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Radio labeling of Hurdle graph and Biregular rooted Trees

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Abstract: *A Radio labeling of a connected graph G is an injective map h: V(G) → {0, 1, 2, …, N} such that for every two distinct vertices x and y of G, d(x, y) + $|h(x) - h(y)|$ ≥ 1 + diam(G). The span of a labeling h is the greatest integer in the range of h. The minimum span taken over all radio labeling of the graph is called radio number of G, and is denoted by rn(G). In this paper, we find the radio number of hurdle graph and radio number of biregular rooted trees.*

Keywords: *Radio labeling, Distance, Eccentricity, Diameter, Hurdle graph, Rooted tree, Biregular rooted trees, Status, Median.*

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I. Introduction and Definitions

Throughout this paper we consider finite, simple, undirected and connected graphs. Let V(G) and E(G) respectively denote the vertex set and edge set of G. Radio labeling, or multilevel distance labeling, is motivated by the channel assignment problem for radio transmitters [4]. Chartrand et al. investigated the upper bound for the radio number of path P_n .The exact value for the radio number of path was given by Liu and Zhu [2]. A wireless network is composed of a set of stations (or transmitters) on which appropriate channels are assigned. The task is to assign a channel to each station such that the interference which is caused by the geographical distance between stations is avoided. The span of a labeling h is the greatest integer in the range of h. The minimum span taken over all radio labelings of the graph is called radio number of G, denoted by rn(G). For standard terminology and notations we follow Harary [5] and Gallian [6].

Definition 1.1 A Radio labeling of a connected graph G is an injective map h: V(G) → {0, 1, 2, …, N} such that for every two distinct vertices x and y of G, d(x, y) + $|h(x) - h(y)|$ ≥ 1 + diam(G). The span of a labeling h is the greatest integer in the range of h. The minimum span taken over all radio labelings of the graph is called radio number of G, denoted by rn(G).

Definition 1.2[3] The distance d(u, v) from a vertex u to a vertex v in a connected graph G is the minimum of the lengths of the u-v paths in G.
Definition 1.3[3] The eccentricity e(v) of a vertex v in a connected graph G is the distance between v and a vertex farthest from v in G.
Definition 1.4[3] The diameter diam(G) of G is the greatest eccentricity among the vertices of G.
Definition 1.5 A graph obtained from a path P_n by attaching a pendant edges to every internal vertices of the path is called Hurdle graph with n-2 hurdles and is denoted by Hd_n.
Definition 1.6 The status of a vertex v in a graph G denoted by $S_G(v)$ or S(v) is the sum of the distance between v and every other vertex in G. That is $S(v) = \sum_{u \in V(G)} d(u,v)$.
Definition 1.7 For a graph G, the median M(G) is the set of vertices with minimum status. A vertex v with minimum status is said to be a median vertex. The minimum status of a graph is denoted as $S(G) = \min\{S(v)/v \in V(G)\}$.
Definition 1.8 [3] A tree in which one vertex is distinguished from all the others is called a rooted tree and the vertex is called the root of the tree.
Definition 1.9 A biregular rooted tree is a tree in which every two vertices on the same side of the partition have same degree as each other.
Existing result 1.10[1] Let T be a tree with n vertices and diameter d. Then

$rn(T) \geq (n-1)(d+1) + 1 - 2S(T)$. Moreover, the equality holds if and only for every weight center v* there exists a radio labeling h with $h(w_1) = 0 < h(w_2) < \ldots < h(w_{n-1})$ for which all following properties hold, for every j with $1 \leq j \leq n - 1$,

(1) w_j and w_{j+1} belong to different branches, unless one of them is v*.

(2) $\{w_1, w_n\} = \{v^*, u\}$ where $u \in V(T)$ such that $d(v^*, u) = 1$

(3) $h(w_{j+1}) = h(w_j) + d+1 - d(v^*, w_j) - d(v^*, w_{j+1})$.

Observation 1.11 Let $S(BR_{n,m})$ be the status of the graph $BR_{n,m}$. Then

$$S(BR_{n,m}) = \begin{cases} \frac{n^2}{4}(m-1) + n(m-1) + 1 & \text{if } n \text{ is even} \\ \frac{(n-1)^2}{4}(m-1) + 3\left(\frac{n-1}{2}\right)(m-1) + m & \text{if } n \text{ is odd} \end{cases}$$

Observation1.12 Let $BR_{n,m}$ denote the biregular rooted tree in which it consists of a path of order n and degree m. Then

$$rn(BR_{n,m}) \geq \begin{cases} (n-1)(d+1) + 1 - 2S(BR_{n,m}) & \text{if } n \text{ is even} \\ (n-1)(d+1) + 1 - 2S(BR_{n,m}) + 1 & \text{if } n \text{ is odd} \end{cases}$$

II. Main Results

Theorem 2.1 Let Hd_n be a hurdle graph on n vertices. Then $rn(Hd_n) = n^2 - 3n + 3$ if n is even, $n \geq 2$.

