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Real Analysis of Real Numbers- Cantor and Dedekind Real Number Structuring

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Abstract: *In this paper, the real number system is reconstructed with the Peano's axioms, to create systems of elements that demonstrate the properties of integer and rational numbers. Construction of two real number systems named Dedekind Real Number System that shows the completeness of the order and Cantor Real Number System that shows the Cauchy completeness are done and studied. During this process of systematic construction of real numbers, the integers and rational numbers can also be obtained. The process of constructing a real number system is made through mathematical concepts.*

Keywords: *Peano's Axiom, Archimedean Property, Order Completeness and Cauchy Completeness.*

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I. Introduction

The real number system possesses the Archimedean property and it is Cauchy complete. The real number system such as the natural number system reveals that the fundamental properties play an essential role in the construction process. This work enables to show that the real number system is logically necessary and gives out the importance of the real number system.

The real number system is reconstructed with the Peano's axioms to demonstrate the properties of integer and rational numbers. The real number system includes all the rational numbers including integers, fractions, irrational numbers, transcendental numbers and the square root of 2 also get included in the real numbers.

Peano's axiom describes the natural number system in an efficient manner. Axioms are used to build a concrete natural system; they also do not pose any problem with reference to the definition of the Integer System and Field of Rational. This can be well explained with a preceding system. Initially real numbers consisted only of the rational numbers because irrational numbers were not derived. A system which is closed with basic mathematical operations is possible by carrying out it from the natural system which is not closed.

Richard Dedekind's (1831-1916) and Georg Cantor's (1845-1918) intuition can be relied upon to help with the construction of real numbers. Therefore, these are classified as the Dedekind Real Number System and Cantor Real Number System. In this paper Peano's axiom, Dedekind Real Number System and Cantor Real Number System are used for reconstruction of real number system. Organization of the paper is with respective sections: Preliminaries, Properties of Dedekind Real Number Systems,

II. Preliminaries

Definition2.1: The five axioms of peano: We assume the existence of a set N with the following properties:

(i) There exists an element $1 \in N$.

(ii) For every $n \in N$, there exists an element $S(n) \in N$ such that $\{(n, S(n)) \mid n \in N\}$ is a function.

(iii) $1 \notin S(N)$.

(iv) S is one-one.

(v) If P is any subset of N such that $1 \in P$ and $S(n) \in P\ \forall n \in P$, then $P=N$.

Definition 2.2: The Dedekind's cut: A subset α of Q is called a cut if the following conditions are satisfied:

(i) $\alpha \neq \emptyset$, $\alpha \neq Q$.

(ii) For every $r \in \alpha$ and $s \in Q\backslash\alpha$, $r < s$.

(iii) Max α does not exist.

Definition 2.3: Rational Cuts: If r ∈ Q, then the set can be defined by α r = {x ∈ Q | x < r} is a cut. We call α_r a rational cut.
Definition 2.3: The negative cut: For any α ∈ R, the set defined by -α = {-s ∈ Q | s ∉ α, s ≠ min (Q\α)} is also a cut. This is called the negative cut of α.

Definition 2.4: The Dedekind Real Number System: An ordered field (R_D, ⊕, ⊙, ≻) is called a Dedekind real number system if
(i) There exist a subfield (Q_D, ⊕, ⊙, ≻) which is isomorphic to (Q, +,>).
(ii) (R_D, ⊕, ⊙, ≻) is order complete.

III. Properties Of Dedekind Real Number Systems

Most of the results proven here are usually regarded as properties fundamental to the real numbers. Order completeness of Dedekind real number systems are invoked in most of the proofs, and this may be how Dedekind stumble upon the idea that order completeness may just be the 'essence' of the real numbers. It is now started off with the Archimedean Property, which incidentally also employs the fact that Dedekind real number systems are order complete.

3.1: Archimedean Property
Theorem 3.1.1: Let x, y ∈ R, x>0. Then there exist n ∈ N such that $nx > y$. Let A = {nx| n ∈ N}.By contradiction assume, A is bounded above by y then Sup A exists (By order completeness). Since $x > 0$, $\sup A - x < supA$ and $\sup A - x$ is not an upper bound for A. Hence m ∈ N such that $\sup A - x < mx$, that is $\sup A < (m+1)x$. But $(m+1)x$ ∈ A and this contradict the fact that Sup A is an upper bound for A. Since the falsity of the claim leads to a contradiction, the claim must be true.

Lemma 3.1.2: Let x, y ∈ R. Then the following holds:
(i) There exist n ∈ N such that $n > y$.
(ii) If $x > 0$, there exist n ∈ N such that $x > \frac{1}{n}$.
(iii) If $y \geq 0$, there exist n ∈ N such that $n - 1 \leq y < n$.
Proof:
(i) This is a special case of Archimedean Property for $x = 1 > 0$.
(ii) By Archimedean Property, for $y = 1$, there exist n ∈ N such that $nx > 1$, that is $x > \frac{1}{n}$.
(iii) Consider the set A = {m ∈ N | y<m}. (i) Ensures that A is not empty.
Hence, since N is well-ordered, Min A = n exist. Then n-1∉ A.

