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Fixed Point Theorem in Fuzzy Metric Space Using Compatibility of Type (β)

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Abstract: *In this paper, a fixed point theorem for six self-maps has been established using the concept of compatible maps of type (β), which generalizes the result of Jain et. al. [5].*
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I. Introduction

The concept of Fuzzy sets was initially investigated by Zadeh [15] as a new way to represent vagueness in everyday life. Subsequently, it was developed by many authors and used in various fields. To use this concept in Topology and Analysis, several researchers have defined Fuzzy metric space in various ways. In this paper we deal with the Fuzzy metric space defined by Kramosil and Michalek [10] and modified by George and Veeramani [3]. Recently, Grabiec [4] has proved fixed point results for Fuzzy metric space. In the sequel, Singh and Chauhan [13] introduced the concept of compatible mappings of Fuzzy metric space and proved the common fixed point theorem. Pathak, Chang and Cho [12] introduced the concept of compatible mapping of type (P). Jain and Singh [6] proved a fixed point theorem for six self maps in a fuzzy metric space.
For the sake of completeness, we recall some definitions and known results in Fuzzy metric space.

II. Preliminaries

Definition 2.1. [11] A binary operation * : $[0,1]\times[0,1]\to[0,1]$ is called a *t-norm* if ([0, 1], *) is an abelian topological monoid with unit 1 such that $a*b\le c*d$ whenever $a\le c$ and $b\le d$ for a, b, c, d $\in$ [0, 1].
Examples of t-norms are a * b = ab and a * b = min {a, b}.
Definition 2.2. [1] The 3-tuple (X, M, *) is said to be a *Fuzzy metric space* if X is an arbitrary set, * is a continuous t-norm and M is a Fuzzy set in $X^2\times[0,\infty)$ satisfying the following conditions: for all x, y, z $\in$ X and s, t > 0

(FM-1) M (x, y, 0) = 0,
(FM-2) M (x, y, t) =1 for all t > 0 if and only if x = y,
(FM-3) M (x, y, t) = M (y, x, t),
(FM-4) M (x, y, t) * M (y, z, s) ≤ M (x, z, t + s),
(FM-5) M (x, y, .) : [0,∞) $\to$ [0, 1] is left continuous,
(FM-6) $\lim_{t\to\infty}$ M (x, y, t) =1.

Note that M(x, y, t) can be considered as the degree of nearness between x and y with respect to t. We identify x = y with M(x, y, t) = 1 for all t > 0. The following example shows that every metric space induces a Fuzzy metric space.
Example 2.1. [11] Let (X, d) be a metric space. Define a * b = min {a, b} and $M(x,y,t)=\dfrac{t}{t+d(x,y)}$ for all $x,y\in X$ and $t>0$. Then (X, M, *) is a Fuzzy metric space. It is called the Fuzzy metric space induced by d.

Definition 2.3. [11] A sequence $\{x_n\}$ in a Fuzzy metric space (X, M, *) is said to be a *Cauchy sequence* if and only if for each ε > 0, t > 0, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all n, m $\geq n_0$.
The sequence $\{x_n\}$ is said to *converge* to a point x in X if and only if for each ε > 0, t > 0 there exists $n_0 \in N$ such that $M(x_n, x, t) > 1 - \varepsilon$ for all n,m $\geq n_0$.
A Fuzzy metric space (X, M, *) is said to be *complete* if every Cauchy sequence in it converges to a point in it.
Definition 2.4. [13] Self mappings A and S of a Fuzzy metric space (X, M, *) are said to be *compatible* if and only if $M(ASx_n, SAx_n, t) \rightarrow 1$ for all t > 0, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Ax_n \rightarrow p$ for some p in X as $n \rightarrow \infty$.
Definition 2.5. [14] Self maps A and S of a Fuzzy metric space (X, M, *) are said to be compatible maps of type (β) if
$M(AAx_n, SSx_n, t) \rightarrow 1$ for all t > 0,
whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Ax_n \rightarrow p$ for some p in X as $n \rightarrow \infty$.
Definition 2.6. [6] Two maps A and B from a Fuzzy metric space (X, M, *) into itself are said to be weakly compatible if they commute at their coincidence points, i.e. Ax = Bx implies ABx = BAx for some $x \in X$.
Definition 2.7. [8] Self maps A and S of Fuzzy metric space (X, M, *) are said to be occasionally weakly compatible (owc) if and only if there is a point x in X which is coincidence point of A and S at which A and S both commute.
Remark 2.1. [14] The concept of compatible maps of type (β) and weak compatibility is more general than the concept of compatible maps in a Fuzzy metric space.
Proposition 2.1. [4] In a fuzzy metric space (X, M, *), limit of a sequence is unique.
Lemma 2.1. [4] Let (X, M, *) be a fuzzy metric space. Then for all x, y $\in$ X, M (x, y, .) is a non-decreasing function.
Lemma 2.2. [11] Let (X, M, *) be a fuzzy metric space. If there exists $k \in (0, 1)$ such that for all x, y $\in$ X, $M(x, y, kt) \geq M(x, y, t)$ $\forall$ t > 0, then x = y.
Lemma 2.3. [11] Let $\{x_n\}$ be a sequence in a fuzzy metric space (X, M, *). If there exists a number k $\in$ (0, 1) such that $M(x_{n+2}, x_{n+1}, kt) \geq M(x_{n+1}, x_n, t)$ $\forall$ t > 0 and n $\in$ N. Then $\{x_n\}$ is a Cauchy sequence in X.
Lemma 2.4. [7] The only t-norm * satisfying $r * r \geq r$ for all r $\in$ [0, 1] is the minimum t-norm, that is a * b = min {a, b} for all a, b $\in$ [0, 1].

