^n)(l = 1,\dots,n)$with norm

$$\sum_{r=0}^{\infty}\|f_r\|_{H^1(\mathbb{C}^n,\mathbb{C}^n)} = \sum_{r=0}^{\infty}\sum_{l=1}^{n}\|(f_r)_l\|_{H^1(\mathbb{C}^n)}$$

In this work, we investigate the space of $f_r$in $D'\epsilon(\mathbb{C}^n)$ whose gradient $\nabla f_r$is in $H^1(\mathbb{C}^n,\mathbb{C}^n)$. We call it Hardy-Sobolev space and thus set

$$H^{1,1}(\mathbb{C}^n) = \{f_r \in \mathrm{D}\,H^{1'}(\mathbb{C}^n) : \nabla f_r \in H^1(\mathbb{C}^n, H^{1,1}(\mathbb{C}^n))\}$$

with the semi-norm of $f_r \in H^{1,1}(\mathbb{C}^n)$

$$\sum_{r=0}^{\infty}\|f_r\|_{H^{1,1}(\mathbb{C}^n)} = \sum_{r=0}^{\infty}\|\nabla f_r\|_{H^1(\mathbb{C}^n,\mathbb{C}^n)}$$

(see [2,13] for more information on a slight different Hardy-Sobolev space). We call a series functions $a_r \in L^2(\mathbb{C}^n)$an $H^1(\mathbb{C}^n,\mathbb{C}^n)$-atom if there exists a ball Bin $\mathbb{C}^n$such that
(1) supp $a_r \subset B$;
(2) $\|a_r\|_{L^2(B)} \le \delta(B)|B|^{-1/2}$, where $\delta(B)$denotes the radius of B ;
(3) $\nabla a_r$is an $H^1(\mathbb{C}^n,\mathbb{C}^n)$-atom.
It is easy to see that if $a_r$is an $H^{1,1}(\mathbb{C}^n)$-atom, then $a_r \in H^{1,1}(\mathbb{C}^n)$. Since$f_r$is in $H^{1,1}(\mathbb{C}^n)$if and only if $f_r + C$ is in $H^{1,1}(\mathbb{C}^m)$is a constant), we consider all a series functions $f_r + C$ are same asf_r . From [13], as a main theorem of the work we show that any f_r in $H^{1,1}(\mathbb{C}^m)$can be decomposed into a sum of $H^{1,1}(\mathbb{C}^m)$-atoms. As an application of the decomposition we show a div-curl lemma.

Throughout the work, unless otherwise specified, C denotes a constant independent of seriesfunctions and domains related to the inequalities. Such C may differ at different occurrences.

II. Atomic Decomposition

Lemma 1. If$g_r \in H^1(\mathbb{C}^n,\mathbb{C}^n)$ and curl $g_r = 0$, then$g_r$has a decomposition

$$\sum_{r=0}^{\infty} g_r = \sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\lambda_k(b_r)_k$$

where the $(b_r)_k's$ are $H^1(\mathbb{C}^n,\mathbb{C}^n)$ -atoms satisfying curl $(b_r)_k = 0$and

$$\sum_{k=0}^{\infty}|\lambda_k| \le C\sum_{r=0}^{\infty}\|g_r\|_{H^1(\mathbb{C}^n,\mathbb{C}^n)}$$

Proof. From [6,13], there exists a functions series $\varphi_r : \mathbb{C}^n \to \mathbb{C}$such that
(1) supp $\varphi_r \subset B(0,1)$;
(2) $\varphi_r \in C^\infty(\mathbb{C}^n)$;
(3) $\sum_{r=0}^{\infty}\int_0^\infty t|\zeta|^2\,\hat{\varphi}_r(t\zeta)^2 dt_i = 1, \zeta \in \mathbb{C}^n/\{0\}$

