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A series Operators for Hardy Spaces on Linear Functional domains of $\mathbb{C}^n$

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Abstract: *We show that there is a linear functional bounded uniformly on all atoms in $H^1(\mathbb{C}^n)$. In this work, we first give a new atomic decomposition, where the decomposition converges in $L^2(\mathbb{C}^n)$ rather than only in the distribution sense. Then using this decomposition, we show that for $\varepsilon \leq 0, T_{r-1}$ is a linear a series operators which is bounded on $L^2(\mathbb{C}^n)$, then T_{r-1} can be extended to a bounded a series operators from $H^{\varepsilon+1}(\mathbb{C}^n)$ to $L^2(\mathbb{C}^n)$ if and only if T_{r-1} is bounded uniformly on all $(\varepsilon+1,2)$-atoms in $L^{\varepsilon+1}(\mathbb{C}^n)$. A similar result from $H^{\varepsilon+1}(\mathbb{C}^n)$ to $H^{\varepsilon+1}(\mathbb{C}^n)$ is also obtained.*

Keword: *Hardy Spaces, atomic decomposition*, quasi-Banach space.

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Introduction

From [7] the atomic decompositions of Hardy spaces play an important role in the boundedness of a series operators on Hardy spaces. The best known examples of a class with this property are Calderon-Zygmund a series operators. Recently, in [1,3,5,7], gave an example of a linear functional series defined on a dense subspace of Hardy space $H^1(\mathbb{C}^n)$, which maps all atoms into bounded scalars, but it cannot be extended to a bounded functional series on the whole space$H^1(\mathbb{C}^n)$. As a consequence of his example, it implies that to show the boundedness of a series operators from Hardy space $H^{\varepsilon+1}(\mathbb{C}^n), \varepsilon \leq 0$, to some other quasi-Banach space, in general it does not suffice to just verify that this series operators maps atoms into bounded elements of this quasi-Banach space. Therefore, it should be very carefully to do this. Maybe this problem is based on the atomic decomposition of Hardy spaces. Since Calderon-Zygmund a series operators are bounded on $L^2(\mathbb{C}^n)$ spaces, the atomic decompositions are converged in the distribution sense . So, the series operators should not be put into each one atom in the series.

In this work, using the Calderon reproducing formula, we give an atomic decomposition of a dense subspace $H^{\varepsilon+1}(\mathbb{C}^n) \cap L^{\varepsilon+1}(\mathbb{C}^n)$ of the Hardy spaces $H^{\varepsilon+1}(\mathbb{C}^n)$, where the decomposition converges also in $L^2(\mathbb{C}^n)$ rather than only in the distribution sense. Then, using this atomic decomposition, we can show the boundedness of linear operators series on Hardy spaces by T_{r-1} is bounded uniformly on all atoms.

Suppose $\psi_r(x)$ satisfying the conditions $\int_0^\infty \sum_r \left|\hat{\psi}_r(t\xi)\right|^2 \frac{dt}{t} = 1$ for all $\xi \in \mathbb{C}^n \setminus \{0\}$ and $\int_{\mathbb{C}^n} \sum_r \psi_r(x) x^\alpha dx = 0$ for all nonnegative multi-indexes α with$|\alpha| \leq [n\left(\frac{-\varepsilon}{\varepsilon+1}\right)]$.

Definition 1: Let $f_r \in S'(\mathbb{C}^n)$, the space of tempered distributions. Supose ψ_r be a series functions as above. We define $S(f_r)$, by

$$\sum_r S(f_r)(x) = \sum_r \left\{ \int_0^\infty \int_{|\mathcal{Y}-x|<t} |(\psi_r)_t * f_r(\mathcal{Y})|^2 \frac{d\mathcal{Y}dt}{t^{n+1}} \right\}^{\frac{1}{2}}, \tag{1}$$

Where$\sum_r (\psi_r)_t(x) = t^{-n} \sum_r \psi_r\left(\frac{x}{t}\right)$.

Definition 2: The Hardy space$H^{\varepsilon+1}(\mathbb{C}^n), \varepsilon \leq 0$, is defined by

$$H^{\varepsilon+1}(\mathbb{C}^n) = \{f_r \in S'(\mathbb{C}^n): S(f_r) \in L^{\varepsilon+1}(\mathbb{C}^n)\}. \tag{2}$$

If $f_r \in H^{\varepsilon+1}(\mathbb{C}^n)$, the norm of f_r is defined by $\|S(f_r)\|_{\varepsilon+1}$ It was known that the definition 2is independent of the choice of the a series functions ψ_r. The usual atomic decomposition of $H^{\varepsilon+1}(\mathbb{C}^n)$ is as follows (see [2, 4, 6,8]).