Lemma 3.1.3: $\frac{\boldsymbol{R}}{\boldsymbol{Q}}$ is not empty.
Proof: Take Q ⊆ R, then Q = R. Hence, (Q, +,>) = (R, +,>) which is order complete.
But (Q, +,>) is not order complete. Hence, $\frac{\boldsymbol{R}}{\boldsymbol{Q}}$ cannot be empty. We call elements in $\frac{\boldsymbol{R}}{\boldsymbol{Q}}$ as irrational points**.**

Theorem3.1.4: Density theorem: Let x, y ∈ R be such that $x < y$. Then there exist r ∈ Q, z ∈ $\frac{\boldsymbol{R}}{\boldsymbol{Q}}$ such that $x < r,\ z\ < y$
Proof: Assume $x > 0$ and if $\frac{1}{(y-x)} > 0$ then $\frac{1}{(y-x)} < n$ (by Archimedean property) there exist n ∈ N. Therefore $ny - nx > 1$. Since $nx > 0$, there exist m ∈ N such that $m - 1 \leq nx + 1 < m$ (by Archimedean property). Then $m \leq nx + 1 < ny$ that is $m < nx < ny$. Hence $\frac{x<m}{n<y}$ and $r = \frac{m}{n}$ ∈ Q. If $x = 0$, then $y > 0$ and by Archimedean property, there exist n ∈ N such that $x = \frac{0<1}{n<y}$ and it is simply let r=1/n ∈ Q. If $x < 0$ and$y > 0$, then simply let $r = 0 \in Q$. Finally, if $x < y \leq 0$, then $-x > -y \geq 0$ and we have proved that there exist r′ ∈ Q such that–x>r′ >-y, that is $x < -r' < y$ and we let $r = -r'\ \in\ Q$. Since $\frac{\boldsymbol{R}}{\boldsymbol{Q}}$ is not empty there exist z′ ∈$\frac{\boldsymbol{R}}{\boldsymbol{Q}}$. By above result there exist r ∈ Q such that $x\ +\ z' < r < y\ +\ z'$, that is $x < r - z' < y$. we claim $z =\ r - z\ ' \notin\ Q$. Suppose not. Then there exist r′ ∈ Q such that $r - z' = r'$, that is $z' = r - r' \in\ Q$, a contradiction. The density theorem is proven.

3.2. Cauchy Completeness of R

Theorem 3.2.1: Let $\{x_n\}$ in R. Then the following holds:

(i) If $\{x_n\}$ is monotonically increasing and bounded by M, then $\{x_n\}$ is convergent.

Proof: Since $| x_n | \leq M \ \forall\ n \in N$, the set $A = \{x_n \mid n \in N\}$ is bounded above by M, $x_1 \in A$ so $A \neq \varphi$. x = SupA exists (by order completeness). Let any $\varepsilon > 0$ be given. Then there exist $xk \in A$ such that $xk > SupA - \varepsilon$. Since $\{x_n\}$ is increasing, we have $xn \geq xk \ \forall\ n \geq k$. Hence $\forall\ n \geq k$, Sup A – ε < $x_k \leq x_n \leq$ Sup A < Sup A + ε, $| x_n - \text{SupA} | < \varepsilon$. Therefore $\{x_n\} \to SupA$.

(ii) If $\{x_n\}$ is monotonically decreasing and bounded by M, then {xn} is convergent.

Proof: Since $| x_n | \leq M\ \forall\ n \in N$, the set $A = \{x_n \mid n \in N\}$ is bounded below by –M. Let x1∈A so A ≠ φ. Then $-A = \{-x_n \mid n \in N\} \neq \varphi$ is bounded above and since $\{-x_n\}$ is increasing, we have $\{-xn\} \to Sup\,(-A)$. $infA = -Sup\,(-A)$. Since lim $\{-x_n\}$ exists, we have $m\, x_n = lim\,((-1)\,\{-x_n\}) - lim\,\{-x_n\} - Sup\,(-A) - (-infA)\ infA$, that is $\{x_n\} \to infA$.

(iii) If $\{x_n\}$ is Cauchy and has a subsequence which converges to x, then we also have $\{x_n\} \to x$.

Proof: Let some subsequence $(x_{n(k)}) \to x$. Let any ε > 0 be given. Since $\{x_n\}$ is Cauchy there exist N ∈ N s.t

$$| x_n - x_m | < \varepsilon/2 \qquad \forall\, n, m \geq N$$

$$Since\ (x_n(k)) \to x,, there\ exist\ T \in N\ such\ that$$

$$| x_n k) - x | < \varepsilon/2 \qquad \forall\, k \geq T$$

If taken as S = max (N, T), note that $nS \geq nN \geq N$. Hence,

$$| x_n - x | = | (x_n - xn(S)) + (x_n(S) - x) |$$
$$\leq | x_n - xn(S)| + | x_n(S) - x |$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\ n \geq N$$ Therefore $\{x_n\} \to x$.

(iv) If $\{x_n\}$ is bounded, ,there exist a subsequence which is convergent (Bolzano – Weierstrass property)

Proof: There exist a subsequence of {xn} which is monotone. Since $\{x_n\}$ is bounded, this subsequence is also bounded. Hence, by (i) and (ii), we can conclude that this subsequence must be convergent.

(v) If $\{x_n\}$ is Cauchy, it is convergent (Cauchy Completeness).

Proof: Now $\{x_n\}$ has a convergent subsequence. Since $\{x_n\}$ is also Cauchy, we can conclude, that $\{x_n\}$ is also convergent.

Result 3.2.2:

(i) $\{r \mid r \in Q, r < x + y\} = \{s + t \mid s, t \in Q, s < x, t < y$.

(ii) $\{r \mid r \in Q, 0 < r < xy\} = \{st \mid s, t \in Q, 0 < s < x, 0 < t < y\}$, x, y > 0.

(iii) $Sup\{r \mid r \in Q, r < x\} = Sup\{r \mid r \in Q, 0 < r < x\}$, x > 0.

3.3 Dedekind Real Number Systems Are Unique

Consider any 2 Dedekind Real number system $(R, +, ., >)$ and $(R', +', .', >')$, by transitivity of isomorphism, $(Q, +, ., >) \simeq (Q', +', .', >')$ and we let $\varphi: Q \to Q'$ be that isomorphism. Define the mapping $\psi: R \to R'$ by $\psi(x) = Sup\,Ax\ \forall\ x \in R$. where $Ax = \{\varphi(r) \mid r < x, r \in Q\}$.

