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The Sum Span of Sum Span and the Product Span of Product Span in an Artex Space over a Bi-Monoid

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***ABSTRACT**: Sum Combination of elements of an Artex Space over a bi-monoid and the sum span of a subset of a completely bounded Artex space over a bi-monoid induced to define the sum span of sum span of a subset of a completely bounded Artex space over a bi-monoid. It is proved that the sum span of sum span of a subset of a completely bounded Artex space over a bi-monoid is the sum span. Product Combination of elements of an Artex Space over a bi-monoid and the product span of a subset of a completely bounded Artex space over a bi-monoid induced to define the product span of product span of a subset of a completely bounded Artex space over a bi-monoid. It is proved that the product span of product span of a subset of a completely bounded Artex space over a bi-monoid is the product span.*

***Keywords:** Bi-monoids, Artex Spaces over bi-monoids, Completely Bounded Artex Spaces over bi-monoids, Sum Combination, Sum span the sum span of sum span the product span of product span.*

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I. Introduction

The algebraic system Bi-semi-group is more general to the algebraic system ring or an associative ring. Artex Spaces over Bi-monoids were introduced. As a development of Artex Spaces over Bi-monoids, SubArtex spaces of Artex spaces over bi-monoids were introduced. From the definition of a SubArtex space, it is clear that not every subset of an Artex space over a bi-monoid is a SubArtex space. We found and proved some propositions which qualify subsets to become SubArtex Spaces. Completely Bounded Artex Spaces over bi-monoids were introduced. It contains the least and greatest elements namely 0 and 1. These elements play a good role in our study. Sum Combination of elements of an Artex Space over a bi-monoid and the sum span of a subset of a completely bounded Artex space over a bi-monoid induced to define the sum span of sum span of a subset of a completely bounded Artex space over a bi-monoid. Product Combination of elements of an Artex Space over a bi-monoid and the product span of a subset of a completely bounded Artex space over a bi-monoid induced to define the product span of product span of a subset of a completely bounded Artex space over a bi-monoid.. As the theory of Artex spaces over bi-monoids is developed from lattice theory, this theory will play a good role in many fields especially in science and engineering and in computer fields. In Discrete Mathematics this theory will create a new dimension.

II. Artex Spaces Over Bi-Monoids

2.1 Artex Space Over a Bi-monoid : Let (M , + , .) be a bi-monoid with the identity elements 0 and 1 with respect to + and **.** respectively. A non-empty set A together with two binary operations ^ and v is said to be an Artex Space Over the Bi-monoid (M , + , .) if

1.(A, Λ , V) is a lattice and

2.for each m∈M , m≠0, and a∈A, there exists an element ma ∈ A satisfying the following conditions :

(i) m(a Λ b) = ma Λ mb

(ii) m(a V b) = ma V mb

(iii) ma Λ na ≤ (m +n)a and ma V na ≤ (m + n)a

(iv) (mn)a = m(na), for all m,n∈M, m≠0, n≠0, and a,b∈A

(v) 1.a = a, for all a ∈ A.

Here, ≤ is the partial order relation corresponding to the lattice (A, Λ , V) .The multiplication ma is called a **bi-monoid multiplication with an artex element** or simply bi-monoid multiplication in A.

Example 2.2.1 Let W = {0,1,2,3,…}.

Then (W , + , .) is a bi-monoid , where + and **.** are the usual addition and multiplication respectively.

Let Z be the set of all integers
Then (Z, ≤) is a lattice in which Λ and V are defined by a Λ b = minimum of {a,b} and a V b = maximum of {a,b} , for all a,b∈Z.
Clearly for each m∈W,m≠0, and for each a∈Z, ma∈Z.
Also,
(i) m(a Λ b) = ma Λ mb
(ii) m(a V b) = ma V mb
(iii) ma Λ na ≤ (m +n)a and ma V na ≤ (m + n)a
(iv) (mn)a = m(na)
(v) 1.a = a , for all m,n∈W, m ≠ 0, n ≠ 0 and a,b∈Z
Therefore, Z is an Artex Space Over the Bi-monoid (W , + , .)
Example 2.2.2 As defined in Example 2.2.1, Q, the set of all rational numbers is an Artex space over W
Example 2.2.3 As defined in Example 2.2.1, R, the set of all real numbers is an Artex space over W.
Example 2.2.4 : Let Q'= Q^+ ∪ {0}, where Q^+ is the set of all positive rational numbers.
Then (Q', + , .) is a bi-monoid. Now as defined in Example 2.2.1, Q, the set of all rational numbers is an Artex space over Q'
Example 2.2.5 : R'= R^+ ∪ {0}, where R^+ is the set of all positive real numbers. Then (R', + , .) is a bi-monoid. As defined in Example 2.2.1, R, the set of all real numbers is an Artex space over R'