Proof Let h be an optimal radio labeling for Hd_n and $\{x_1, x_2, \ldots, x_p\}$ be the ordering of $V(Hd_n)$ such tha $0 = h(x_1) < h(x_2) < \ldots < h(x_p)$. Then $d(x_i, x_{i+1}) + |h(x_{i+1}) - h(x_i)| \geq 1 + diam(Hd_n) = n$, $1 \leq i \leq p - 1$.

Let $n = 2a$, $a \geq 2$. In this case diameter $d = 2a - 1$ and $p = 2n - 2$.

Let $v_1, v_2, \ldots, v_n$ denote the vertices of P_n from which the Hurdle graph Hd_n is obtained, by v'_{i-1} the terminal vertex of the pendent edges attached to v_i for $2 \leq i \leq a$ and by v'_{i+1} the terminal vertex of the pendent edges attached to v_i for $a + 1 \leq i \leq 2a - 1$.

By result (1.10), $rn(Hd_n) \geq (p - 1)(d + 1) + 1 - 2S(Hd_n)$. (2.1.1)

First we compute the status function of Hd_n. In this case Hd_n has two weight centres namely v_a and v_{a+1}, $a \geq 2$.

we have $S(Hd_n) = S_{Hd_n}(v_a)$

$$= \sum_{u \in V(Hd_n)} d(u, v_a)$$

$$= 3.1 + 4(2 + \ldots + a - 1) + 2.a$$

$$= 3 + 4\left(\frac{a(a-1)}{2} - 1\right) + 2a$$

$$= 2a^2 - 1 = 2\left(\frac{n}{2}\right)^2 - 1$$

$$= \frac{n^2 - 2}{2} \qquad (2.1.2)$$

Substituting (2.1.2) in (2.1.1) we get

$$rn(Hd_n) \geq (p-1)(d+1) + 1 - 2\left(\frac{n^2-2}{2}\right)$$

$$= (2n - 2 - 1)(n - 1 + 1) + 1 - 2\left(\frac{n^2-2}{2}\right)$$

$$= (2n - 3)(n) + 1 - (n^2 - 2)$$

$$= n^2 - 3n + 3$$

Therefore $rn(Hd_n) \geq n^2 - 3n + 3$

Let $\{x_1, x_2, \ldots, x_p\}$ be the ordering of the vertices of Hd_n.

Label the vertices $x_1, x_2, \ldots, x_p$ as in the following procedure

$v_a \rightarrow v'_{2a} \rightarrow v_{a-1} \rightarrow v'_{2a-1} \rightarrow v_{a-2}$

$v'_{2a-2} \rightarrow v_{a-3} \rightarrow v'_{2a-3} \rightarrow v_{a-4}$

$\ldots \quad \ldots \quad \ldots \quad \ldots$

$v_2 \rightarrow v'_{a+2} \rightarrow v_1 \rightarrow v_{2a} \rightarrow v'_{a-1}$

$v_{2a-1} \rightarrow v'_{a-2} \rightarrow v_{2a-2} \rightarrow v'_{a-3}$

$v_{2a-3} \rightarrow \ldots \rightarrow v'_1 \rightarrow v_{a+1}$.

Define a function h: $V(Hd_n) \rightarrow \{0, 1, 2, \ldots, n^2 - 3n + 3\}$ by $h(x_1) = 0$ and
$h(x_{i+1}) = h(x_i) + d + 1 - d(x_{i+1}, x_i)$ for $1 \leq i \leq p - 1$
Thus it is possible to assign labels to the vertices of Hd_n with span equal to the lower bound.
Therefore $rn(Hd_n) \leq n^2 - 3n + 3$
Hence $rn(Hd_n) = n^2 - 3n + 3$, $n = 2a$, $a \geq 2$
Example 2.1 In Table 1, Figure 1, Figure 2 and Figure 3 an ordering of the vertices, ordering version, renamed version and optimal radio labeling for Hd_8 are shown.
Table 1