Theorem 3. Let $f_r \in H^{\epsilon+1}(\mathbb{C}^n)$. Then there is a sequence of $(\epsilon+1,2)$-atoms $\{(a_r)_j\}$ and a sequence of scalars $\{\lambda_j\}$ with $\sum_j |\lambda_j|^{\epsilon+1} \le C\|f_r\|_{H^{\epsilon+1}}^{\epsilon+1}$ such that $f_r = \sum_j \lambda_j (a_r)_j$, where the series converges to f_r in the sense of tempered distributions. Conversely, if f_r is a tempered distribution such that $\sum_j \lambda_j (a_r)_j$ in the sense of tempered distributions with $\sum_j |\lambda_j|^{\epsilon+1} < \infty$, and the $(a_r)_j$'s being $(\epsilon+1,2)$-atoms, then $f_r \in H^{\epsilon+1}(\mathbb{C}^n)$ and $\|f_r\|_{H^{\epsilon+1}}^{\epsilon+1} \le C\sum_j |\lambda_j|^{\epsilon+1}$.

Here a series functions $a_r(x)$ is said to be an $(\epsilon+1,2)$-atom of $H^{\epsilon+1}(\mathbb{C}^n)$, $\epsilon \le 0$, if $a_r(x)$ is supported in a cube Q; $\|a_r\|_2 \le |Q|^{\frac{1}{2}-\frac{1}{\epsilon+1}}$; and finally, $\int a_r(x)x^{\alpha_r}\,dx = 0$ for all nonnegative multi-indexes α_r with $|\alpha_r| \le [n\left(\frac{1}{\epsilon+1}-1\right)]$.

Corollary 4. Let $f_r \in L^2(\mathbb{C}^n) \cap H^{\epsilon+1}(\mathbb{C}^n)$. Then there is a sequence of $(\epsilon+1,2)$-atoms $\{(a_r)_j\}$ and a sequence of scalars $\{\lambda_j\}$ with $\sum_j |\lambda_j|^{\epsilon+1} \le C\|f_r\|_{H^{\epsilon+1}}^{\epsilon+1}$ such that $f_r = \sum_j \lambda_j (a_r)_j$, where the series converges to f_r in $L^2(\mathbb{C}^n)$.

Proof : Let ψ_r be a series functions mentioned above. Then the following Calderon reproducing formula holds

$$\sum_r f_r(x) = \int_0^\infty \sum_r (\psi_r)_t * (\psi_r)_t * f_r(x)\frac{dt}{t}, \qquad (3)$$

Where the integral converges in $L^2(\mathbb{C}^n)$. Now, suppose $f_r \in L^2 \cap H^{\epsilon+1}$. Let $\Omega_k = \{x \in \mathbb{C}^n : S(f_r)(x) > 2^k\}$ and $B_k = \{Q :$ dyadic cubes such that $|Q \cap \Omega_k| > \frac{1}{2}|Q|$ and $|Q \cap \Omega_{k+1}| \le \frac{1}{2}|Q|\}$. For each dyadic cube Q, denote $\hat{Q} = \{(\mathcal{Y},t): \mathcal{Y} \in Q \ and \ \sqrt{ne}\,(Q) \le t < 2\sqrt{ne}\,(Q)\}$,

Where $e(Q)$ is the side length of . We claim that

$$\sum_r f_r(x) = \sum_k \sum_{\tilde{Q}\in B_k} \sum_{Q\subseteq \tilde{Q}\cap B_k} \sum_r \int_{\hat{Q}} (\psi_r)_t(x-\mathcal{Y})(\psi_r)_t * f_r(\mathcal{Y})\frac{d\mathcal{Y}dt}{t}, \quad (4)$$

Where $\tilde{Q} \in B_k$ are maximal dyadic cubes in B_k, and the series converges in $L^2(\mathbb{C}^n)$.