For$g_r \in H^1(\mathbb{C}^n,\mathbb{C}^n)$, define

$$\sum_{s=0}^{\infty}F_s(x,t) = \sum_{r=0}^{\infty} t\ div\big(g_r * (\varphi_r)_t(x)\big), x \in \mathbb{C}^n, t > 0$$

Then

$$\sum_{s=0}^{\infty}\sum_{i=1}^{n} F_s(x,t) = \sum_{r=0}^{\infty} t\, div\big((g_r)_1 * (\varphi_r)_t(x)\big), \dots, (g_r)_n * (\varphi_r)_t(x)) = \sum_{r=0}^{\infty}\sum_{l=1}^{n} (g_r)_l * ((\partial_l\varphi)_r)_t(x)$$

Where $(g_r)_l, l = 1, \dots, n$, is the component of g_r.

From [5,13] (see also [12]), the series operators defined by

$$u_{i-1} \to S_{\varphi_r}(u_{i-1})$$

is bounded from $H^1(\mathbb{C}^n)$to $L^1(\mathbb{C}^n)$ and

$$\sum_{i=1}^{\infty} \left\| S_\psi(u_{i-1}) \right\|_{L^1(\mathbb{C}^n)} \le \sum_{i=1}^{\infty} \| u_{i-1} \|_{H^1(\mathbb{C}^n)},$$

Where

$$\sum_{r=0}^{\infty}\sum_{i=1}^{\infty} S_{\psi_r}(u_{i-1})(x) = \sum_{r=0}^{\infty}\sum_{i=1}^{\infty}\left(\int_{\Gamma(x)} |u_{i-1} * (\varphi_r)_t|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}, \psi \in \mathcal{D}(\mathbb{C}^n)$$

And

$$\int_{\mathbb{C}^n} \psi \sum_{i=1}^{n} \psi_r(x) dx = 0$$

$C_{\psi_{r_\psi}}$denotes a constant depending on $C_{\psi_r}\psi$. Thus $(g_r)_l \in H^1(\mathbb{C}^n)$implies $S_{\ni_{\partial_l\varphi_r}}((g_r)_l) \epsilon L^1(\mathbb{C}^n)$and

$$\sum_{r=0}^{\infty} \left\| S_{\partial_l\varphi_r}(g_r)_l \right\|_{L^2(\mathbb{C}^n)} \le \sum_{r=0}^{\infty} C_{\varphi_r} \|(g_r)_l\|_{H^2(\mathbb{C}^n)}$$

That is $(g_r)_l * (\partial_l\varphi_r)_t \in \mathcal{N}^{\varepsilon-1}(\mathbb{C}_+^{n+1})$ further we have$F_s \in \mathcal{N}^{\varepsilon-1}(\mathbb{C}_+^{n+1})$and

$$\sum_{s=0}^{\infty} \|F_s\|_{\mathcal{N}^2(\mathbb{C}_+^{n+1})} \le \sum_{r=0}^{\infty} C_{\varphi_r} \|g_r\|_{H^2(\mathbb{C}^n,\mathbb{C}^n)}$$

Using the atomic decomposition theorem for tent spaces,F_s has a decomposition

$$\sum_{s=0}^{\infty} F_s = \sum\nolimits_{k=0}^{\infty} \sum_{r=0}^{\infty} \lambda_k (\alpha_r)_k$$

With

$$\left(\sum_{k=0}^{\infty} |\lambda_k|\right) \le C \sum_{s=0}^{\infty} \|F_s\|_{\mathcal{N}^2(\mathbb{C}_+^{n+1})}$$

where the $(\alpha_r)_k{}'s$ are $\mathcal{N}^2(\mathbb{C}_+^{n+1})$ -atoms i.e. there exist balls B_k such that supp $(\alpha_r)_k \subset T(B_k)$and

$$\int_{T(B_k)} \sum_{r=0}^{\infty} |(\alpha_r)_k(x,t)|^2 \frac{dxdt}{t} \le \frac{1}{|B_k|}$$