Example 2.1. Let (X, d) be a metric space. Define a * b = min{a, b} and $M(x, y, t) = \frac{t}{t + d(x, y)}$ for all x, y $\in$ X and all t > 0. Then (X, M, *) is a Fuzzy metric space. It is called the Fuzzy metric space induced by d.
Define self maps I and L as follows :

$$I(x) = x \quad \text{for all } x \in X \quad \text{and} \quad L(x) = \begin{cases} x, & \text{if } 0 \leq x < \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} \leq x \end{cases}$$

Taking $x_n = \frac{1}{2} - \frac{1}{n}$, we get $Ix_n = x_n = \frac{1}{2} - \frac{1}{n}$ and $Lx_n = \frac{1}{2} - \frac{1}{n}$.

Thus, $Lx_n \rightarrow \frac{1}{2}$ as $n \rightarrow \infty$ and $Ix_n \rightarrow \frac{1}{2}$, as $n \rightarrow \infty$.

Hence, $x = \frac{1}{2}$

Therefore, $IIx_n = I\left(\frac{1}{2} - \frac{1}{n}\right) = \frac{1}{2} - \frac{1}{n}$

and $LLx_n = L\left(\frac{1}{2}-\frac{1}{n}\right) = \frac{1}{2}-\frac{1}{n}$.

Consider $\lim_{n\to\infty} M(IIx_n, LLx_n, t) = \lim_{n\to\infty} M\left(\frac{1}{2}-\frac{1}{n}, \frac{1}{2}-\frac{1}{n}, t\right) = 1$ for $t > 0$.

Therefore, by definition, (I, L) is compatible mapping of type (β).

Now, $\lim_{n\to\infty} M(ILx_n, Lx, t) = \lim_{n\to\infty} M\left(\frac{1}{2}-\frac{1}{n}, 1, t\right) < 1$ for $t > 0$.

Therefore, (I, L) is not semi-compatible mapping. Thus the pair (I, L) of self maps is compatible of type (β) but not semi-compatible.

Remark 2.2. In view of above example, it follows that the concept of compatible maps of type (β) is more general than that of semi-compatible maps.

III. Main Result

Theorem 3.1. Let (X, M,*) be a complete Fuzzy Metric space. Let A, B, S, T, P and Q be self-mappings from X into itself such that the following conditions are satisfied:

(3.1.1) $P(X) \subset ST(X), Q(X) \subset AB(X)$;

(3.1.2) $AB = BA, ST = TS, PB = BP, QT = TQ$;

(3.1.3) either P or AB is continuous;

(3.1.4) the pair (P, AB) is compatible type of (β) and (Q, ST) is occasionally weak compatible;

(3.1.5) there exists $k \in (0,1)$ such that $\forall x, y \in X$ and $t > 0$,

$$M(Px, Qy, kt) \geq \min\{M(Qy, STy, t), M(ABx, STy, t), M(Px, ABx, t)\}.$$

Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point.

As $P(X) \subset ST(X), Q(X) \subset AB(X)$ then there exists $x_1, x_2 \in X$ such that $Px_0 = STx_1$ and $Qx_1 = ABx_2$.

