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# On certain continued fractions involving two basic hypergeometric series

Dr. Jayprakash Yadav[1], Vidhi Bhardwaj[2]
[1]*Prahladrai Dalmia Lions College of Commerce and Economics, Mumbai, India.*
[2]*D.S.J. College, Mumbai, India*

**Abstract:** *The object of this paper is to establish continued fraction for the ratio of two* ${}_4\phi_3$ *series by the application of some known transformation formula in basic hyper geometric series.*
**Keywords and phrases:** *Basic hyper geometric series, Continued fractions and some known results in basic hyper geometric series.*

## I. Definitions And Notations

The basic hyper geometric series is defined as

$$ {}_r\phi_s\left[\begin{matrix}(a);z\\(b)\end{matrix}\right]=\sum_{n=0}^{\infty}\frac{((a);q)_n z^n}{((b);q)_n(q;q)_n}, \tag{1}$$

where for the convergence of the series, we have $0<|q|<1$ and $|z|<1$ if r = s+1, and for any z if $r \le s$.

$$\begin{aligned}((a);q)_n&=(a_1;q)_n(a_2;q)_n\ldots(a_r;q)_n,\\((b);q)_n&=(b_1;q)_n(b_2;q)_n\ldots(b_s;q)_n.\end{aligned} \tag{2}$$

For real or complex, a, $q$ <1, the $q$ -shifted factorial is defined by

$$(a,q)_n=\begin{cases}1 & ;if \quad n=0\\(1-q)(1-aq)\ldots(1-aq^{n-1}) & ;if \quad n\in N.\end{cases} \tag{3}$$

*We will also use the following results in our analysis*:

$$ {}_4\phi_3\left[\begin{matrix}q^{-n},a,b,c;q\\d,e,f\end{matrix}\right]=\frac{(e/a,f/a)_n}{(e,f)_n}a^n\,{}_4\phi_3\left[\begin{matrix}q^{-n},a,d/b,d/c;q\\d,aq^{1-n}/e,aq^{1-n}/f\end{matrix}\right] \tag{4}$$

and

$$ {}_8\phi_7\left[\begin{matrix}\mu,q\mu^{\frac{1}{2}},-q\mu^{\frac{1}{2}},a,b,c,dq^n,eq^n;\frac{de}{abc}\\ \mu^{\frac{1}{2}},-\mu^{\frac{1}{2}},\mu q/a,\mu q/b,\mu q/c,e,d\end{matrix}\right]=\frac{(aq/f,bq/f,cq/f,abc/f;q)_\infty}{(abq/f,acq/f,bcq/f,q/f;q)_\infty}\,{}_4\phi_3\left[\begin{matrix}q^{-n},a,b,c;q\\d,e,f\end{matrix}\right] \tag{5}$$

where $\mu=abc/f,\ def=abcq^{1-n}$ **[** Gasper and **R**ahman [2]; Appiii(iii.20) **]**

## II. Main Results

$$(a)\;{}_4\phi_3\left[\begin{matrix} q^{-n},a,b,c:q \\ d,e,f \end{matrix}\right] \Big/ {}_4\phi_3\left[\begin{matrix} q^{-n},aq,b,c:q \\ dq,e,f \end{matrix}\right] = 1+\frac{\alpha_0}{\beta_0-}\frac{\gamma_0}{1+}\frac{\alpha_1}{\beta_1-}\frac{\gamma_1}{1+}\frac{\alpha_2}{\beta_2-}\frac{\gamma_2}{1+}\ldots\ldots, \qquad (6)$$

Where $\alpha_i=\dfrac{dq^{2i+1}(1-q^{i-n})(1-bq^i)(1-cq^i)}{(1-dq^{2i})(1-dq^{2i+1})(1-eq^i)(1-fq^i)}$, $\beta_i=\dfrac{a^2q^2}{(aq-e)(aq-f)(1-eq^i)(1-fq^i)}$

And $\gamma_i=\dfrac{ef}{bc}\dfrac{(1-dq^{i+n+1})(1-aq^{i+1})(b-dq^{i+1})(c-dq^{i+1})q^{i-2n+1}}{(1-dq^{2i+1})(1-dq^{2i+1})(e-aq^{2-n+i})(f-aq^{2-n+i})(1-eq^i)(1-fq^i)}$.