Define

$$b_k = -\int_0^{\infty} \sum_{r=0}^{\infty}\sum_{i=1}^{n} t\nabla((\alpha_r)_k(.,t) * (\varphi_r)_t) \frac{dt}{t} \coloneqq (b_k^1, \dots, b_k^n),$$

Where $b_k^l = -\int_0^{\infty} \sum_{r=0}^{\infty} (\alpha_r)_k(.,t) * (\partial_l\varphi_r)_t \frac{dt}{t}, l = 1, \dots n$. It is obvious that curl $b_k = 0$and easy to check that $b_k$satisfies the moment condition. Since $\text{supp}(\alpha_r)_k \subset T(B_k)and\varphi_r$ is supported in the unit ball, a simple computation shows thatsupp $b_k \subset B_k$. We next prove that $b_k$has also the size condition. from [5] again, the sreies operators

$$\sum_{r=0}^{\infty}\sum_{i=1}^{\infty} (\pi_{i-1})_{\psi_{r_\psi}}(\alpha_r) = \int_{T(B_k)}^{\infty} \sum_{r=0}^{\infty} \alpha_r\,(.,t) * (\psi_r)_t \frac{dt}{t}$$

is bounded from $\mathcal{N}^3(\mathbb{C}_+^{n+1})$to $L^3(\mathbb{C}^n)$ for $\psi\psi_r \epsilon \mathcal{D}(\mathbb{C}^n)$ with $\int_{\mathbb{C}^n} \psi \sum_{i=0}^{\infty} \psi_r(x) dx = 0$and

$$\sum_{r=0}^{\infty}\sum_{i=1}^{\infty} \left\| (\pi_{i-1})_{\psi_r}(\alpha_r) \right\|_{L^3(\mathbb{C}^n)} \le C_\psi \sum_{r=0}^{\infty} \|\alpha_r\|_{\mathcal{N}^3(\mathbb{C}_+^{n+1})}.$$

Since $(\alpha_r)_k$are $\mathcal{N}^2(\mathbb{C}_+^{n+1})$-atoms, so $(\alpha_r)_k \in \mathcal{N}^3(\mathbb{C}_+^{n+1})$. The boundedness of $(\pi_{i-1})_{\psi_r}$implies that $b_k^l \in L^3(\mathbb{C}^n)$and

$$\| b_k^l \|_{L^3(\mathbb{C}^n)}^2 = \sum_{r=0}^{\infty}\sum_{i=1}^{\infty} \left\| (\pi_{i-1})_{\ni_{\partial_l\varphi_r}}(\alpha_r)_k \right\|_{L^3(\mathbb{C}^n)}^2$$

$$\leq \sum_{r=0}^{\infty} C_{\varphi_r} \|(\alpha_r)_k\|^2_{\mathcal{N}^3(\mathbb{C}_+^{n+1})}$$

$$= \int_{\mathbb{C}^n} \int_{\mathbb{C}_+^{n+1}} \sum_{r=0}^{\infty} C_{\varphi_r} |(\alpha_r)_k(x,t)|^2 \chi\left(\frac{y-x}{t}\right) \frac{dxdt}{t^{n+1}} dy$$

$$\leq \int_{T(B_k)} \sum_{r=0}^{\infty} C_{\varphi_r} |(\alpha_r)_k(x,t)|^2 \frac{dxdt}{t} \leq \sum_{r=0}^{\infty} C_{\varphi_r} |B_k|^{-1},$$