To prove: ψ is well-defined.

Proof: For any $x \in R$, the Archimedean property for R demands that there exist $n \in N$ (hence in Q) such that $n > x$. Hence, $\varphi(r) \in Ax \Rightarrow r < x < n \Rightarrow \varphi(r) <' \varphi(n)$ (by isomorphism). Hence, A_x is bounded above $(by\ \varphi(n))$ and so by order completeness of R′, Sup A_x exists, i.e. $\psi(x) \in R'$. Also, for any $x, y \in R$ such that $x = y$, we have $\psi(x) = Sup\{\varphi(r) \mid r < x, r \in Q\}$

$= \quad Sup\{\varphi(r) \mid r < x = y, r \in Q\}$ (note that suprema is unique)

$= \quad \psi(y)$

Hence ψ is well-defined.

Suppose now that there exist $x, y \in R$ such that $\psi(x) = \psi(y)$ but $x \neq y$. Without loss of generality, it may be assumed $x < y$. By density theorem for R there exist $r_1 \in Q$ such that $x < r1 < y$. Applying the density theorem for R on r_1 and y, we obtain some $r_2 \in Q$ such that $x < r_1 < r_2 < y$. Now $\varphi(r) \in Ax \Rightarrow r < x < r_1 < r_2 \Rightarrow \varphi(r) <' \varphi(r1) <' \varphi(r_2)$ (by isomorphism). Hence, both $\varphi(r_1)$ and $\varphi(r_2)$ are upper bound for A_x. Since $\varphi(r1) <' \varphi(r_2)$, we cannot have $\varphi(r_2) = Sup\,Ax$ and hence $\psi(x) <' \varphi(r_2)$. But $\varphi(r_2) \in Ay$, so $\psi(x) <' \varphi(r_2) \leq' \psi(y)$, contradicting $\psi(x) = \psi(y)$

Hence ψ must be one-one.

Now take any $x' \in R'$. Consider the element $x = Sup\{r\ \epsilon\ Q \mid \varphi(r) < x'\}$.

To claim: $x \in R$.

By Archimedean property for R′, there exist $n' \in N'$ (hence in Q′) such that $n' >' x'$, as φ is onto, there exist $n \in Q$ such that $n' = \varphi(n)$. Then $\varphi(r) <' x' \Rightarrow \varphi(r) <' x' <' \varphi(n) \Rightarrow r < n$ (by isomorphism). Hence n is an upper bound for $\{r \in Q \mid \varphi(r) <' x'\}$ and so by order completeness of R, x exists

To claim: $Sup\,A_x = x'$.

If x′ is not an upper bound for A_x, then there exist $\varphi(r) \in Ax$ such that $\varphi(r) >' x'$. Then $r \notin \{r \in Q \mid \varphi(r) <' x'\}$. Since $r < x$, r is not an upper bound for $\{r \in Q \mid \varphi(r) <' x'\}$ and so there exist $r_1 \in \{r \in Q \mid \varphi(r) <' x'\}$ such that $r < r_1$. By isomorphism, $\varphi(r) <' \varphi(r_1)$, i.e. $\varphi(r) <' \varphi(r1) <' x'$ and so $r \in \{r \in Q \mid \varphi(r) <' x'\}$, a contradiction! Hence x′ is an upper bound for A_x. Now take any $y' \in R'$ such that $y' <' x'$. Then by density theorem for R′, there exist $r1' \in Q'$ such that $y' <' r1' <' x'$. Applying the density theorem on r_1' and x′, we obtain $r_2' \in Q'$ such that $y' <' r1' <' r2' <' x'$. Since φ is onto there exist $r_1, r_2 \in Q$ such that $\varphi(r_1) = r_1', \varphi(r_2) = r_2'$, i.e. $r_1, r_2 \in \{r \in Q \mid \varphi(r) <' x'\}$ and so $r_1, r_2 \leq x$. By isomorphism $\varphi(r_1) <' \varphi(r_2) \Rightarrow r_1 < r_2$ and so we have $r_1 < x$. Then $r_1' = \varphi(r_1) \in Ax$ and $y' <' r_1' <' x'$. Hence $x' = Sup\,Ax$, i.e. $\psi(x) = x'$. Hence ψ is onto.

Take $x, y \in R$. $\forall\, n \geq k$.

(i) $\psi(x + y) = Sup\{\varphi(r) \mid r < x + y, r \in Q\}$

$= Sup\{\varphi(s + t) \mid s < x, t < y, s, t \in Q\}$

$= Sup\{\varphi(s) +' \varphi(t) \mid s < x,\ t < y, s, t \in Q\}$ (by isomorphism)

$= Sup\{\varphi(s) \mid s < x, s \in Q\} +' Sup\{\varphi(t) \mid t < y, t \in Q\}$

$= \psi(x) +' \psi(y)$.

(ii) Assume $x, y > 0$. Then $\Psi(x.y) = Sup\{\varphi(r) \mid r < x.y, r \in Q\}$

$= Sup\{\varphi(r) \mid 0 < r < x.y, r \in Q\}$

$= Sup\{\varphi(s.t) \mid 0 < s < x, 0 < t < y, s, t \in Q\}$

$= Sup\{\varphi(s) .' \varphi(t) \mid 0 < s < x, 0 < t < y, s, t \in Q\}$ (by isomorphism)

$= Sup\{\varphi(s) \mid 0 < s < x, s \in Q\} .' Sup\{\varphi(t) \mid 0 < t < y, t \in Q\}$(

Note that by isomorphism, φ(s), φ(t)>′ 0′)

$= Sup\{\varphi(s) \mid s < x, s \in Q\} *' Sup\{\varphi(t) \mid t < y, t \in Q\}$

$= \psi(x) .' \psi(y)$.

To claim: One of x, y is zero. Without loss of generality (due to commutative), it is assumed x = 0.