$$(b)\;\frac{{}_8\phi_7\left[\begin{matrix} \mu,q\mu^{\frac12},-q\mu^{\frac12},a,b,c,dq^n,eq^n;\dfrac{de}{abc} \\ \mu^{\frac12},-\mu^{\frac12},\mu q/a,\mu q/b,\mu q/c,e,d \end{matrix}\right]}{{}_8\phi_7\left[\begin{matrix} \vartheta,q\vartheta^{\frac12},-q\vartheta^{\frac12},aq,b,c,dq^{n+1},eq^n;\dfrac{de}{abc} \\ \vartheta^{\frac12},-\vartheta^{\frac12},\vartheta q/a,\vartheta q/b,\vartheta q/c,e,dq \end{matrix}\right]} = \frac{(f-aq)(f-abcq)}{(f-abq)(f-acq)}\left\{1+\frac{\alpha_0}{\beta_0-}\frac{\gamma_0}{1+}\frac{\alpha_1}{\beta_1}\frac{\gamma_1}{1+}\ldots\right\}$$

(7)

$\mu=abc/f$, $def=abcq^{1-n}$ , $\vartheta=abcq/f$ , $def=abcq^{1-n}$.

Where $\alpha_i=\frac{dq^{2i+1}(1-q^{i-n})(1-bq^i)(1-cq^i)}{(1-dq^{2i})(1-dq^{2i+1})(1-eq^i)(1-fq^i)}$, $\beta_i=\frac{a^2q^2}{(aq-e)(aq-f)(1-eq^i)(1-fq^i)}$

and $\gamma_i=\frac{(1-dq^{i+n+1})(1-aq^{i+1})(b-dq^{i+1})(c-dq^{i+1})}{(1-dq^{2i+1})(1-dq^{2i+2})(1-eq^i)(1-fq^i)(e-aq^{i-n+i})(f-aq^{2-n+i})bc}$

**Proof of (a):** For $i\ge o$,

Let $H_i={}_4\phi_3\left[\begin{matrix} q^{-n+i},aq^i,bq^i,cq^i:q \\ dq^{2i},eq^i,fq^i \end{matrix}\right]$ and $F_i={}_4\phi_3\left[\begin{matrix} q^{-n+i},aq^{i+1},bq^i,cq^i:q \\ dq^{2i+1},eq^i,fq^i \end{matrix}\right]$

So,

$$H_i-F_i=\frac{dq^{2i+1}(1-q^{i-n})(1-bq^i)(1-cq^i)}{(1-dq^{2i})(1-dq^{2i+1})(1-eq^i)(1-fq^i)}H_{i+1}$$

This gives $\dfrac{H_i}{F_i}=1+\dfrac{\alpha_i}{F_i/H_{i+1}}$, (8)

Where $\alpha_i=\dfrac{dq^{2i+1}(1-q^{i-n})(1-bq^i)(1-cq^i)}{(1-dq^{2i})(1-dq^{2i+1})(1-eq^i)(1-fq^i)}$.

Let us transform $F_i$ and $H_i$ by means of the transformation formula (4)
we get:

$$F_i = \frac{(e/aq)_n (f/aq)_n}{(eq^i)_n (fq^i)_n} (aq^{i+1})^n \; {}_4\phi_3 \left[ \begin{matrix} q^{i-n}, aq^{i+1}, dq^{i+1}/b, dq^{i+1}/c; q \\ dq^{2i+1}, aq^{2-n+1}/e, aq^{2-n+}/f \end{matrix} \right], \tag{9}$$

$$H_i = \frac{(e/aq)_n (f/a)_n}{(eq^i)_n (fq^i)_n} (aq^i)^n \; {}_4\phi_3 \left[ \begin{matrix} q^{i-n}, aq^i, dq^i/b, dq^i/c; q \\ dq^{2i}, aq^{1-n+i}/e, aq^{1-n+i}/f \end{matrix} \right], \tag{10}$$

$$H_{i+1} = \frac{(e/aq)_n (f/a)_n}{(eq^{i+1})_n (fq^{i+1})_n} (aq^{i+1})^n \; {}_4\phi_3 \left[ \begin{matrix} q^{i-n+1}, aq^{i+1}, dq^{i+1}/b, dq^{i+1}/c; q \\ dq^{2i+2}, aq^{2-n+i}/e, aq^{2-n+i}/f \end{matrix} \right], \tag{11}$$

So,

$$\frac{H_{i+1}}{(1-e/aq)(1-f/aq)} - (eq^i)(fq^i)F_i$$

$$= \frac{(1-aq^{i+1})(1-dq^{i+1}/b)(1-dq^{i+1}/c)(e/aq)_n (f/aq)_n (aq^{i+2})^n (1-dq^{i+n+1})q^{i+1}}{q^{2n}(1-dq^{2i+1})(1-dq^{2i+2})(1-aq^{2-n+i}/e)(1-aq^{2-n+i}/f)(eq^{i+1})_n (fq^{i+1})_n (1-dq^{2i+1})} \times \tag{12}$$

$$\sum_{r=0}^{\infty} \frac{(q^{i-n+1})_{r-1} (aq^{i+1}, dq^{i+1}/b, dq^{i+1}/c)_r q^r}{(q)_{r-1} (dq^{2i+2})_r (aq^{2-n+i}/e, aq^{2-n+i}/f)_r},$$