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$Px_{2n} = STx_{2n+1} = y_{2n+1} \text{ and } Qx_{2n+1} = ABx_{2n+2} = y_{2n+2}, \tag{1}$$

for n = 0, 1, 2, …

Step I. Now put $x = x_{2n}$ and $y = x_{2n+1}$ in (3.1.5) we have,

$$M(Px_{2n}, Qx_{2n+1}, kt) \geq \min\{M(Qx_{2n+1}, STx_{2n+1}, t), M(ABx_{2n}, STx_{2n+1}, t), M(Px_{2n}, ABx_{2n}, t)\}$$

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, kt) &\geq \min\{M(y_{2n+2}, y_{2n+1}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n}, t)\} \\ &\geq \min\{M(y_{2n+1}, y_{2n+2}, t), M(y_{2n}, y_{2n+1}, t)\}. \end{aligned} \tag{2}$$

From Lemma 2.1 and 2.2 we have,

$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n+1}, y_{2n+2}, t).$$

Thus we have,

$$M(y_{2n+1}, y_{2n+2}, t) \geq \min\{M(y_{2n+1}, y_{2n+2}, t/k), M(y_{2n}, y_{2n+1}, t/k)\} \tag{3}$$

By (2) and (3) we have,

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, t) &\geq \min\{M(y_{2n+1}, y_{2n+2}, t/k), M(y_{2n}, y_{2n+1}, t/k), M(y_{2n}, y_{2n+1}, t)\} \\ &= \min\{M(y_{2n+1}, y_{2n+2}, t/k), M(y_{2n}, y_{2n+1}, t)\} \\ &\geq \min\{M(y_{2n+1}, y_{2n+2}, t/k^2), M(y_{2n}, y_{2n+1}, t/k^2), M(y_{2n}, y_{2n+1}, t)\} \\ &= \min\{M(y_{2n+1}, y_{2n+2}, t/k^2), M(y_{2n}, y_{2n+1}, t)\} \\ &\geq \dots \dots \dots \end{aligned}$$

$$\geq \min\{M(y_{2n+1}, y_{2n+2}, t/k^n), M(y_{2n}, y_{2n+1}, t)\}.$$

Taking limit as $n \to \infty$, we have

$$M(y_{2n+1}, y_{2n+2}, t) \geq M(y_{2n}, y_{2n+1}, t), \quad \forall \; t > 0.$$

Similarly, we also have

$$M(y_{2n+2}, y_{2n+3}, t) \geq M(y_{2n+1}, y_{2n+2}, t), \quad \forall \; t > 0.$$

Thus for all $n, t > 0$,

$$M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t).$$

Therefore,

$$M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, t/k)$$
$$\geq M(y_{n-2}, y_{n-1}, t/k^2)$$
$$\geq M(y_{n-3}, y_{n-2}, t/k^3)$$
$$\geq \ldots \; \ldots \; \ldots$$
$$\geq M(y_0, y_1, t/k^n).$$

Hence, $\lim_{n\to\infty} M(y_n, y_{n+1}, t) = 1, \; \forall \; t > 0.$

Now for any integer p, we have,

$$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, t/p) * M(y_{n+1}, y_{n+2}, t/p) * M(y_{n+2}, y_{n+3}, \ldots$$
$$* \ldots * M(y_{n+p-1}, y_{n+p}, t/p).$$

Therefore

$$\lim_{n\to\infty} M(y_n, y_{n+p}, t) = 1 * 1 * 1 * \ldots * 1$$

i.e. $\lim_{n\to\infty} M(y_n, y_{n+p}, t) = 1$.

This shows that $\{y_n\}$ is a Cauchy sequence in X. Since (X, M,*) is complete, $\{y_n\}$ converges to some point $z \in X$. Also its subsequences converge to the same point i.e. $z \in X$.

i.e. $\{Px_{2n}\} \to z$ and $\{Qx_{2n+1}\} \to z$ (4)

$\{ABx_{2n}\} \to z$ and $\{STx_{2n+1}\} \to z$ (5)

Case I: Suppose AB is continuous.
Since AB is continuous, we have

$$AB(ABx_{2n}) \to ABz \text{ and } AB(Px_{2n}) \to ABz.$$

As (P, AB) is compatible pair of type (β), we have

$M(PPx_{2n}, AB(AB)x_{2n}, t) = 1$, for all $t > 0$

or $M(PPx_{2n}, ABz, t) = 1.$

Therefore, $PPx_{2n} \to ABz.$

Step II: We shall prove that ABz = z.