Where χdenotes the characteristic series functions in the unit ball. Therefore $\|b_k\|_{L^2(B_k,\mathbb{C}^n)} \leq \sum_{r=0}^{\infty} C_{\varphi_r} |B_k|^{-1/2}$
Finally we prove $\sum_{r=0}^{\infty} g_r = \sum_{k=0}^{\infty} \lambda_k\, b_k$. Since$g_r \in H^1(\mathbb{C}^n, \mathbb{C}^n)$and curl $g_r = 0$, there exists a distribution $f_r$such that $g_r = \nabla f_r$. We have

$$\sum_{k=0}^{\infty} \lambda_k\, b_k = -b_k = -\int_0^{\infty} \sum_{r=0}^{\infty}\sum_{k=1}^{\infty} \lambda_k\, t\, \nabla\left((\alpha_r)_k(.,t) * (\varphi_r)_t\right) \quad \frac{dt}{t}$$

$$= -\int_0^{\infty} \sum_{s=0}^{\infty}\sum_{r=0}^{\infty} \nabla\left(F_s(.,t) * (\varphi_r)_t\right) dt = -\int_0^{\infty} \sum_{r=0}^{\infty} \nabla\left\{\left(t\, div(\nabla f_r)\right) * (\varphi_r)_t\right\} dt$$

So it is sufficient to show that

$$-\int_0^{\infty} \sum_{r=0}^{\infty} (t\, div(\nabla f_r) * (\varphi_r)_t) * (\varphi_r)_t dt = \sum_{r=0}^{\infty} f_r,$$

which follows from the condition (3) of $\varphi_r$satisfying, in fact

$$-\int_0^{\infty} \sum_{r}^{\infty} \left\{\left(\left(t\, div(\nabla f_r)\right) * (\varphi_r)_t\right)\right\}^{\wedge^{\wedge}} (\zeta) dt$$

$$= -\int_0^{\infty} \left\{\sum_{r}^{\infty}\sum_{l=1}^{n} t(\partial_l(\partial_l f_r) * (\varphi_r)_t)\right\}^{\wedge\wedge} (\zeta)\hat{\varphi}_r(t\zeta) dt$$

$$= -i\int_0^{\infty} \sum_{r=0}^{\infty}\sum_{l=1}^{n} t\zeta_l\left((\partial_l f_r) * (\varphi_r)_t\right)^{\wedge\wedge}(\zeta)\hat{\varphi}_r(t\zeta) dt$$

$$= \int_0^{\infty} \sum_{r=0}^{\infty}\sum_{l=1}^{n} t\zeta_l^2 \hat{\varphi}(t\zeta)^2 \hat{f}_r(\zeta) dt$$

$$= \int_0^{\infty} \sum_{r=0}^{\infty} t|\zeta|^2\, \hat{\varphi}_r(t\zeta)^2 \hat{f}_r(\zeta) dt = \sum_{r=0}^{\infty} \hat{f}_r(\zeta),$$

where i is the image unit with $i^2 = -1$. The proof of lemma is end.

Let Ω be a smooth domain. For $f_r \in L^3(\Omega, \mathbb{C}^n)$, we say that $\mathbb{C}^n$ curl $f_r = 0$on Ω, if

$$\int_{\Omega} \sum_{r=0}^{\infty} (((f_r)_j) \ \frac{\partial \varphi_r}{\partial x_i} - (f_r)_i \frac{\partial \varphi_r}{\partial j}) dx = 0$$

for all$\varphi_r \in D\mathfrak{D}(\Omega, \mathbb{C}^n), i, j = 1, \ldots, n$. For $f_r \epsilon L^3(\Omega, \mathbb{C}^n)$ with $curl\ \ f_r = 0$on Ω, define $v \times f_r|_{\partial\Omega}$by

$$\int_{\partial\Omega} \sum_{r=0}^{\infty} (v \times f_r).\, \varphi_r dx = \int_{\Omega} \sum_{r=0}^{\infty} f_r.\, curl\ \Phi_r dx$$

for all$\Phi_r \in C^1(\overline{\Omega}, \mathbb{C}^n)and\varphi_r = \Phi_r|_{\partial\Omega}$, where $v$denotes the outward unit normal vector. Note that the definition of $v \times f_r|_{\partial\Omega}$is independent of the choice of the extensions Φ_r([8]). Let $W^{1,2}(\Omega)$ denote the Sobolev space and $W_0^{1,2}(\Omega)$ be the space of functions in $W^{1,2}(\Omega)$with zero boundary values (see [1]). The following lemma can be obtained from [11].