$v_4 \rightarrow v'_8 \rightarrow v_3 \rightarrow v'_7 \rightarrow v_2$

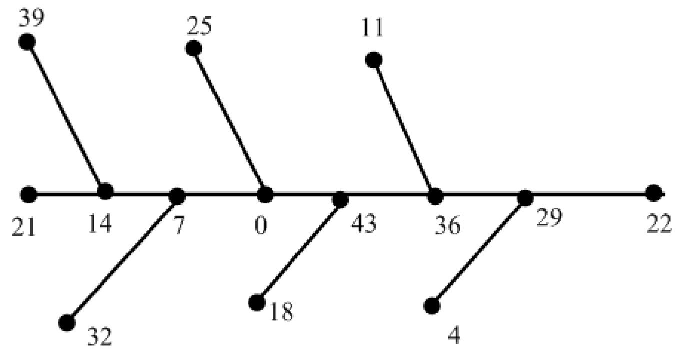
$v'_6 \rightarrow v_1 \rightarrow v_8 \rightarrow v'_3$

$v_7 \rightarrow v'_2 \rightarrow v_6 \rightarrow v^1_1$

v_5

Figure 1

Figure 2



rn $(Hd_8) = 43$

Figure 3

Theorem 2.2 Let Hd_n be a hurdle graph on n vertices. Then $rn(Hd_n) = n^2 - 3n + 4$ if n is odd, $n \geq 5$.

Proof. Let h be an optimal radio labeling for Hd_n and $\{x_1, x_2, ..., x_p\}$ be the ordering of $V(Hd_n)$ such that $0 = h(x_1) < h(x_2) < ... < h(x_p)$. Then $d(x_i, x_{i+1}) + | h(x_{i+1}) - h(x_i) | \geq 1 + d$, $1 \leq i \leq p - 1$

Let $n = 2a + 1$, $a \geq 2$. In this case diameter $d = 2a$ and $p = 2n - 2$.

Let $v_1, v_2, \ldots, v_n$ denote the vertices of P_n from which the Hurdle graph Hd_n is obtained by v_i', the terminal vertex of the pendent edges attached to v_i, $2 \leq i \leq 2k$.

By result (1.10), $rn(Hd_n) \geq (p - 1)(d + 1) + 1 - 2\, S(Hd_n)$ (2.2.1)

First we compute the status function of Hd_n. In this case Hd_n has one weight centre v_{a+1}.

We have $S(Hd_n) = S_{Hd_n}(v_{a+1})$

$$= \sum_{u \in V(Hd_n)} d(u, v_{a+1})$$

$$= 3.1 + 4\,(2 + \ldots + a)$$

$$= 3 + 4\left(\frac{a(a+1)}{2} - 1\right)$$

$$= 2a^2 + 2a - 1$$

$$= 2\left(\frac{n-1}{2}\right)^2 + 2\left(\frac{n-1}{2}\right) - 1$$

$$= \frac{n^2 - 3}{2} \qquad (2.2.2)$$

Substituting (2.2.2) in (2.2.1) we get

$rn(Hd_n) \geq (p-1)(d+1) + 1 - 2\left(\frac{n^2-3}{2}\right)$

$= (2n-2-1)(n-1+1) + 1 - 2\left(\frac{n^2-3}{2}\right)$

$= (2n-3)\,n + 1 - (n^2 - 3)$

$= n^2 - 3n + 4$

Therefore $rn(Hd_n) \geq n^2 - 3n + 4$

Let $\{x_1, x_2, ..., x_p\}$ be the ordering of the vertices of Hd_n.

Label the vertices $x_1, x_2, \ldots, x_p$ as in the following procedure

$v_{a+1} \to v_1 \to v'_{a+1} \to v'_{2a} \to v'_a$

$v'_{2a-1} \to v'_{a-1} \to v'_{2a-2} \to v'_{a-2}$

$\ldots \quad \ldots \quad \ldots \quad \ldots$

$v'_{a+2} \to v'_2 \to v_{2a+1} \to v_a$

$v_{2a} \to v_{a-1} \to v_{2a-1} \to v_{a-2}$

$v_{2a-2} \to \ldots \to v_2 \to v_{a+2}$.

Define a function h: $V(Hd_n) \to \{0, 1, 2, \ldots, n^2 - 3n + 4\}$ by $h(x_1) = 0$ and $h(x_{i+1}) = h(x_i) + d + 1 - d(x_{i+1}, x_i)$ for $1 \leq i \leq p-1$.

Thus it is possible to assign labels to the vertices of Hd_n with span equal to the lower bound.

Therefore $rn(Hd_n) \leq n^2 - 3n + 4$

Hence $rn(Hd_n) = n^2 - 3n + 4$, $n = 2a+1$, $a \geq 2$.