2.3 Properties

Properties 2.3.1 : We have the following properties in a lattice (L , Λ,V)

1.a Λ a = a	1'.a V a = a
2.a Λ b = b Λ a	2'.a V b = b V a
3.(a Λ b)Λ c = a Λ(b Λ c)	3'.(a V b) V c = a V(b V c)
4.a Λ (a V b) = a	4'.a V (a Λ b) = a, for all a,b,c ∈ L

Therefore, we have the following properties in an Artex Space A over a bi-monoid M.

(i)	m(a Λ a) = ma	(i)'.m(a V a) = ma
(ii)	(m(a Λ b) =m(b Λ a)	(ii)'.m(a V b) = m(b V a)
(iii)	m((a Λ b)Λ c)=m(a Λ(b Λ c))	(iii)'.m((a V b) V c) = m(a V(b V c))
(iv)	m(a Λ (a V b)) = ma	(iv)'.m(a V (a Λ b)) = ma,

for all m∈M, m ≠ 0 and a,b,c ∈ A

2.4 SubArtex Space : Let (A, Λ ,V) be an Artex space over a bi-monoid. (M , + , .). Let S be a nonempty subset of A. Then S is said to be a SubArtex Space of A if (S, Λ ,V) itself is an Artex Space over M.

2.4.1 Example : As defined in Example 2.2.1, Z is an Artex Space over W = {0,1,2,3,…..} and W is a subset of Z. Also W itself is an Artex space over W under the operations defined in Z. Therefore, W is a SubArtex space of Z.

2.5 Complete Artex Space over a bi-monoid : An Artex space A over a bi- monoid M is said to be a Complete Artex Space over M if as a lattice, A is a complete lattice, that is each nonempty subset of A has a least upper bound and a greatest lower bound.

2.5.1 Remark : Every Complete Artex space must have a least element and a greatest element.
The least and the greatest elements, if they exist, are called the bounds or units of the Artex space and are denoted by 0 and 1 respectively.

2.6 Lower Bounded Artex Space over a bi-monoid : An Artex space A over a bi-monoid M is said to be a Lower Bounded Artex Space over M if as a lattice, A has the least element 0.

2.6.1 Example : Let A be the set of all constant sequences (x_n) in [0,∞)
Let W = {0,1,2,3,…}.
Define ≤ ', an order relation, on A by for (x_n), (y_n) in A, (x_n) ≤ ' (y_n) means $x_n \leq y_n$, for each n
where ≤ is the usual relation " less than or equal to "
Therefore, A is an Artex space over W.
The sequence (0_n), where 0_n is 0 for all n, is a constant sequence belonging to A
Also (0_n) ≤ ' (x_n) , for all the sequences (x_n) belonging to in A
Therefore, (0_n) is the least element of A.
That is, the sequence 0,0,0,…… is the least element of A
Hence A is a Lower Bounded Artex space over W.

2.7 Upper Bounded Artex Space over a bi-monoid : An Artex space A over a bi-monoid M is said to be an Upper Bounded Artex Space over M if as a lattice, A has the greatest element 1.

2.7.1 Example : Let A be the set of all constant sequences (x_n) in (-∞,0] and let W = {0,1,2,3,…}.
Define ≤' ,an order relation, on A by for (x_n), (y_n) in A, (x_n) ≤' (y_n) means $x_n \leq y_n$, for n = 1,2,3,…, where ≤ is the usual relation " less than or equal to "
A is an Artex space over W.

Now, the sequence (1_n), where 1_n is 0, for all n, is a constant sequence belonging to A
Also $(x_n) \leq' (1_n)$, for all the sequences (x_n) in A
Therefore, (1_n) is the greatest element of A.
That is, the sequence 0,0,0,….. is the greatest element of A
Hence A is an Upper Bounded Artex Space over W.

2.8 Bounded Artex Space over a bi-monoid : An Artex space A over a bi-monoid M is said to be a Bounded Artex Space over M if A is both a Lower bounded Artex Space over M and an Upper bounded Artex Space over M.

2.9 Completely Bounded Artex Space over a bi-monoid: A Bounded Artex Space A over a bi-monoid M is said to be a Completely Bounded Artex Space over M if (i) 0.a = 0, for all a ϵ A (ii) m.0 = 0, for all m ϵ M.