To show the claim, it suffices to show that for any positive integer N,

$$\left\|\sum_r \sum_{k>N} \sum_{Q\in B_k} \int_{\hat{Q}} (\psi_r)_t(x-\mathcal{Y})(\psi_r)_t * f_r(\mathcal{Y})\frac{d\mathcal{Y}dt}{t}\right\|_2 \to 0, as\ N \to \infty.$$

First let $\tilde{\Omega}_k = \left\{x \in \mathbb{C}^n: M(x\Omega_k)(x) > \frac{1}{2}\right\}$, where M is the Hardy-Littlewood maximal a series function. Then $\Omega_k \subseteq \tilde{\Omega}_k$, and by the maximal theorem, $|\tilde{\Omega}_k| \le C|\Omega_k|$.

Let $\chi(x,\mathcal{Y},t)$ be the characterization of $\{(x,\mathcal{Y},t): x \in \tilde{\Omega}_k \setminus \Omega_{k+1}, |x-\mathcal{Y}| < t\}$. For any $x \in Q \in B_k$, since $|Q \cap \Omega_k| \ge \frac{1}{2}|Q|$, one has $x \in \tilde{\Omega}_k$, thus if $(\mathcal{Y},t) \in \hat{Q}$, then

$$\int_{\mathbb{C}^n} \chi(x,\mathcal{Y},t)dx \ge |Q \cap (\tilde{\Omega}_k \setminus \Omega_k)|$$

$$|Q \cap \tilde{\Omega}_k| - |Q \cap \Omega_{k+1}| \ge |Q| - \frac{|Q|}{2} = C't^n.$$

Therefore

$$C2^{2k}|\Omega_k| \ge 2^{2k}|\tilde{\Omega}_k| \ge \sum_r \int_{\tilde{\Omega}_k\setminus\Omega_{k+1}} (Sf_r)^2(x)dx$$

$$= \sum_r \int_{\mathbb{C}^n}\int_0^\infty \int_{\mathbb{C}^n} |(\psi_r)_t * f_r(\mathcal{Y})|^2 \chi(x,\mathcal{Y},t)\frac{d\mathcal{Y}dtdx}{t^{n+1}}$$

$$\ge \sum_{Q\in B_k} \int_{\hat{Q}}\int_{\mathbb{C}^n} |(\psi_r)_t * f_r(\mathcal{Y})|^2 \chi(x,\mathcal{Y},t)\frac{d\mathcal{Y}dtdx}{t^{n+1}}$$

$$\ge C'\left\{\sum_{Q\in B_k}\sum_r \int_{\hat{Q}} |(\psi_r)_t * f_r(\mathcal{Y})|^2 \frac{d\mathcal{Y}dt}{t}\right\}. \qquad (5)$$

Now by duality argument and Hölder's inequality, we have

$$\left\|\sum_{k>N}\sum_{Q\in B_k}\sum_{r}\int_{\tilde{Q}}(\psi_r)_t(x-\mathcal{Y})\psi_t * f_r(\mathcal{Y})\frac{d\mathcal{Y}dt}{t}\right\|_2$$

$$= sup_{\|g_r\|_2\le 1}\left|<\sum_r\sum_{k>N}\sum_{Q\in B_k}\int_{\tilde{Q}}(\psi_r)_t(x-\mathcal{Y})(\psi_r)_t * f_r(\mathcal{Y})\frac{d\mathcal{Y}dt}{t}, g_r>\right|$$

$$\le sup_{\|g_r\|_2\le 1}\sum_r\sum_{k>N}\sum_{Q\in B_k}\int_{\hat{Q}}(\psi_r)_t(x-\mathcal{Y})(\psi_r)_t * f_r(\mathcal{Y})\frac{d\mathcal{Y}dt}{t}$$

$$\le sup_{\|g_r\|_2\le 1}\left\{\sum_r\sum_{k>N}\sum_{Q\in B_k}\int_{\hat{Q}}|(\psi_r)_t * g_r(\mathcal{Y})|^2\frac{d\mathcal{Y}dt}{t}\right\}^{\frac{1}{2}}$$

$$\left\{\sum_r\sum_{k>N}\sum_{Q\in B_k}\int_{\hat{Q}}|(\psi_r)_t * f_r(\mathcal{Y})|^2\frac{d\mathcal{Y}dt}{t}\right\}^{\frac{1}{2}}$$

$$\le sup_{\|g_r\|_2\le 1}\sum_r\left\{\int_{\mathbb{C}_+^{n+1}}|(\psi_r)_t * f_r(\mathcal{Y})|^2\frac{d\mathcal{Y}dt}{t}\right\}^{\frac{1}{2}}\left\{\sum_r\sum_{k>N}\sum_{Q\in B_k}\int_{\hat{Q}}|(\psi_r)_t * f_r(\mathcal{Y})|^2\frac{d\mathcal{Y}dt}{t}\right\}^{\frac{1}{2}}$$

$$\le C\left\{\sum_r\sum_{k>N}\sum_{Q\in B_k}\int_{\hat{Q}}|(\psi_r)_t * f_r(\mathcal{Y})|^2\frac{d\mathcal{Y}dt}{t}\right\}^{\frac{1}{2}}, \qquad (6)$$