Put $x = ABx_{2n}$ and $y = x_{2n+1}$ in (3.1.5), we have

$$M(P(AB)x_{2n}, Qx_{2n+1}, kt) \geq \min\{M(Qx_{2n+1}, STx_{2n+1}, t), \; M(AB(AB)x_{2n}, STx_{2n+1}, t),$$
$$M(P(AB)x_{2n}, AB(AB)_{2n}$$
$$M(AB(P)x_{2n}, Qx_{2n+1}, kt) \geq \min\{M(Qx_{2n+1}, STx_{2n+1}, t), M(AB(AB)x_{2n}, STx_{2n+1}, t),$$
$$M(AB(P)x_{2n}, AB(AB)x_{2n}, t)\}.$$

Taking limit as $n \to \infty$ and using (4) and (5) we have,

$$M(ABz, z, kt) \geq \min\{M(z, z, t), M(ABz, z, t), M(ABz, ABz, t)\}$$
$$= \min\{M(ABz, ABz, t), M(ABz, z, t), M(z, z, t)$$

i.e. $M(ABz, z, kt) \geq M(ABz, z, t)$.

Therefore, by using Lemma 2.2, we have

$$ABz = z. \tag{6}$$

Step III: We shall prove that Pz = z.

Put x = z and $y = x_{2n+1}$ in (3.1.5) we have,

$$M(Pz, Qx_{2n+1}, kt) \geq \min\{M(Qx_{2n+1}, STx_{2n+1}, t), M(ABz, STx_{2n+1}, t), M(Pz, ABz, t)\}.$$

Taking limit as $n \to \infty$ and using (4) and (5), we have

$$M(Pz, z, kt) \geq \min\{M(z, z, t), M(ABz, z, t), M(Pz, ABz, t)\}$$
$$= \min\{M(z, z, t), M(z, z, t), M$$
$$= \min\{M(Pz, z, t), M($$

i.e. $M(Pz, z, kt) \geq M(Pz, z, t)$

Therefore, by using Lemma 2.2, we have

$$Pz = z.$$

Therefore,

$$ABz = Pz = z. \tag{7}$$

Step IV: We shall prove that Bz = z.

Put $x = Bz$ and $y = x_{2n+1}$ in (3.1.5) we have,

$$M(PBz, Qx_{2n+1}, kt) \geq \min\{M(Qx_{2n+1}, STx_{2n+1}, t), M(ABBz, STx_{2n+1}, t),$$
$$M(PBz, ABBz, t)\}.$$

As $BP = PB$ and $AB = BA$, we have

$$P(Bz) = B(Pz) = Bz \text{ and}$$
$$(AB)(Bz) = (BA)(Bz) = B(ABz) = Bz.$$

Taking limit as $n \to \infty$ and using (4) and (5) we have,

$$M(Bz, z, kt) \geq \min\{M(z, z, t), M(Bz, z, t), M(Bz, Bz, t)\}$$
$$= \min\{M(Bz, Bz, t), M(Bz, z, t), M(z, z, t)\}$$

i.e. $MBz, z, kt) \geq M(Bz, z, t)$

Therefore, by using Lemma 2.2, we have

$$Bz = z. \tag{8}$$

Also from (7),

$$ABz = z.$$

Then $Az = z$ (9)

Therefore, from (7), (8) and (9), we have

$$Az = Bz = Pz = z. \tag{10}$$

Step V: We shall prove that $STz = Qz$

As $P(X) \subset ST(X)$, there exists $u \in X$ such that $z = Pz = STu.$

Put $x = x_{2n}$ and $y = u$ in (3.1.5) we have,

$$M(Px_{2n}, Qu, kt) \quad \min\{M(Qu, STu, t), M(ABx_{2n}, STu, t), M(Px_{2n}, ABx_{2n}, t)\},$$

Taking limit as $n \to \infty$ and using (4) and (5), we have

$$M(z, Qu, kt) \quad \min\{M(Qu, z, t), M(z, z, t)\}$$
$$\geq M(Qu, z, t)$$

i.e. $M(z, Qu, kt) \geq M(z, Qu, t)$

Therefore, by using Lemma 2.2, we have

$$Qu = z.$$

Hence, $STu = z = Qu$.