Then $\psi(x.y) = \psi(0.y)$

$= \psi(0)$

$= 0'$

$= 0' .' \psi(y)$

$= \psi(x) .' \psi(y)$

(b) $x, y < 0$ $\quad \psi(x.y) = \psi\big((-x).(-y)\big)$

$= \psi(-x) *' \psi(-y)$ $(\because -x, -y > 0)$

$= \big(-\psi(x)\big) *' (-\psi(y))$

$= \psi(x) *' \psi(y)$

(c) $x > 0, y < 0$

$= -\psi(x.y) = \psi\big(-(x.y)\big) = \psi(x.(-y))$

$= \psi(x).' \psi(-y) (\because x, -y > 0) = \psi(x) .' (-\psi(y)) - \big(\psi(x).' \psi(y)\big) i.e. \psi(x.y)$

$= \psi(x).' \psi(y)$

IV. Cantor Real Number System

Definition 4.1: Let C denote the set of all Cauchy sequences in Q. It is $(r_n), (s_n) \in C$ are equivalent and we write $(r_n) \sim (s_n)$ if given any rational $\varepsilon > 0$, there exist $k \in N$ such that $|(r_n) - s_n| < \boldsymbol{\varepsilon}$.

Theorem 4.2: The relation ~ is an equivalence relation on C.

Proof: Take any$(r_n) \in C$. Given any $\varepsilon > 0$, take $1 \in N$.Then $|(r_n), - (r_n),| = 0 < \varepsilon\ \forall\, n \geq 1$.Hence, $(r_n) \sim (r_n)$. Hence ~ is reflexive.

Take any $(r_n), (s_n) \in C$ such that $(r_n), \sim (s_n)$. Given any$\varepsilon > 0$, there exist $k \in N\ s.t\ |r_n - s_n| < \varepsilon\ \forall\, n \geq k$ i.e. $|s_n - (r_n),| = |(r_n), - s_n| < \varepsilon\ \forall\, n \geq k$. Hence, $(s_n) \sim (r_n)$. hence, ~ is symmetric.

Take any $(r_n), (s_n), (t_n) \in C$ such that $(r_n) \sim (s_n)$ and $(s_n) \sim (t_n)$. Then given any > 0 , there exist $k1, k2 \in N$s.t$|s_n - t_n| < \varepsilon/2\ \forall\, n \geq k1\ |s_n - t_n| < \varepsilon/2 \forall\, n \geq k2$.

Take $k = max(k_1, k_2)$. Then$|r_n - s_n| < \varepsilon/2, |s_n - t_n| < \varepsilon/2\ \forall\, n \geq k$. But

$$|r_n - t_n| = |((r_n) - s_n) + (s_n - t_n)|$$
$$\leq |r_n - s_n| + |s_n - t_n|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \forall\, n \geq k$$

$= \varepsilon\ \forall\, n \geq k$.

Hence $(r_n) \sim (t_n)$. Hence, ~ is transitive. Hence, ~ is an equivalence relation on C.

Definition 4.3: Rational Convergence Point: Let $(r_n) \in C$ be such that$(r_n) \to r \in Q$. Then $(r_n) \sim (s_n)$ only if $(s_n) \to r$. It is called $[(r_n)]$ a rational convergence point. In this case, it is denoted $[(r_n)]$ simply as [r].

Theorem 4.4: $(r_n) \sim (s_n)$.
Proof: Suppose$(r_n) \sim (s_n)$. Let any ε > 0 be given. Then there exist $k_1 \in N$ such that$\left| r_n - s_n \right| < \frac{\varepsilon}{2} \forall n \geq k1$. Since $(r_n) \to r$, there exist $k2 \in N$ s.t $\left| r_n - r \right| < \frac{\varepsilon}{2} \forall n \geq k2$.
Take $k = max(k1, k2)$ and we have $| r_n - s_n| < \varepsilon/2, | r_n - r | < \varepsilon/2, \forall n \geq k$

But $| sn - r | = | (s_n - r_n) + (r_n - r) |$
$\leq | s_n - r_n | + | r_n - r |$
$< \varepsilon/2 + \varepsilon/2 \quad \forall n \geq k$
$= \varepsilon \ \forall n \geq k.$

Hence, $(s_n) \to r$.
Now suppose $(s_n) \to r$. Note that this means$(s_n) \in C$. Then given $\varepsilon > 0$, there exist $k1 \in N$ s.t $| s_n - r | < \varepsilon/2 \ \forall n \geq k1$ Since $(r_n) \to r$ there exist $k2 \in N$ s.t $| r_n - r | < \varepsilon/2 \ \forall n \geq k2$
Take $k = max(k1, k2)$ and we have $| r_n - r | < \varepsilon/2, | s_n - r | < \varepsilon/2 \ \forall n \geq k$
But $|r_n - sn | = | (r_n - r) + (r - s_n) |$
$\leq | r_n - r | + | r - s_n|$
$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \forall n \geq k$
$= \varepsilon \ \forall n \geq k.$
Hence $(rn) \sim (sn)$.

Definition 4.5: **Order on R:** For any $(r_n) \in C$, we say that (r_n) is a positive sequence if ,there exist some rational $r > 0$ and $a\ k \in N$ such that $r_n > r \ \forall n \geq k$**.**

Definition 4.6: The subset PR of R by $PR = \{ [(r_n)] \in R | (r_n)\ is\ a\ positive\ sequence \}$

Theorem 4.7: The set PR is a well-defined set**.**
Proof: Let$[(r_n)] \in PR$. It must be shown that if $(s_n) \sim (r_n)$, then $[(s_n)] \in PR$ also, that is (s_n) is a positive sequence. Since (r_n) is a positive sequence there exist some rational $r > 0$ and $k_1 \in N$ such that $r_n > r \ \forall n \geq k_1$ Also as $(s_n) \sim (r_n)$, for $\varepsilon = r/2 > 0$ there exist $k_2 \in N$ s.t$\forall n \geq k_2$. Take$k = max(k_1, k_2)$. Then$\forall n \geq k$, have got$| r_n - s_n | < r/2 , r_n > r \Rightarrow r_n - s_n < r/2 , r_n > r \Rightarrow r - r/2 < r_n - r/2 < s_n$, that is $0 < r/2 < s_n$. Hence, (s_n) is also a positive sequence. Hence, PR is a well-defined set.