Where the last inequality follows from the L^2 estimates of the Littlewood-Paley square series function

$$\left\{\sum_r\int_{\mathbb{C}_+^{n+1}}|(\psi_r)_t * f_r(\mathcal{Y})|^2\frac{d\mathcal{Y}dt}{t}\right\}^{\frac{1}{2}} \le \sum_r C\|g_r\|_{L^2(\mathbb{C}^n)}.$$

Then the estimate in (5) implies that

$$\left\|\sum_r\sum_{k>N}\sum_{Q\in B_k}\int_{\hat{Q}}(\psi_r)_t(x-\mathcal{Y})(\psi_r)_t * f_r(\mathcal{Y})\frac{d\mathcal{Y}dt}{t}\right\|_2 \le C(\sum_{k>N}2^{2k|\Omega_k|})^{\frac{1}{2}}.$$

The last term tends to zero as N goes to infinity is because of

$$\sum_k 2^{2k}|\Omega_k| \le \sum_r C\|S(f_r)\|_2^2 \le \sum_r C\|f_r\|_2^2 < \infty.$$

Thus (4) hold, and the series converges in $L^2(\mathbb{C}^n)$.

Moreover,(4) gives an atomic decomposition of $H^{\varepsilon+1}(\mathbb{C}^n)$. To see this, we denote

$$\sum_r(\mathrm{b}_r)_{\tilde{Q}}(x) = \sum_r\sum_{Q\subseteq\tilde{Q}\cap B_k}\int_{\hat{Q}}(\psi_r)_t(x-\mathcal{Y})(\psi_r)_t * f_r(\mathcal{Y})\frac{d\mathcal{Y}dt}{t},$$

then it is easy to see that $(b_r)_{\tilde{Q}}(x)$ is supported in $5\tilde{Q}$(the same center and n times side length of$\tilde{Q}$). By Hölder's inequality,

$$\sum_r\left\|(b_r)_{\tilde{Q}}(x)\right\|_{\varepsilon+1}^{\varepsilon+1} = \left\|\sum_r\sum_{Q\subseteq\tilde{Q}\cap B_k}\int_{\tilde{Q}}(\psi_r)_t(x-\mathcal{Y})(\psi_r)_t * f_r(\mathcal{Y})\frac{d\mathcal{Y}dt}{t}\right\|_{\varepsilon+1}^{\varepsilon+1}$$

$$\le \left|n\tilde{Q}\right|^{\left(1-\frac{\varepsilon+1}{2}\right)}\left\|\sum_r\sum_{Q\subseteq\tilde{Q}\cap B_k}\int_{\tilde{Q}}(\psi_r)_t(x-\mathcal{Y})(\psi_r)_t * f_r(\mathcal{Y})\frac{d\mathcal{Y}dt}{t}\right\|_2^{\varepsilon+1}. \qquad (7)$$

Using duality argument again, we obtain

$$\left\|\sum_r\sum_{Q\subseteq\tilde{Q}\cap B_k}\int_{\tilde{Q}}(\psi_r)_t(x-\mathcal{Y})(\psi_r)_t * f_r(\mathcal{Y})\frac{d\mathcal{Y}dt}{t}\right\|_2$$

$$= sup_{\sum_r\|g_r\|_2\le 1}\left|<\sum_r\sum_{Q\subseteq\tilde{Q}\cap B_k}\int_{\tilde{Q}}(\psi_r)_t(x-\mathcal{Y})(\psi_r)_t * f_r(\mathcal{Y})\frac{d\mathcal{Y}dt}{t}, g_r>\right|$$

$$\leq sup_{\sum_r \|g_r\|_2 \leq 1} \sum_r \sum_{Q\subseteq\tilde{Q}\cap B_k} \int_{\hat{Q}} |(\psi_r)_t * g_r(\mathcal{Y})(\psi_r)_t * f_r(\mathcal{Y})| \frac{d\mathcal{Y}dt}{t}$$