Since (Q, ST) is occasionally weak compatible, therefore we have

$$QSTu = STQu.$$

Thus, $Qz = STz$. (11)

Step VI: We shall prove that $Qz = z$

Put $x = x_{2n}$ and $y = z$ in (3.1.5) we have,

$M(Px_{2n}, Qz, kt) \geq \min\{M(Qz, STz, t), M(ABx_{2n}, STz, t), M(Px_{2n}, ABx_{2n}, t)\}$.

Taking limit as $n \to \infty$ and using (4) and (5) and step 5, we have

$M(z, Qz, kt) \geq \min\{M(Qz, Qz, t), M(z, Qz, t), M(z, z, t)\}$

i.e. $M(z, Qz, kt) \geq M(z, Qz, t)$.

Therefore, by using Lemma 2.2, we have

$Qz = z$. (12)

Step VII: We shall prove that Tz = z.

Put $x = x_{2n}$ and $y = Tz$ in (3.1.5), we have

$M(Px_{2n}, QTz, kt) \geq \min\{M(QTz, STTz, t), M(ABx_{2n}, STTz, t),$
$M(Px_{2n}, ABx_{2n}, t)\}$.

As QT = TQ and ST = TS, we have

$QTz = TQz = Tz$ and $ST(Tz) = T(STz) = T(Qz) = Tz$.

Taking limit as $n \to \infty$ and using (4) and (5), we have

$M(z, Tz, kt) \geq \min\{M(Tz, Tz, t), M(z, Tz, t), M(z, z, t)\}$

i.e. $M(z, Tz, kt) \geq M(z, Tz, t)$.

Therefore, by using Lemma 2.2, we have

$Tz = z$. (13)

Now $STz = Tz = z$ implies $Sz = z$. (14)

Hence by (10), (12), (13) and (14), we have

$Az = Bz = Qz = Tz = Sz = z$.

Hence, z is a common fixed point of A, B, S, T, P and Q.

Case II: Suppose P is continuous.

As P is continuous,

$P(Px_{2n}) = P^2x_{2n} \to Pz$

and $P(ABx_{2n}) \to Pz$.

As (P, AB) is compatible pair of type (β),

$\lim_{n\to\infty} M(PPx_{2n}, (AB)(AB)x_{2n}, t) = 1$, for all $t > 0$.

or $M(Pz, (AB)(AB)x_{2n}, t) = 1$.

Therefore $AB^2x_{2n} \to Pz$

Step VIII: We shall prove that Pz = z.

Put $x = Px_{2n}$ and $y = x_{2n+1}$ in (3.1.5) we have,

$M(PPx_{2n}, Qx_{2n+1}, kt) \geq \min\{M(Qx_{2n+1}, STx_{2n+1}, t), M(ABPx_{2n}, STx_{2n+1}, t),$
$M(PPx_{2n}, ABPx_{2n}, t)\}$

$= \min\{M(Qx_{2n+1}, STx_{2n+1}, t), M(P(AB)x_{2n}, STx_{2n+1}, t),$
$M(P(P)x_{2n}, P(AB)x_{2n}, t)\}$.

Taking limit as $n \to \infty$ and using (4) and (5), we have

$M(Pz, z, kt) \geq \min\{M(z, z, t), M(Pz, z, t), M(Pz, Pz, t)\}$

$= \min\{M(Pz, Pz, t), M(Pz, Pz, t), M(z, z, t)\}$.

i.e. $M(Pz, z, kt) \geq M(Pz, z, t)$.

Therefore, by using Lemma 2.2, we have

$Pz = z$. (15)

Step IX: Put $x = ABx_{2n}$ and $y = x_{2n+1}$ in (3.1.5), we have

$M(PABx_{2n}, Qx_{2n+1}, kt) \geq \min\{M(Qx_{2n+1}, STx_{2n+1}, t), M(AB(AB)x_{2n}, STx_{2n+1}, t),$
$M(P(AB)x_{2n}, AB(AB)x_{2n}, t)\}$

$= \min\{M(Qx_{2n+1}, STx_{2n+1}, t), M(AB(AB)x_{2n}, STx_{2n+1}, t),$
$M(AB(P)x_{2n}, AB(AB)x_{2n}, t)\}$

Taking limit as $n \to \infty$ and using (4) and (5), we have

$M(ABz, z, kt) \geq \min\{M(z, z, t), M(ABz, z, t), M(ABz, ABz, t)\}$

i.e. $M(ABz, z, kt) \geq M(ABz, z, t)$.