Theorem 4.8: For any $[(r_n)] \in R$, one and only one of the following holds: $[(r_n)] = [0], [(r_n)] \in PR, -[(r_n)] \in PR$. PR is closed under ⊕ and ⊙.
Proof:
Case 1: Take any $[(r_n)] \in R$. It is first shown that one of the cases must hold. If $[(r_n)] \neq [0]$, then it cannot have$(rn) \to 0$. Hence, there exist a rational $r > 0$ and $k_1 \in N$ such that$|r_n| \geq r \ \forall n \geq k_1$ For $\varepsilon = r/2 > 0$, there exist $k_2 \in N$ s.t $| r_n - r_m | < r/2 \ \forall n, m \geq k_2$. Take$k = max(k_1, k_2)$. Then it is $| r_n| \geq r| r_n - r_m| < r/2 \ \forall n, m \geq k$**.** In particular, have got $| r_k| \geq r\left| r_n - r_k \right| < \frac{r}{2} \forall n \geq k$.Hence $\forall n \geq k$, it is $-(r/2) < r_n - r_k < r/2 \ r_k - r/2 < r_n < r_k + r/2$
Case 2: $| r_k| \geq r$
(a) $r_k \geq r$
Then $r/2 = r - r/2 \leq r_k - r/2 < rn$.This means $r_n > r/2 \ \forall n \geq k$ and so (r_n) is a positive sequence, i.e. $[(r_n)] \in PR$.
(b) $r_k \leq -r$
Then$r_n < r_k + r/2 \leq -r + r/2 = -(r/2)$.This means $-r_n > r/2 \ \forall n \geq k$ and so $(-r_n)$ is a positive sequence.
Hence$-[(r_n)] = [(r_n)] \in PR$. If $[(r_n)] \in PR$, then, there exist $r > 0, k \in N$ such that $r_n > r \ \forall n \geq k$.i.e. $-r_n < -r < 0 \ \forall n \geq k$ and hence it is impossible that $-[(r_n)] = [(r_n)] \in PR$.By symmetry, it can be claimed that $[(r_n)] \in PR\ and\ -[(rn)] \in PR$ never be held together. If $[(r_n)] = [0]$, then$(r_n) \to 0$. Hence for any rational $r > 0$, there exist $k \in N$ such that$| r_n| < r \ \forall n \geq k$ i.e.$r_n \leq | r_n | < r \ \forall n \geq k$ and hence it is impossible that $[(r_n)] \in PR$
Hence $[(r_n)] = [0]\ and\ [(r_n)] \in PR$ never be held together. Since $[(r_n)] = [0] \Leftrightarrow -[(r_n)] = [0]$, it may be claimed from preceding result that $[(r_n)] = [0]\ and - [(r_n)] \in PR$ never hold together also. Hence, only

one of the cases is true. Take any $[(r_n)], [(s_n)] \in PR$. Then there exist rational $r, s > 0$ such that, there exist $k_1 k_2 \in N$ where $r_n > r > 0 \forall n \geq k_2 s_n) > s > 0 \forall n \geq k_2$
Take $k = max(k_1 k_2)$ and have got $r_n > r > 0, s_n > s > 0 \forall n \geq k$ i.e. $r_n + s_n > r + s > 0, r_n s_n > rs > 0 \forall n \geq k$.Hence, $(r_n + s_n)$), $(r_n s_n)$) are both positive sequence.
Then $[(r_n)] \oplus [(s_n)] = [(r_n + s_n)] \in PR$and $[(r_n)] \odot [(s_n)] = [(r_n s_n)] \in PR$.
Hence, PR is closed under $\oplus$ and $\odot$.

V. The Cantor Real Number System

Theorem 5.1: (RQ, ⊕, ⊙, ≻) is a subfield of (R, ⊕, ⊙, ≻). Furthermore, (RQ, ⊕, ⊙, ≻) ≃ (Q, +, ., >).
Take any [r], [s]∈ RQ. Then
$[r] \oplus (-[s]) = [(r)] \oplus (-[(s)])$
$= [(r)] \oplus [(-s)]$
$= [(r - s)]$
$= [r - s] (\in RQ)$

$[r] \odot ([s]) - 1 = [(r)] \odot ([(s)]) - 1$
$= [(r)] \odot [(1/s)] \forall [s] \neq [0]$, note this means $s \neq 0$
$= [(r/s)]$
$= \left[\frac{r}{s}\right] (\in RQ)$
$[1] \in RQ$. Hence $(RQ,\oplus,\odot,\succ)$ is a subfield of $(R,\oplus,\odot,\succ)$.
Consider the mapping $\varphi: RQ \rightarrow Q$ *given by* $\varphi([r]) = r \forall [r] \in RQ$.
$[r_1] = [r_2] => lim(r_1) = lim(r_2) => r_1 = r_2$. Hence, φ is well-defined.
Let $r_1 = r_2$. Then $lim(r_1) = lim(r_2)$ and so $(r_1) \sim (r_2)$, i.e., $[r_1] = [r_2]$. Hence, φ is one – one.
For any$r \in Q$, take $[r] \in RQ$ and we will have $\varphi([r]) = r$. Hence φ is onto.
Hence φ is bijective.
For any $[r], [s] \in RQ$, $\varphi([r] \oplus [s]) = \varphi([(r)] \oplus [(s)])$
$= \varphi([(r + s)])$
$= \varphi([r + s])$
$= r + s$
$= \varphi([r]) + \varphi([s])$
$\varphi([r] \odot [s]) = \varphi([(r)] \odot [(s)])$
$= \varphi([(r.s)])$
$= \varphi([r.s])$
$= r.s$
$= \varphi([r]).\varphi([s])$
$[r] \succ [s] => [(r)] \succ [(s)]$
$=> (r - s)$ is a positive sequence.
$=> r_n - s_n > t \forall n \geq k$ For some $t \in Q, t > 0, k \in N$.
$=> r - s > t$ Since $r_n = r, s_n) = s \forall n \in N$.
⇨ $r > t + s$, i.e. $r > s$ since $t > 0$.
Hence φ is an isomorphism from RQ to Q and so (RQ, ⊕, ⊙, ≻) ≃ (Q, +, ., >).