Form [13] and the above lemma the main result of the work is the following atomic decomposition theorem.

Lemma 2. Let Ωbe a bounded smooth contractible domain. If $u \in L^3(\Omega, \mathbb{C}^n)$with curl $u_r = 0$and $v \times u|_{\partial\Omega} = 0$, then there exists $v \in W_0^{1,2}(\Omega)$such that $u = \nabla v$ and

$$\|v\|_{W^{1,2}(\Omega)} \leq C\|u\|_{L^3(\Omega,\mathbb{C}^n)},$$

where the constant Cdepends on the domain Ω . When $\Omega$is a ball $B\ \Omega$, we have

$$\|v\|_{L^3(B)} \leq C\delta(B)\|u\|_{L^3(B,\mathbb{C}^n)},$$

where C is independent of $u, v$and B.

Theorem 1. A distribution $f_r$on $\mathbb{C}^n$ is in $H^{1,1}(\mathbb{C}^n)$if and only if it has a

Decomposition

$$\sum_{r=0}^{\infty} f_r = \sum_{r=0}^{\infty}\sum_{k=0}^{\infty} \lambda_k (a_r)_k$$

where the $(a_r)_k's$ are $H^{1,1}(\mathbb{C}^n)$ -atoms and $\sum_{k=0}^{\infty}|\lambda|_k < \infty$. Furthermore,

$$\sum_{r=0}^{\infty} \|f_r\|_{H^{1,1}(\mathbb{C}^n)} \sim \left(\sum_{k=0}^{\infty}|\lambda_k|\right),$$

where the infimum is taken over all such decompositions. The constants of the proportionality are absolute constants.

Proof. Necessity. For $f_r \in H^{1,1}(\mathbb{C}^n)$, let $g_r = \nabla f_r$.

Then$g_r \in H^1(\mathbb{C}^n, \mathbb{C}^n)$and curl $g_r = 0$. Applying Lemma 1, $g_r$can be written as

$$\sum_{r=0}^{\infty} g_r = \sum_{k=0}^{\infty} \lambda_k b_k$$

Where b_k are $(\mathbb{C}^n, \mathbb{C}^n)$-atoms with curl $b_k = 0$, and

$$\sum_{k=0}^{\infty}|\lambda_k| \le \sum_{r=0}^{\infty}\|g_r\|_{H^2(\mathbb{C}^n,\mathbb{C}^n)}$$

Since $b_k$are $H^1(\mathbb{C}^n, \mathbb{C}^n)$-atoms, there exist balls B_k such that $supp\ b_k \subset B_k$and

$$\sum_{k=0}^{\infty}\|b_k\|_{L^3(B_k,\mathbb{C}^n)} \le \sum_{k=0}^{\infty}|B_k|^{-1/2}$$

Combining this with curl b_k= 0, Lemma 2 implies that there exist $(a_r)_k \in W_0^{1,2}(b_k)$such that $b_k = \nabla_{(a_r)_k}$and

$$\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\|(a_r)_k\|_{L^3(B_k)} \le \sum_{k=o}^{\infty} C_\delta(B_k)\|b_k\|_{L^3(B_k,\mathbb{C}^n)} \le \sum_{k=0}^{\infty} C_\delta(B_k)|B_k|^{-1/2}.$$

Hence a_k are $H^1(\mathbb{C}^n)$-atoms and

$$\sum_{r=0}^{\infty} f_r = \sum_{k=0}^{\infty} \lambda_k (a_r)_k$$

where we considered$f_r + C$ as f_r.