2.9.1 Note : While the least and the greatest elements of the Complemented Artex Space is denoted by 0 and 1, the identity elements of the bi-monoid (M , + , .) with respect to addition and multiplication are, if no confusion arises, also denoted by 0 and 1 respectively.

III. The Sum Span Of A Sub Set Of An Artex Space Over A Bi-Monoid

3.1 Sum Combination : Let (A, Λ ,V) be an Artex Space over a bi-monoid (M , + , .). Let $a_1, a_2, a_3, \ldots a_n$ ϵ A. Then any element of the form $m_1a_1Vm_2a_2Vm_3a_3V \ldots V m_na_n$, where $m_i \in M$, is called a Finite Sum Combination or Finite Join Combination of $a_1, a_2, a_3, \ldots a_n$.

3.2 The Sum Span of a sub set of an Artex Space over a Bi-monoid : Let (A, Λ ,V) be a Completely Bounded Artex Space over a bi-monoid (M , + , .) and W be a nonempty subset of A. Then the Sum Span of W or Join Span of W denoted by S[W] is defined to be the set of all finite sum combinations of elements of W. That is, $S[W] = \{m_1a_1Vm_2a_2Vm_3a_3V \ldots Vm_na_n / m_i \in M \text{ and } a_i \in W\}$.

3.3 PROPOSITIONS

Proposition 3.3.1: Let (A, Λ ,V) be a Completely Bounded Artex Space over a bi-monoid (M , + , .) and W be a nonempty subset of A. Then W ⊆ S [W]

Proposition 33.2 : Let (A, Λ ,V) be a Completely Bounded Artex Space over a bi-monoid (M , + , .). Let W and V be any two nonempty subsets of A. Then W ⊆ V implies S [W] ⊆ S [V].

Proposition 3.3.3 : Let (A, Λ ,V) be a Completely Bounded Artex Space over a bi-monoid (M , + , .). Let W and V be any two nonempty subsets of A. Then S [W∪V] = S [W] V S [V].

IV. The Sum Span Of Sum Span Of A Sub Set Of An Artex Space Over A Bi-Monoid

4.1 The Sum Span of Sum Span of a sub set of an Artex Space over a Bi-monoid : Let (A, Λ ,V) be a Completely Bounded Artex Space over a bi-monoid (M , + , .) and W be a nonempty subset of A. Then the Sum Span of Sum Span of W or Join Span of Join Span of W denoted by S[S[W]] is defined to be the set of all finite sum combinations of elements of S[W].

Note : $S^n[W] = S[S[S[S\ldots S[W]\ldots]]] = S[W]$ (Sum Span taken n times)

Proposition 4.1.1: Let (A, Λ ,V) be a Completely Bounded Artex Space over a bi-monoid (M , + , .) and W be a nonempty subset of A. Then S[S[W]] = S [W]

Proof : Let (A, Λ ,V) be a Completely Bounded Artex Space over a bi-monoid (M , + , .)
Let W be a nonempty subset of A
Let U = S[W]
Let x ϵ S[S[W]] = S[U]
Then, by the definition of S[U], $x = m_1s_1Vm_2s_2V \ldots Vm_is_iV \ldots Vm_ns_n$ where $m_i \in M$ and $s_i \in U = S[W]$
Now, by the definition of S[W], $s_i = m_{i1}a_{i1}Vm_{i2}a_{i2}V\ldots Vm_{ij}a_{ij}V \ldots Vm_{ik}a_{ik}$ where $m_{ij} \in M$ and $a_{ij} \in W$
Therefore $x = m_1s_1Vm_2s_2V \ldots Vm_is_iV\ldots Vm_ns_n$

$= m_1(m_{11}a_{11}Vm_{12}a_{12}V \ldots Vm_{1k}a_{1k})Vm_2(m_{21}a_{21}Vm_{22}a_{22}V \ldots Vm_{2m}a_{2m})V\ldots \ldots$
$\ldots Vm_i(m_{i1}a_{i1}Vm_{i2}a_{i2}V \ldots Vm_{ir}a_{ir})\ldots Vm_n(m_{n1}a_{n1}Vm_{n2}a_{n2}V \ldots Vm_{nt}a_{nt})$