Theorem 5.2: Denseness of rational:
Let $[(s_n)], [(t_n)] \in R$ be such that$[(s_n)] \prec [(t_n)]$.Then there exist $[r] \in RQ$ such that$[(s_n))] \prec [r] \prec [(t_n)]$.
since $[(s_n)] \prec [(t_n)], (t_n - s_n)$ is a positive sequence and so there exist a rational $r' > 0$ and a $k_1 \in N$ s.t $t_n - s_n > r' \forall n \geq k_1$. As $(s_n), (t_n)$ are Cauchy, ,there exist $k_2, k_3 \in N$ $s.t$
$| t_n - t_m | < r'/3 \ \forall n, m \geq k_2. | s_n - s_m | < r'/3 \ \forall n, m \geq k_3$.
Take $k = max(k_1, k_2, k_3)$ and have got, $\forall n \geq k, | t_n - t_k | < r'/3, | s_n - s_k| < r'/3, i.e. tk - r'/3 < t_n < t_k + r'/3 \, t_n - s_n) > r', s_k - r'/3 < s_n) < s_k + r'/3, t_n - s_n) > r'$
Now,$(t_k + s_k)/2 - s_n = ½ ((t_k - s_n) + (s_k - s_n))$
$>$ $½ ((t_k - s_k - r'/3) - r'/3)$
$>$ $½ \left(r' - \frac{r'}{3} - \frac{r'}{3}\right) = \frac{r'}{6}$
$>$ $t_n - (t_k + s_k)/2 = ½ ((t_n - t_k) + (t_n - s_k))$
$>$ $½ (-r'/3 + (t_k - r'/3 - s_k))$
$>$ $½ (- r'/3 + (- r'/3 + r')) = r'/6$

Let $r = (t_k + s_k)/2$. Then both $(r - s_n)$ and $(t_n - r)$ are positive sequence, i.e. $[r] \succ [(s_n)]$ and $[(t_n)] \succ [r]$. Hence $[(s_n)] \prec [r] \prec [(t_n)]$ for some $[r] \in RQ$.

Theorem 5.3: Let $r_n, r \in Q$. Then the following holds:
(i) $|[r]| = [|r|]$
(ii) $([r_n])$ is Cauchy iff (r_n) is Cauchy.
(iii) $([r_n]) \to [r]$ iff $(r_n) \to r$
(iv) $(r_n) \in [r]$ iff $([r_n]) \to [r]$
(v) Let $\alpha \in R$ and $(r_n) \in \alpha$. Then $([r_n])$ is Cauchy and furthermore, $([r_n]) \to \alpha$.
By isomorphism, $[r] \succ [0] \Rightarrow r > 0, i.e. |r| = r$; $[r] \prec [0] \Rightarrow r < 0, i.e. |r| = -r$ $[r] = [0] \Rightarrow r = 0, i.e. |r| = r$. Hence $|[r]| = [r] = [|r|]$ if $[r] \succcurlyeq [0], -[r] = [-r] = [|r|], if$ $[r] \prec [0], i.e. |[r]| = [|r|]$.
Suppose$(r_n) \sim (s_n)$. Let any $\varepsilon > 0$ be given. Then there exist $k_1 \in N$ such that $|r_n - s_n| < \frac{\varepsilon}{2} \forall n \geq k_1$. Since$(r_n) \to r$, ,there exist $k_2 \in N$ s.t $|r_n - r| < \frac{\varepsilon}{2} \forall n \geq k_2$.Take $k = max(k_1, k_2)$ and we have $|r_n - s_n| < \varepsilon/2, |r_n - r| < \varepsilon/2, \forall n \geq k$
But $|s_n - r| = |(s_n) - r_n) + (r_n - r)|$

$$\leq |s_n) - r_n| + |r_n - r|$$
$$< \varepsilon/2 + \varepsilon/2 \quad \forall n \geq k$$
$$= \varepsilon \ \forall n \geq k. (s_n) \to r.$$

Now suppose $(s_n)) \to r$. Note that this means$(s_n)) \in C$. Then given $\varepsilon > 0$ there exist $k_1 \in N$ s.t $|s_n - r| < \varepsilon/2 \quad \forall n \geq k_2$Since $(r_n) \to r$ there exist $k_2 \in N$ s.t $|r_n - r| < \varepsilon/2 \forall n \geq k_2$Take $k = max(k_1, k_2)$ and we have $|r_n - r| < \varepsilon/2, |s_n) - r| < \varepsilon/2 \ \forall n \geq k$
But $|r_n - sn| = |(r_n - r) + (r - s_n))|$

$$\leq |r_n - r| + |r - s_n)|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \forall n \geq k$$
$$= \varepsilon \forall n \geq k. \text{ Hence } (r_n) \sim (s_n).$$