Example 2.2 In Table 2, Figure 4, Figure 5 and Figure 6 an ordering of the vertices, ordering version, renamed version and optimal radio labeling for Hd_9 are shown.

Table 2

$v_5 \to v_1 \to v'_5 \to v'_8 \to v'_4$

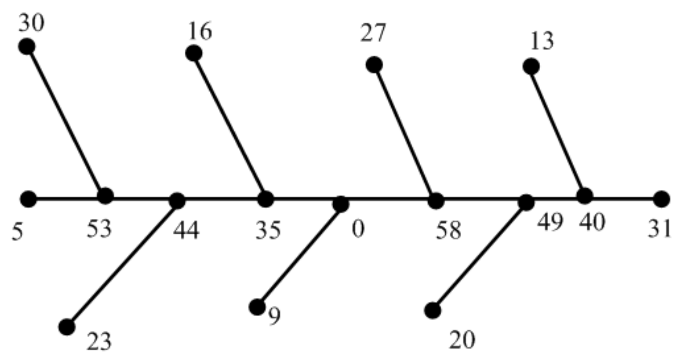
$v'_7 \to v'_3 \to v'_6 \to v'_2$

$v_9 \to v_4 \to v_8 \to v_3$

$v_7 \to v_2 \to v_6$.

Figure 4

Figure 5



$rn(Hd_9) = 58$

Figure 6

Theorem 2.3 Let $BR_{n,\ m}$ denote the biregular rooted tree in which it consists of a path of order n and degree m. Then $rn(BR_{n,\ m}) = \frac{1}{2}\left[n^2(m-1) + m + 1\right] + n + 1$, if n is odd and $m \geq 3$.

Proof. Let h be an optimal radio labeling for $BR_{n,\ m}$ and $\{x_1, x_2, ..., x_p\}$ be the ordering of $V(BR_{n,\ m})$ such that $0 = h(x_1) < h(x_2) < ... < h(x_p)$. Then $d(x_i, x_{i+1}) + | h(x_{i+1}) - h(x_i) | \geq 1 + d$, $1 \leq i \leq p - 1$.
In $BR_{n,\ m}$, the total number of vertices = p = nm - n + 2 and diameter d = n + 1.
If we choose x_1 as the median vertex then x_p must not be adjacent to x_1. Choose the vertex x_i such that x_i and x_{i+1} belong to different branches.
By (1.12), $rn(BR_{n, m}) \geq (p - 1)(d + 1) + 1 - 2S(BR_{n, m}) + 1$, (2.3.1)
where $S(BR_{n, m})$ is the status of the graph $BR_{n, m}$. From 1.11 we have

$S(BR_{n,m}) = \frac{(n-1)^2}{4}(m-1) + 3\left(\frac{n-1}{2}\right)(m-1) + m$ (2.3.2)

Substituting (2.3.2) in (2.3.1) we get

$rn(BR_{n,m}) \geq (p-1)(d+1) + 1 - 2\left(\frac{(n-1)^2}{4}(m-1) + 3\left(\frac{n-1}{2}\right)(m-1) + m\right) + 1$

$= (nm - n + 1)(n + 2) + 1 - 2\left(\frac{(n-1)^2}{4}(m-1) + 3\left(\frac{n-1}{2}\right)(m-1) + m\right) + 1$

$= \frac{1}{2}\left[n^2(m-1) + m + 1\right] + n + 1$

Hence $rn(BR_{n,m}) \geq \frac{1}{2}\left[n^2(m-1) + m + 1\right] + n + 1$ if n is odd.

Assume that m is odd

Define a function $h: V(BR_{n,m}) \rightarrow \{0, 1, 2, \ldots, \frac{1}{2}\left[n^2(m-1) + m + 1\right] + n + 1\}$ by $h(x_1) = 0$ and

$h(x_{i+1}) = h(x_i) + d + 1 - d(x_{i+1}, x_i)$ for $1 \leq i \leq p-1$.

Thus it is possible to assign labels to the vertices of $BR_{n,m}$ with span equal to the lower bound. Therefore

$rn(BR_{n,m}) \leq \frac{1}{2}\left[n^2(m-1) + m + 1\right] + n + 1$.

Hence $rn(BR_{n,m}) = \frac{1}{2}\left[n^2(m-1) + m + 1\right] + n + 1$ when m is odd.

The case when m is even follows similarly.