$= (m_1m_{11}a_{11}Vm_1m_{12}a_{12}V \ldots Vm_1m_{1k}a_{1k})V(m_2m_{21}a_{21}Vm_2m_{22}a_{22}V \ldots Vm_2m_{2m}a_{2m})V\ldots \ldots$
$\ldots V(m_im_{i1}a_{i1}Vm_im_{i2}a_{i2}V\ldots Vm_im_{ir}a_{ir})\ldots V (m_nm_{n1}a_{n1}Vm_nm_{n2}a_{n2}V \ldots Vm_nm_{nt}a_{nt})$

$= (m_1m_{11}) a_{11}V(m_1m_{12}) a_{12}V \ldots V(m_1m_{1k}) a_{1k}V(m_2m_{21}) a_{21}V(m_2m_{22}) a_{22}V \ldots V(m_2m_{2m}) a_{2m}V\ldots \ldots$
$\ldots V(m_im_{i1}) a_{i1}V(m_im_{i2}) a_{i2}V\ldots V(m_im_{ir}) a_{ir}V\ldots V (m_nm_{n1}) a_{n1}V(m_nm_{n2}) a_{n2}V \ldots V(m_nm_{nt}) a_{nt}$

It is a finite sum combination of elements of W.
Therefore, x ϵ S[W]
S[S[W]] ⊆ S [W]
Conversely, suppose x ϵ S[W]
Then $x = m_1a_1Vm_2a_2Vm_3a_3V \ldots Vm_na_n$ where $m_i \in M$ and $a_i \in W$
$x = 1.(m_1a_1) V 1.(m_2a_2) V 1.(m_3a_3) V \ldots V 1.(m_na_n)$ where $m_i \in M$ and $a_i \in W$

Each m_ia_i is a finite sum combination of elements of W
Therefore, each $m_ia_i \in S[W]$
Since $1 \in M$, $x = 1.(m_1a_1) \vee 1.(m_2a_2) \vee 1.(m_3a_3) \vee \ldots \vee 1.(m_na_n) \in S[S[W]]$
Therefore, $S[W] \subseteq S[S[W]]$
Hence $S[S[W]] = S[W]$

Corollary 4.1.2: Let (A, Λ ,V) be a Completely Bounded Artex Space over a bi-monoid (M , + , .) and W be a nonempty subset of A. Then $S[S[S[W]]] = S[W]$

Corollary 4.1.3: Let (A, Λ ,V) be a Completely Bounded Artex Space over a bi-monoid (M , + , .) and W be a nonempty subset of A. Then $S^n[W] = S[W]$

V. The Product Span Of A Sub Set Of An Artex Space Over A Bi-Monoid

5.1 Product Combination : Let (A, Λ ,V) be an Artex Space over a bi-monoid (M , + , .). Let $a_1, a_2, a_3, \ldots, a_n \in A$. Then any element of the form $m_1a_1 \wedge m_2a_2 \wedge m_3a_3 \wedge \ldots \wedge m_na_n$, where $m_i \in M$, is called a Finite Product Combination or Finite Meet Combination of $a_1, a_2, a_3, \ldots, a_n$.

5.2 The Product Span of a sub set of an Artex Space over a Bi-monoid : Let (A, Λ ,V) be a Completely Bounded Artex Space over a bi-monoid (M , + , .) and W be a nonempty subset of A. Then the Product Span of W or Meet Span of W denoted by P[W] is defined to be the set of all finite product combinations of elements of W. That is, $P[W] = \{ m_1a_1 \wedge m_2a_2 \wedge m_3a_3 \wedge \ldots \wedge m_na_n / m_i \in M \text{ and } a_i \in W\}$.

5.3 The Product Span of Product Span of a sub set of an Artex Space over a Bi-monoid : Let (A, Λ ,V) be a Completely Bounded Artex Space over a bi-monoid (M , + , .) and W be a nonempty subset of A. Then the Product Span of Product Span of W or Meet Span of Meet Span of W denoted by P[P[W]] is defined to be the set of all finite product combinations of elements of P[W].

Note : $P^n[W] = P[P[P[P\ldots P[W]\ldots]]] = P[W]$ (Product Span taken n times)

Proposition 5.3.1: Let (A, Λ ,V) be a Completely Bounded Artex Space over a bi-monoid (M , + , .) and W be a nonempty subset of A. Then $P[P[W]] = P[W]$.