$$\leq sup_{\sum_r \|g_r\|_2 \leq 1} \left\{\sum_r \sum_{Q\subseteq\tilde{Q}\cap B_k} \int_{\hat{Q}} |(\psi_r)_t * g_r(\mathcal{Y})|^2 \frac{d\mathcal{Y}dt}{t}\right\}^{\frac{1}{2}} \cdot \left\{\sum_r \sum_{Q\subseteq\tilde{Q}\cap B_k} \int_{\hat{Q}} |(\psi_r)_t * f_r(\mathcal{Y})|^2 \frac{d\mathcal{Y}dt}{t}\right\}^{\frac{1}{2}}$$

$$\leq C \left\{\sum_r \sum_{Q\subseteq\tilde{Q}\cap B_k} \int_{\hat{Q}} |(\psi_r)_t * f_r(\mathcal{Y})|^2 \frac{d\mathcal{Y}dt}{t}\right\}^{\frac{1}{2}},$$

Where the last inequality also follows from the L^2 estimate of the littlewood-Paley square series function as the same as in (6) used. Hence, together with the cancellation condition of ψ_r, it is easy to see that if we set

$$\sum_r (a_r)_{\tilde{Q}}(x) = C\left|n\tilde{Q}\right|^{(\frac{1}{2}-\frac{1}{\varepsilon+1})} \left\{\sum_r \sum_{Q\subseteq\tilde{Q}\cap B_k} \int_{\hat{Q}} |(\psi_r)_t * f_r(\mathcal{Y})|^2 \frac{d\mathcal{Y}dt}{t}\right\}^{-\frac{1}{2}}$$
$$\times \sum_r \sum_{Q\subseteq\tilde{Q}\cap B_k} \int_{\hat{Q}} (\psi_r)_t(x-\mathcal{Y})(\psi_r)_t * f_r(\mathcal{Y}) \frac{d\mathcal{Y}dt}{t}$$

For a suitable constant C,then $(a_r)_{\tilde{Q}}(x)$ is an $(\varepsilon+1,2)$-atom. Finally, by (5), we obtain

$$\sum_k \sum_{\tilde{Q}\in B_k} |\lambda_{\tilde{Q}}|^{\varepsilon+1} = \sum_k \sum_{\tilde{Q}\in B_k} \left|5\tilde{Q}\right|^{(1-\frac{\varepsilon+1}{2})} \left\{\sum_r \sum_{Q\subseteq\tilde{Q}\cap B_k} \int_{\hat{Q}} |(\psi_r)_t * f_r(\mathcal{Y})|^2 \frac{d\mathcal{Y}dt}{t}\right\}^{\frac{\varepsilon+1}{2}}$$
$$\sum_k |\Omega_k|^{(1-\frac{\varepsilon+1}{2})} 2^{2k} |\Omega_k|^{\frac{\varepsilon+1}{2}} \leq \sum_r C\|S(f_r)\|_{\varepsilon+1}^{\varepsilon+1} = C\|f_r\|_{H^{\varepsilon+1}}^{\varepsilon+1}.$$

Therefore, we have the new atomic decomposition of $H^{\varepsilon+1}(\mathbb{C}^n)$

$$\sum_r f_r(x) = \sum_k \sum_{\tilde{Q}\in B_k} \lambda_{\tilde{Q}}(a_r)_{\tilde{Q}}(x) \qquad (8)$$

Which converges in $L^2(\mathbb{C}^n)$ .This ends the proof of **Corollary** 4.

Corollary 5. fix $\varepsilon \leq 0.$ *Let* T_{r-1} be a linear a series operators which is bounded on $L^2(\mathbb{C}^n)$. $(i)T_{r-1}$ can be extended to a bounded a series operators from $H^{\varepsilon+1}(\mathbb{C}^n)$ to $L^{\varepsilon+1}(\mathbb{C}^n)$ if and only if $\|T_{r-1}a_r\|_{\varepsilon+1} \leq C$ for all $(\varepsilon+1,2)$-atoms, where the constant C is independent of a; $(ii)T_{r-1}$ can be extended to a bounded a series operator from $H^{\varepsilon+1}(\mathbb{C}^n)$ to $H^{\varepsilon+1}(\mathbb{C}^n)$ if and only if $\|T_{r-1}a_r\|_{H^{\varepsilon+1}} \leq C$ for all $(\varepsilon+1,2)$-atoms, where the constant C is also independent of a_r.