Theorem 5.4: $(R,\oplus,\odot,\succ)$ is Cauchy Complete**.**
Proof: Let (α_n) be any Cauchy sequence in R. Now, by denseness of rational, for each$n \in N$, there exist $[r_n] \in RQ$ s.t
$\alpha_n \oplus (-[1/n]) \prec [r_n] \prec \alpha_n \oplus [1/n]$, i.e. $|[r_n] \oplus (-\alpha_n)| \prec [1/n]$.
Let any real $\boldsymbol{\varepsilon} \succ [0]$ be given. Since (α_n) is Cauchy there exist $k1 \in N$ such that $|\alpha_n \oplus (-\alpha_m)| \prec \boldsymbol{\varepsilon}/3 \forall n \geq k$. By denseness of rational, there exist $[\varepsilon] \in RQ$ such that$[0] \prec [\varepsilon] \prec \boldsymbol{\varepsilon}/3$. By Archimedean property for Q there exist $k_2 \in N$ s.t $1/k_2 < \varepsilon$, hence $1/n < \varepsilon \forall n \geq k_2$. By isomorphism, It can then be claimed $[1/n] \prec [\varepsilon] \prec \boldsymbol{\varepsilon}/3 \forall n \geq k_2$.
Take $k = max(k_1, k_2)$. Then

$$|[r_n] \oplus (-[r])| = \left|\left([r_n] \oplus (-\alpha_n)\right) \oplus \left(\alpha_n \oplus (-\alpha_m)\right) \oplus \left(\alpha_m \oplus (-[r_m])\right)\right|$$
$$\preccurlyeq |[r_n] \oplus (-\alpha_n)| \oplus |\alpha_n \oplus (-\alpha_m)| \oplus |\alpha_m \oplus (-[rm])|$$
$$\prec [1/n] \oplus |\alpha n \oplus (-\alpha_m)| \oplus [1/m]$$
$$\prec \boldsymbol{\varepsilon}/3 \oplus \boldsymbol{\varepsilon}/3 \oplus \boldsymbol{\varepsilon}/3 \qquad \forall n,m \geq k$$
$$= \boldsymbol{\varepsilon} \qquad \forall n,m \geq k$$

Hence $([r_n])$ and (r_n) is a Cauchy sequence in Q. $\alpha = [(r_n)]$ will be in R, therefore $([r_n]) \to \alpha$.
Now let any real $\boldsymbol{\varepsilon} \succ [0]$ be given. There exist $k_1 \in N$ s.t $|[r_n] \oplus (-\alpha)| \prec \boldsymbol{\varepsilon}/2 \forall n \geq k_1$.There exist $k_2 \in N$ s.t
$[1/n] \prec \boldsymbol{\varepsilon}/2 \forall n \geq k_2$. Take $k = max(k_1, k_2)$ and it is

$$|\alpha n \oplus (-\alpha)| = \left|\left(\alpha_n \oplus (-[r_n])\right) \oplus \left([r_n] \oplus (-\alpha)\right)\right|$$
$$\preccurlyeq |\alpha_n \oplus (-[r_n])| \oplus |[r_n] \oplus (-\alpha)|$$
$$\prec \left[\frac{1}{n}\right] \oplus |[r_n] \oplus (-\alpha)|$$
$$\prec \frac{\varepsilon}{2} \oplus \frac{\varepsilon}{2} \forall n \geq k$$
$$= \boldsymbol{\varepsilon} \forall n \geq k. \text{ Hence} (\alpha_n) \to \alpha.$$

$(R,\oplus,\odot,\succ)$ is Cauchy Complete.

Theorem 5.5: Archimedean Property
For every$[(r_n)]$, $[(s_n)]$ $\in$R such that$[(r_n)] \succ [0]$ there exist $[n] \in RN$ such that$[n] \odot [(r_n)] \succ [(s_n)]$.If$[(r_n)] \succ [(s_n)]$, then since $[1] \in$RN, there is nothing to prove. Hence, it can be

assumed that $[(s_n)] \succcurlyeq [(rn)] \succ [0]$. By denseness of rational there exist $[r]$, $[s] \in RQ$ such that $[(s_n)] \oplus [1] \succ [s] \succ [(s_n)] \succcurlyeq [(r_n)] \succ [r] \succ [0]$.Under isomorphism, it is invoked that the Archimedean property of Q so ,there exist $[n] \in RN$ such that $[n] \odot [r] \succ [s]$. It is then $[n] \odot [(rn)] \succ [n] \odot [r] \succ [s] \succ [(s_n)]$ $i.e. [n] \odot [(r_n)] \succ [(s_n)]$. Hence, the Archimedean property holds.

Definition 5.6: For any ordered field $(RC, \oplus, \odot, \succ)$, we say that it is a Cantor Real Number System if
(i) there exist a subfield $(QC, \oplus, \odot, \succ)$ that is isomorphic to $(Q, +, ., >)$.
(ii) $(RC, \oplus, \odot, \succ)$ is Cauchy Complete
(iii) $(R_C, \oplus, \odot, \succ)$ has the Archimedean property.
Hence it is Cantor Real Number System and so Cantor Real Number System does exist.