Example 2.3 For the graph $BR_{3,5}$ in Figure 7, $rn(BR_{3,5}) = 25$

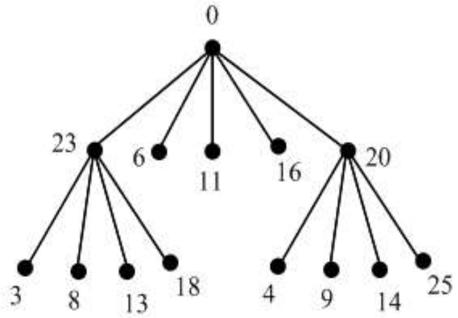


Figure 7

Observations (i) $rn(BR_{n,m}) = rn(BR_{n,m-1}) + 5$

(ii) $diam(BR_{n,m}) = diam(BR_{n,m-1})$

Theorem 2.4 Let $BR_{n,m}$ denote the biregular rooted tree in which it consists of a path of order n and degree m. Then $rn(BR_{n,m}) = \frac{1}{2}\left[n^2(m-1)\right] + n + 1$ if n is even and $m \geq 3$.

Proof. Let h be an optimal radio labeling for $BR_{n,m}$ and $\{x_1, x_2, \ldots, x_p\}$ be the ordering of $V(BR_{n,m})$ such that $0 = h(x_1) < h(x_2) < \ldots < h(x_p)$. Then $d(x_i, x_{i+1}) + |h(x_{i+1}) - h(x_i)| \geq 1 + d$, $1 \leq i \leq p-1$.

In $BR_{n,m}$, the total number of vertices = p = nm - n + 2 and diameter d = n + 1.

Choose the first vertex x_1 as the median vertex. Choose the next vertex x_2 such that x_1 and x_2 belongs to different branches. Proceeding like this, choose the vertex x_p such that x_{p-1} and x_p belongs to different branches.

By (1.12), $rn(BR_{n,m}) \geq (p-1)(d+1) + 1 - 2S(BR_{n,m})$ (2.4.1)

where $S(BR_{n,m})$ is the status of the graph $BR_{n,m}$. From 1.11 we have

$S(BR_{n,m}) = \frac{n^2}{4}(m-1) + n(m-1) + 1$ (2.4.2)

Substituting (2.4.2) in (2.4.1) we get

$$rn(BR_{n,\,m}) \geq (p - 1)(d + 1) + 1 - 2\left(\frac{n^2}{4}(m - 1) + n(m - 1) + 1 \right)$$

$$= (nm - n - 1)(n + 2) + 1 - 2\left(\frac{n^2}{4}(m - 1) + n(m - 1) + 1 \right)$$

$$= \frac{1}{2}\left[n^2 (m - 1) \right] + n + 1$$

Hence $rn(BR_{n,\,m}) \geq \frac{1}{2}\left[n^2 (m - 1) \right] + n + 1$

Assume that m is odd

Define a function h: $V(BR_{n,\,m}) \rightarrow \{0, 1, 2, \ldots, \frac{1}{2}\left[n^2 (m - 1) \right] + n + 1\}$ by $h(x_1) = 0$ and

$h(x_{i+1}) = h(x_i) + d + 1 - d(x_{i+1}, x_i)$ for $1 \leq i \leq p - 1$

Thus it is possible to assign labels to the vertices of $BR_{n,\,m}$ with span equal to the lower bound.

Therefore $rn(BR_{n,\,m}) \leq \frac{1}{2}\left[n^2 (m - 1) \right] + n + 1$

Hence $rn(BR_{n,\,m}) = \frac{1}{2}\left[n^2 (m - 1) \right] + n + 1$ when m is odd.

The case when m is even follows similarly.

Example 2.4 For the graph $BR_{4,\,5}$ in Figure 8, rn $(BR_{4,\,5}) = 37$.

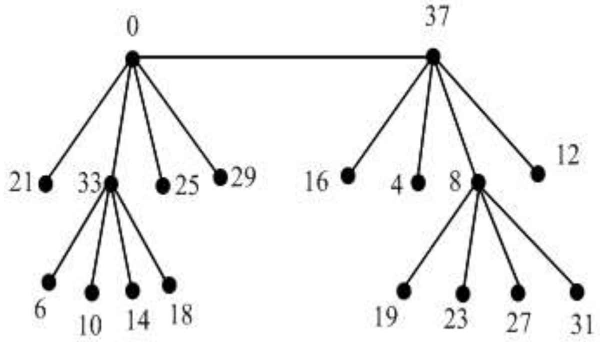


Figure 8

Observations: (i) rn $(BR_{n,\,m})$ = rn $(BR_{n,\,m-1}) + 8$ for all $m \geq 3$

(ii) diam $(BR_{n,\,m})$ = diam $(BR_{n,\,m-1})$

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