Therefore, by using Lemma 2.2, we have

$$ABz = z.$$

Therefore from (15),

$$ABz = Pz = z. \quad (16)$$

Now apply step IV to get

$$Bz = z \text{ and so } Az = Bz = Pz = z. \quad (17)$$

Further applying Step V, VI and VII, we have

$$Qz = Sz = Tz = z. \quad (18)$$

Using (17) and (18), we get

$$Az = Bz = Pz = Qz = Sz = Tz = z$$

i.e. *z* is a common fixed point of the six maps A, B, S, T, P and Q in this case also.

Uniqueness: Let z_0 be another common fixed point of the maps A, B, S, T, P and Q, then

$$z = Az = Bz = Sz = Tz = Pz = Qz \quad \text{and}$$

$$z_0 = Az_0 = Bz_0 = Sz_0 = Tz_0 = Pz_0 = Qz_0.$$

Now put $x = z$ and $y = z_0$ in (3.1.5), we have

$M(Pz, Qz_0, kt) \geq \min\{M(Qz_0, STz_0, t), M(ABz, STz_0, t), M(Pz, ABz, t)\}$.

Therefore from (7) and (11), we have

$M(Pz, Qz_0, kt) \geq \min\{M(Qz_0, Qz_0, t), M(Pz, Qz_0, t), M(Pz, Pz, t)\}$

$M(z, z_0, kt) \geq \min\{M(z_0, z_0, t), M(z, z_0, t), M(z, z, t)\}$

i.e. $M(z, z_0, kt) \geq M(z, z_0, t)$.

Therefore, by using Lemma 2.2, we have

$$z = z_0.$$

Hence, z is the unique common fixed point of the six self maps A, B, S, T, P and Q.

This completes the Proof.

Remark 3.1. If we take B = T = I, the identity maps on *X* in Theorem 3.1, then condition (3.1.2) is satisfied trivially and we get the following corollary.

Corollary 3.1. Let (X, M,*) be a complete Fuzzy Metric space. Let A, S, P and Q be self-mappings from X into itself such that the following conditions are satisfied:

(3.2.1) $P(X) \subset S(X)$, $Q(X) \subset A(X)$;

(3.2.2) either A or P is continuous;

(3.2.3) the pair (P, A) is compatible type of (β) and (Q, S) is occasionally weak compatible;

(3.2.4) there exists $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$

$$M(Px, Qy, kt) \geq \min\{M(Qy, Sy, t), M(Ax, Sy, t), M(Px, Ax, t)\}.$$

Then A, S, P and Q have a unique common fixed point in X.

Proof. The proof is similar to the proof of theorem 3.1

IV. Conclusion

In view of remark 3.1, corollary 3.1 is a generalization of the result of Jain et. al. [5] in the sense that condition of semi-compatibility and occasionally weak compatibility of the pairs of self maps has been restricted to compatibility of type (β) and occasionally weakly compatible respectively and only one map of the first pair is needed to be continuous.

If we take Q = P in Corollary 3.2, we get the following corollary for three self maps.

Corollary 3.2. Let (X, M,*) be a complete Fuzzy Metric space. Let A, S and P be self-mappings from X into itself such that the following conditions are satisfied:

(3.2.5) $P(X) \subset A(X) \cap S(X)$;

(3.2.6) either A or P is continuous;

(3.2.7) the pair (P, A) is compatible type of (β) and (P, S) is occasionally weak compatible;

(3.2.8) there exists $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$

$$M(Px, Py, kt) \geq \min\{M(Py, Sy, t), M(Ax, Sy, t), M(Px, Ax, t)\}.$$

Then A, S and P have a unique common fixed point in X.

Proof. The proof is similar to the theorem 3.1.

If we take A = S = I, the identity map in Corollary 3.3, then the conditions (3.2.5), (3.2.6) and (3.2.7) are satisfied trivially and we get the following application.

4. An Application:

Theorem 4.1. Let P be a self map on a complete fuzzy metric space *(X, M, *)* such that for some $k \in (0, 1)$,

$$M(Px, Py, kt) \geq M(x, y, t) \text{ for all } x, y \in X, t > 0.$$

Then *A* has a unique common fixed point in *X*.

Proof. On taking only one factor in R.H.S. of the contraction (3.2.8), we obtain the Grabeic's [4] Banach contraction principle in fuzzy metric space.

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