Proof. Suppose that a linear series operators T_{r-1} is bounded on $L^2(\mathbb{C}^n)$ and $\|T_{r-1}(a_r)\|_{\varepsilon+1} \leq C$ uniformly on all $(\varepsilon+1,2)$-atoms. By Corollary 4, for any $f_r \in H^{\varepsilon+1}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n),\ \varepsilon \leq 0,$ we obtain

$$\sum_r \|T_{r-1}f_r\|_{\varepsilon+1}^{\varepsilon+1} = \left\|\sum_r \sum_k \sum_{\tilde{Q}\in B_k} T_{r-1}\left(\sum_{Q\subseteq\tilde{Q}\cap B_k} \int_{\hat{Q}} (\psi_r)_t(0-\mathcal{Y})(\psi_r)_t * f_r(\mathcal{Y}) \frac{d\mathcal{Y}dt}{t}\right)\right\|_{\varepsilon+1}^{\varepsilon+1}$$
$$\leq C \sum_k \sum_{\tilde{Q}\in B_k} \left|n\tilde{Q}\right|^{(1-\frac{\varepsilon+1}{2})} \left\{\sum_r \sum_{Q\subseteq\tilde{Q}\cap B_k} \int_{\hat{Q}} |(\psi_r)_t * f_r(\mathcal{Y})|^2 \frac{d\mathcal{Y}dt}{t}\right\}^{\frac{\varepsilon+1}{2}}$$
$$\leq C \sum_k |\Omega_k|^{(1-\frac{\varepsilon+1}{2})} 2^{2k} |\Omega_k|^{\frac{\varepsilon+1}{2}} \leq \sum_r C\|S(f_r)\|_{\varepsilon+1}^{\varepsilon+1} = \sum_r C\|f_r\|_{H^{\varepsilon+1}}^{\varepsilon+1}.$$

Where the equality follows from the fact that the L^2 convergence of the series implies the convergence for almost everywhere, and the first inequality then follows from the uniform boundedness of T_{r-1} on all $(\varepsilon+1,2)$-atoms in $L^{\varepsilon+1}(\mathbb{C}^n)$ and same estimate as (7).

Similarly, since the decomposition in (4) (or in (8)) converges in $L^2(\mathbb{C}^n)$, as a consequence, it also converges in S'. Applying Lusin series functions and taking the pth-power of $L^{\varepsilon+1}$norm to the both sides in (4) yield

$$\sum_r \|T_{r-1}f_r\|_{H^{\varepsilon+1}}^{\varepsilon+1} = \sum_r \sum_k \sum_{\tilde{Q}\in B_k} \left\|T_{r-1}\left(\sum_{Q\subseteq\tilde{Q}\cap B_k} \int_{\hat{Q}} (\psi_r)_t(0-\mathcal{Y})(\psi_r)_t * f_r(\mathcal{Y}) \frac{d\mathcal{Y}dt}{t}\right)\right\|_{\varepsilon+1}^{\varepsilon+1}.$$

Using the fact that T_{r-1} is bounded uniformly on all $(\varepsilon+1,2)$-atoms in $H^{\varepsilon+1}$ and repeating the same estimate above give

$$\sum_r \|T_{r-1}f_r\|_{H^{\varepsilon+1}}^{\varepsilon+1} \le C\sum_r\sum_k\sum_{\tilde{Q}\in B_k}\left|n\tilde{Q}\right|^{(1-\frac{\varepsilon+1}{2})}\left\{\sum_{Q\subseteq\tilde{Q}\cap B_k}\int_{\hat{Q}}|(\psi_r)_t * f_r(\mathcal{Y})|^2\frac{d\mathcal{Y}dt}{t}\right\}^{\frac{\varepsilon+1}{2}}$$
$$\le C\sum_k|\Omega_k|^{(1-\frac{\varepsilon+1}{2})}2^{2k}\,|\Omega_k|^{\frac{\varepsilon+1}{2}} \le \sum_r C\|S(f_r)\|_{\varepsilon+1}^{\varepsilon+1} = \sum_r C\|f_r\|_{H^{\varepsilon+1}}^{\varepsilon+1}.$$

Since $L^2\cap H^{\varepsilon+1}$ is dense in $H^{\varepsilon+1}(\mathbb{C}^n)$, the parts of Corollary 5 are showed, and hence the proof of Corollary 5 is complete.

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