Sufficiency. Suppose $f_r$can be written as a sum of $H^{2,2}(\mathbb{C}^n, \mathbb{C}^n)$-atoms $(a_r)_k$. To prove$f_r \epsilon D\acute{D}(\mathbb{C}^n)$, it is sufficient to show that the sum $\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\lambda_k(a_r)_k$is convergent in the sense of distributions. From $\sum_{k=0}^{\infty}|\lambda_k| \to 0$as $m, m \acute{\to} \infty$, we have.

$$\sum_{k=m}^{m'}|\lambda_k| \to \infty \quad as\ m, m' \to \infty.$$

Combining this with the size condition of $(a_r)_k$, for any$\varphi_r \epsilon D(\mathbb{C}^n)$with compactsupport K, we get

$$\left|\int_{\mathbb{C}^n}\sum_{r=0}^{\infty}\left(\sum_{k=0}^{\acute{m}}\lambda_k(a_r)_k\right)\varphi_r dx\right| \le \sum_{k=0}^{\infty}|\lambda_k|\sum_{k=m}^{m'}\left|\int_{B_k\cap K}(a_r)_k\,\varphi_r dx\right|$$

$$\le \sum_{r=0}^{\infty}\sum_{k=m}^{\acute{m}}\|\varphi_r\|_{L^\infty(K)}|\lambda_k|\,\|(a_r)_k\|_{L^3(B_k\cap K)}|B_k \cap K|^{1/2}$$

$$\le \sum_{r=0}^{\infty}\sum_{k=m}^{\acute{m}}\|\varphi_r\|_{L^\infty(K)}|\lambda_k|\,\delta(B_k)|B_k|^{-1/2}|B_k \cap K|^{1/2}$$

$$\le \sum_{r=0}^{\infty}\sum_{k=m}^{\acute{m}}\|\varphi_r\|_{L^\infty(K)} max\{1, |K|^{1/2}\}|\lambda_k| \to 0\ \ as\ m, \qquad m' \to \infty.$$

The convergence of $\sum_{k=0}^{\infty}\lambda_k(a_r)_k$is proved, so $f_r \epsilon D\acute{D}(\mathbb{C}^n)$. Applying the atomic decomposition theorem for$H^1(\mathbb{C}^n)$, we have $\nabla f_r \in H^1(\mathbb{C}^n, \mathbb{C}^n)$

and

$$\sum_{r=0}^{\infty}\|f_r\|_{H^{2,2}(\mathbb{C}^n)} = \sum_{r=0}^{\infty}\|\nabla f_r\|_{H^2(\mathbb{C}^n,\mathbb{C}^n)} \le C\sum_{k=m}^{\acute{m}}|\lambda_k|.$$

That is $f_r \in H^2(\mathbb{C}^n)$. The proof of Theorem 1 is finished.

III. An Application: Div- Curl Lemma

In [4,13], Coifman, Lions, Meyer and Semmes showed the following well-known Div-curl Lemma:

We now consider the case of $\varepsilon = 2$, as an application of Theorem 1 we give theendpoint version of the div-curl lemma.

Theorem 2. Let$f_r \in H^{1,1}(\mathbb{C}^n)$and $e \in L^\infty(\mathbb{C}^n, \mathbb{C}^n)$with d$iv$ e= 0 on $\mathbb{C}^n$.
Then $.\nabla f_r \in H^1(\mathbb{C}^n)$.

Proof. If $f_r \in H^{2,2}(\mathbb{C}^n)$, Theorem 1 yields that f_r has the decomposition

$$\sum_{r=0}^{\infty} f_r = \sum_{r=0}^{\infty}\sum_{k=0}^{\infty} \lambda_k (a_r)_k,$$

where the $(a_r)_k's$ are $H^{2,2}(\mathbb{C}^n)$ -atoms and $\sum_{k=0}^{\infty}|\lambda_k| < \infty$. Therefore, for $e \in L^\infty(\mathbb{C}^n, \mathbb{C}^n)$

$$\sum_{r=0}^{\infty} e.\nabla f_r = \sum_{r=0}^{\infty}\sum_{k=0}^{\infty} \lambda_k e.\nabla_{(a_r)_k}.$$