Theorem 5.7: *($\mathbb{R}, +, ., >$) is order complete.*

Take any non-empty subset of A of $\mathbb{R}$ that is bounded above by some u_0.Let U = $\{u \in \mathbb{R}$ | u is an upper bound of A}. Since $A \neq \varphi, \exists a_0 \in A. \exists m \in \mathbb{N}\ s.t\ m > -a_0 i.e. a0 < -m\ so\ -m \notin U$. (By Archimedean property) it is defined 2 sequences (x_n), (yn) as such: $x1 = -m\ (\notin U)\ y_1 = u_0\ (\in U)$. Suppose that $x_n \notin U\ and\ y_n \in U$. Define $x_{n+1} = ½(x_n + y_n)$ if $½(x_n + y_n) \notin U$ x_n otherwise $y_n + 1 = ½(x_n + y_n)$ if $½(x_n + y_n) \in U$ y_n otherwise. By definition, note that then will be $x_n \notin U, y_n \in U\ \forall\ n \in N$. Hence, for every n, there exist $an \in A$ such that $xn < an \leq y_n)$, i.e. $x_n < y_n$. Let $N = y1 - x1 > 0$. First a few observations shall be made. If $½(x_n + y_n) \in S$, then $y_n) + 1 - x_n + 1 = ½(x_n + y_n)) - x_n = ½(y_n) - x_n)$. If $½(x_n + y_n) \notin S$, then $y_n + 1 - x_n + 1 = y_n - ½(x_n + y_n)) = ½(y_n - x_n)$. Hence $yn + 1 - x_n + 1 = ½(x_n + y_n) \forall\ n \in N$.Hence $y_n - x_n = ½(x_n - 1 + y_n) - 1) = ½(½(x_n - 2 + y_n) - 2)) = N/2n - 1\ \forall\ n \in N$.

For every $n \in N$, either $y_n = y_n + 1\ or\ y_n - y_n + 1 = y_n - ½(x_n + y_n)$
= $½(y_n - x_n)$
= $N/2^n$
> 0 i.e. $y_n + 1 \leq y_n$ and so (y_n) is decreasing.

For every $n \in N$, either $x_n = x_n + 1\ or x_n + 1 - x_n = ½(x_n + y_n) - x_n$
= $½(y_n - x_n)$
= $N/2n$
> 0 i.e. $x_n + 1 \geq x_n$ and so $\{x_n\}$ is increasing. If m, n ∈ N are such that $m < n$, have got $0 < y_m - y_n < y_m - x_n\ (\because y_n\ x_n < y_m - xm\ (\because x_n > xm) = N/2m - 1$ i.e. $|\ y_m - y_n\ | < N/2m - 1$. Let any $\varepsilon > 0$ be given by Archimedean property, there exist $k' \in N$ such that $N < k'\varepsilon$.By the exponentiation version of Archimedean property for N , there exist $k \in N$ such that $2k > k'$.Hence, $\forall\ n \geq k$, have got $2n \geq 2k > k'$ i.e. $\varepsilon 2n \geq \varepsilon 2k > \varepsilon k' > N, i.e. N/2n < \varepsilon$ Hence $|\ y_m - y_n\ | < N/2m - 1$ (by symmetry of absolute order, it can always be assumed $m < n$. m = n, $|\ y_m - yn\ | = 0 < N/2m - 1) < \varepsilon \forall\ n, m \geq k + 1$. Hence, it is shown that (y_n) is Cauchy. By Cauchy Completeness, (yn) converges to some $y \in R$. Suppose $y \notin U$. Then there exist $a \in A$ such that $a > y$. By density theorem there exist $z \in R$ such that $a - y > z > 0$. Since it is always been $yn \geq a$, it will be $y_n - y \geq a - y > z > 0$. But $(y_n) \to y$, i.e. for there exist $k \in N$ s.t $|\ yn - y\ | < z/2 \forall\ n \geq k$. In particular, $y_k - y \leq |\ yn - y\ | < z/2 < z < y_k - y$, a contradiction! Hence, $y \in U$.Now suppose there exist $u \in U$ such that $u < y$. Note first that similar to above, by Archimedean property and its exponentiation version for N, given any ε > 0 there exist $k \in N$ such that $N/2n < \varepsilon\ \forall\ n \geq$ i.e. $\left|\ (y_n - x_n) - 0\ \right| = y_n - xn\ (\because y_n - x_n > 0) = \frac{N}{2n} - 1 < \varepsilon \forall\ n\ k + 1$ i.e. $(y_n - x_n) \to 0$. If there exist y_k such that $y_k < y$, then it is $y_n \leq y_k < y\ \forall\ n \geq k$ as (y_n) is decreasing. But $(y_n) \to y$ so for $\varepsilon = y - y_k > 0$ there exist N ∈ N such that $|\ y_n - y\ | < y - y_k\ \forall\ n \geq N$.For $N' = max(k, N)$, have got in particular $|\ y'_N - y\ | = -(y'_N - y), |\ y'_N - y\ | < y - y_k \Rightarrow -(y'_N - y) < y - y_k \Rightarrow y'_N > y_k$, contradicting (y_n) being decreasing. Hence, we always have $y_n - y \geq 0$. since $(y_n - xn) \to 0$, for ε = y – u > 0 there exist $k \in N$ such that $y_n - x_n = \left|\ y_n - x_n\ \right| < y - u, i.e. y_n - y < xn - u\ \forall\ n \geq k$ .In particular $0 \leq y_k - y < x_k - u, i.e. x_k > u$. But this means $xk > u \geq a\ \forall\ a \in A$, making xk an upper bound, which is a contradiction! Hence, $y \leq u\ \forall\ u \in U$ and so $Sup\ A = y$ exists. Since $(R, +, ., >)$ has the least upper bound property, it is hence order complete.

VI. Conclusion

Peano's axiom allows creating natural number system, integer system, field of rational, and the real number system. The existence of the real number system was proceeded to pursue the instinct of Richard Dedekind and Georg Cantor. These two approaches congregate distinctive number system. Dedekind built a system that is not only reliable with normal operation but also free from the predicament of the immeasurable triangle from the rational numbers. Cantor with his natural approach resulted with the same system that has given more confidence on this observation. The real number system is a complete ordered field that is therefore Dedekind real number system.

This paper enunciates two things about the system, the real number system is distinctive and any complete ordered field is the real number system. An ordered field is Cauchy complete and possesses the Archimedean property only if the order is complete. Through this paper, a better insight on how the number system actually works. The relationship between the fundamental subsets such as the integer and the natural numbers could be seen well.

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