This gives the following equation:

$$\frac{F_i}{H_{i+1}} = \frac{1}{(1-e/aq)(1-f/aq)(1-eq^i)(1-fq^i)} -$$

$$\frac{(1-aq^{i+1})(1-dq^{i+1}/b)(1-dq^{i+1}/c)(1-dq^{i+n+1})q^{i+1-2n}}{(1-dq^{2i+1})(1-dq^{2i+2})(1-aq^{2-n+i}/e)(1-aq^{2-n+i}/f)(eq^i)_n (fq^i)_n} \frac{F_{i+1}}{H_{i+1}}, \tag{13}$$

This can be written as $\frac{F_i}{H_{i+1}} = \beta_i - \frac{\gamma_i}{H_{i+1}/F_{i+1}}$, where $\beta_i$ *and* $\gamma_i$ are mentions above.

Now using (1) and (2) repeatedly, for $i \geq 0$ we get (6).

**Proof of (b):**

$${}_8\phi_7 \left[ \begin{matrix} \mu, q\mu^{\frac{1}{2}}, -q\mu^{\frac{1}{2}}, a, b, c, dq^n, eq^n; \frac{de}{abc} \\ \mu^{\frac{1}{2}}, -\mu^{\frac{1}{2}}, \mu q/a, \mu q/b, \mu q/c, e, d \end{matrix} \right] = \frac{\left(\frac{aq}{f}, \frac{bq}{f}, \frac{cq}{f}, \frac{abcq}{f}\right)_\infty}{\left(\frac{abq}{f}, \frac{acq}{f}, \frac{bcq}{f}, \frac{q}{f}\right)_\infty} \; {}_4\phi_3 \left[ \begin{matrix} q^{-n}, a, b, c; q \\ d, e, f \end{matrix} \right] \tag{15}$$

Where $\mu = \frac{abc}{f}, def = abcq^{1-n}$.

Let us replace $a$ *by* $aq$ *and* $d$ *by* $dq$ *in the above equation, we get the following*:

$$ {}_8\phi_7\left[\begin{matrix} \vartheta, q\vartheta^{\frac{1}{2}}, -q\vartheta^{\frac{1}{2}}, aq, b, c, dq^{n+1}, eq^{n}; \dfrac{de}{abc} \\ \vartheta^{\frac{1}{2}}, -\vartheta^{\frac{1}{2}}, \vartheta q/a, \vartheta q/b, \vartheta q/c, e, dq \end{matrix}\right] = \frac{\left(\frac{aq^2}{f}, \frac{bq}{f}, \frac{cq}{f}, \frac{abcq^2}{f}\right)_\infty}{\left(\frac{abq^2}{f}, \frac{acq^2}{f}, \frac{bcq}{f}, \frac{q}{f}\right)_\infty} {}_4\phi_3\left[\begin{matrix} q^{-n}, aq, b, c; q \\ dq, e, f \end{matrix}\right] \qquad (16) $$

$Now\ dividing\ equation\ (15)\ by\ equation\ (16)\ we\ get\ the\ following\ equation:$

$$ \frac{{}_8\phi_7\left[\begin{matrix} \mu, q\mu^{\frac{1}{2}}, -q\mu^{\frac{1}{2}}, a, b, c, dq^{n}, eq^{n}; \dfrac{de}{abc} \\ \mu^{\frac{1}{2}}, -\mu^{\frac{1}{2}}, \mu q/a, \mu q/b, \mu q/c, e, d \end{matrix}\right]}{{}_8\phi_7\left[\begin{matrix} \vartheta, q\vartheta^{\frac{1}{2}}, -q\vartheta^{\frac{1}{2}}, aq, b, c, dq^{n+1}, eq^{n}; \dfrac{de}{abc} \\ \vartheta^{\frac{1}{2}}, -\vartheta^{\frac{1}{2}}, \vartheta q/a, \vartheta q/b, \vartheta q/c, e, dq \end{matrix}\right]} = \frac{(f-aq)(f-abcq)}{(f-abq)(f-acq)} \frac{{}_4\phi_3\left[\begin{matrix} q^{-n}, a, b, c; q \\ d, e, f \end{matrix}\right]}{{}_4\phi_3\left[\begin{matrix} q^{-n}, aq, b, c; q \\ dq, e, f \end{matrix}\right]} $$

Now using result (6) on the right hand side we get the result (7).

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