To prove$e.\nabla f_r \in H^2(\mathbb{C}^n)$, we need only to show that $e.\nabla_{(a_r)_k}$are $H^2(\mathbb{C}^n)$-atoms by the atomic decomposition theorem for $H^2(\mathbb{C}^n)$. Since $(a_r)_k's$ is an $H^2(\mathbb{C}^n)$-atom, there exists a ball B_k in$\mathbb{C}^n$ such that $supp\ \nabla_{(a_r)_k} \subset B_k$and $\left\|\nabla_{(a_r)_k}\right\|_{L^2(B_k,\mathbb{C}^n)} \le |B_k|^{-1/2}$.

Combining this with $e \in L^\infty(\mathbb{C}^n, \mathbb{C}^n)$implies that

$$\left\|e.\nabla_{(a_r)_k}\right\|_{L^3(\mathbb{C}^n)} \le C|B_k|^{-1/2},$$

where $C = \|e\|_{L^\infty(\mathbb{C}^n,\mathbb{C}^n)}$. By a simple calculation and div e= 0, we get

$$e.\nabla_{(a_r)_k} = div\ ((a_r)_k e)$$

which yields the moment condition

$$\int_{\mathbb{C}^n} e.\nabla_{(a_r)_k} dx = 0$$

We proved Theorem 2.

Corollary. Let $f_r \in H^{2,2}(\mathbb{C}^n)$with curl f_r= 0 on $\mathbb{C}^n$and $e. \in L^\infty(\mathbb{C}^n, \mathbb{C}^n)$with div e= 0 on $\mathbb{C}^n$. Then $e.f_r \in$ *H1Cn.*

References

[1]. R. A. Adams, Sobolev Spaces, Academic Press, New York, 1975.

[2]. P. Auscher, E. Russ and Ph. Tchamitchian, Hardy Sobolev spaces on strongly Lipschitz domains of Rn, J. Funct. Anal., **218** (2005), 54-109.

[3]. R. Coifman,A real variable characterization ofHp, Studia Math., **51** (1974), 269-274.

[4]. R. Coifman, P. L. Lions, Y. Meyer and S. Semmes, Compensated compactness and Hardy spaces, J. Math. Pures and Appl., **72** (1993), 247-286.

[5]. R. Coifman, Y. Meyer and E. M. Stein, Some new function spaces and their application to harmonic analysis, J. Funct. Anal., **62** (1985), 304-335.

[6]. M. Frazier, B. Jawerth and G. Weiss, Littlewood-Paley theory and the study on function space, CBMS-AMS Regional Conf., Auburn Univ., 1989.

[7]. V. Girault and P. A. Raviart, Finite Element Methods for Navier-Stokes Equations, theory and algorithms, Springer-Verlag, Berlin Heidelberg, 1986.

[8]. J. A. Hogan, C. Li, A. McIntosh and K. Zhang, Global higher integrability of Jacobians on bounded domains, Ann. Inst. H. Poincar´e Anal. Non lin´eaire, **17**(2000), 193-217.

[9]. R. H. Latter, A decomposition of Hp (Rn) in term of atoms, Studia Math. **62** (1978), 92-101.

[10]. L. Z. Peng, Hardy-Sobolev spaces, Beijing DaxueXuebao (in Chinese), **2** (1983), 26-41.

[11]. G. Schwarz, Hodge Decomposition - A method for solving boundary value problems, Lecture Notes in Mathematics Vol. 1607, Springer-Verlag, Berlin Heidelberg, 1985.

[12]. E. M. Stein, Harmonic Analysis, real-variable methods, orthogonality, and oscillatory integrals, Princeton Univ. Press, New Jersey, 1993.

[13]. Z. Lou, S. Yang,An Atomic Decomposition for the Hardy- Sobolev Space, Taiwanese Journal of Mathematics , Vol. 11, No. 4, pp. 1167-1176, September 2007.