Proof : Let (A, Λ ,V) be a Completely Bounded Artex Space over a bi-monoid (M , + , .)
Let W be a nonempty subset of A
Let $U = P[W]$
Let $x \in P[P[W]] = P[U]$
Then, by the definition of P[U], $x = m_1s_1 \wedge m_2s_2 \wedge \ldots \wedge m_is_i \wedge \ldots \wedge m_ns_n$ where $m_i \in M$ and $s_i \in U = P[W]$
Now, by the definition of P[W], $s_i = m_{i1}a_{i1} \wedge m_{i2}a_{i2} \wedge \ldots \wedge m_{ij}a_{ij} \wedge \ldots \wedge m_{ik}a_{ik}$ where $m_{ij} \in M$ and $a_{ij} \in W$
Therefore $x = m_1s_1 \wedge m_2s_2 \wedge \ldots \wedge m_is_i \wedge \ldots \wedge m_ns_n$

$= m_1(m_{11}a_{11} \wedge m_{12}a_{12} \wedge \ldots \wedge m_{1k}a_{1k}) \wedge m_2(m_{21}a_{21} \wedge m_{22}a_{22} \wedge \ldots \wedge m_{2m}a_{2m}) \wedge \ldots \ldots \ldots \wedge m_i(m_{i1}a_{i1} \wedge m_{i2}a_{i2} \wedge \ldots \wedge m_{ir}a_{ir}) \ldots \wedge m_n(m_{n1}a_{n1} \wedge m_{n2}a_{n2} \wedge \ldots \wedge m_{nt}a_{nt})$

$= (m_1m_{11}a_{11} \wedge m_1m_{12}a_{12} \wedge \ldots \wedge m_1m_{1k}a_{1k}) \wedge (m_2m_{21}a_{21} \wedge m_2m_{22}a_{22} \wedge \ldots \wedge m_2m_{2m}a_{2m}) \wedge \ldots \ldots \wedge (m_im_{i1}a_{i1} \wedge m_im_{i2}a_{i2} \wedge \ldots \wedge m_im_{ir}a_{ir}) \ldots \wedge (m_nm_{n1}a_{n1} \wedge m_nm_{n2}a_{n2} \wedge \ldots \vee m_nm_{nt}a_{nt})$

$= (m_1m_{11})a_{11} \wedge (m_1m_{12})a_{12} \wedge \ldots \wedge (m_1m_{1k})a_{1k} \wedge (m_2m_{21})a_{21} \wedge (m_2m_{22})a_{22} \wedge \ldots \wedge (m_2m_{2m})a_{2m} \wedge \ldots \ldots \wedge (m_im_{i1})a_{i1} \wedge (m_im_{i2})a_{i2} \wedge \ldots \wedge (m_im_{ir})a_{ir} \wedge \ldots \wedge (m_nm_{n1})a_{n1} \wedge (m_nm_{n2})a_{n2} \wedge \ldots \wedge (m_nm_{nt})a_{nt}$

It is a finite product combination of elements of W.
Therefore, $x \in P[W]$
$P[P[W]] \subseteq P[W]$
Conversely, suppose $x \in P[W]$
Then $x = m_1a_1 \wedge m_2a_2 \wedge m_3a_3 \wedge \ldots \wedge m_na_n$ where $m_i \in M$ and $a_i \in W$
$x = 1.(m_1a_1) \wedge 1.(m_2a_2) \wedge 1.(m_3a_3) \wedge \ldots \wedge 1.(m_na_n)$ where $m_i \in M$ and $a_i \in W$
Each m_ia_i is a finite product combination of elements of W
Therefore, each $m_ia_i \in P[W]$
Since $1 \in M$, $x = 1.(m_1a_1) \wedge 1.(m_2a_2) \wedge 1.(m_3a_3) \wedge \ldots \wedge 1.(m_na_n) \in P[P[W]]$
Therefore, $P[W] \subseteq P[P[W]]$
Hence $P[P[W]] = P[W]$

Corollary 5.3.2 : Let (A, Λ ,V) be a Completely Bounded Artex Space over a bi-monoid (M , + , .) and W be a nonempty subset of A. Then $P[P[P[W]]] = P[W]$

Corollary 5.3.3 : Let (A, Λ ,V) be a Completely Bounded Artex Space over a bi-monoid (M , + , .) and W be a nonempty subset of A. Then $P^n[W] = P[W]$

VI. Conclusion

The Finite Product Combination and the Finite Sum Combination of elements of a subset of an Artex Space over a Bi-monoid, the Product Span of Product Span of a sub set of an Artex Space over a Bi-monoid and the Sum Span of Sum Span of a sub set of an Artex Space over a Bi-monoid will create a dimension in the theory of Artex spaces over bi-monoids. Interested researcher can do further research in this area of research.

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