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The Finite Generalized Hankel-Clifford Transformation of Certain Spaces of Ultra distributions with Applications To Heat Equations

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Abstract: *In this paper certain spaces of testing functions contained in spaces are introduced. The elements of the dual spaces are ultradistributions. The finite generalized Hankel-Clifford transform is a continuous linear operator in spaces of these type. The finite generalized Hankel-Clifford transformation is defined as a continuous linear mapping between the dual spaces. The developed theory is applied to find the general solutions for a Cauchy problem.*

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I. Introduction

Finite Hankel and Hankel type transform of classical functions were first introduced by I. N. Sneddon [11] and later studied by other authors [3, 6]. Recently J. N. Pandey and R. S. Pathak [8], R. S. Pathak [9] and Malgonde and Lakshmi Gorty [5] extended these transforms to certain spaces of distributions as a special case of the general theory on orthonormal series expansions of generalized functions. L. S. Dube [7], R. S. Pathak and O. P. Singh [10], Malgonde and Lakshmi Gorty [5], investigated finite Hankel transformations and their generalizations in other spaces of distributions through a procedure quite different from that one which was in [1,4]. All previous authors employ a method usually known as the kernel method.Specifically, Malgonde andGorty [5] investigated finite generalized Hankel-Clifford transformation of the first kind given by

$$(\hbar_{\alpha,\beta} f)(n) = F_{\alpha,\beta}(n) = \int_0^1 x^{-(\alpha+\beta)} \mathcal{J}_{\alpha,\beta}(\lambda_n x) f(x)\,dx,\ n = 1,2,3\ldots \qquad (1.1)$$

for $(\alpha - \beta) \geq -\frac{1}{2}$, where $\mathcal{J}_{\alpha,\beta}(z) = z^{(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\sqrt{z}\right)$, $J_{\alpha-\beta}(z)$ denotes the Bessel function of first kind and order $(\alpha-\beta)$ and $\lambda_n, n = 0,1,2,\ldots,$ represent the positive roots of $\mathcal{J}_{\alpha,\beta}(\lambda_n x) = 0$ arranged in the ascending order of magnitude [8].

II. Preliminary results and operational calculus

Property 2.1: The operator $\Delta_{\alpha,\beta} = x^{-\beta} D P_{\alpha,\beta}$

where $D = \frac{d}{dx}; D_{\beta} = x^{-\beta} D; P_{\alpha,\beta} = x^{\alpha-\beta+1} D x^{-\alpha}$, and

$$\Delta_{\alpha,\beta} = x^{\beta} D x^{\alpha-\beta+1} D x^{-\alpha} = xD^2 + \left(1-\alpha-\beta\right)D + x^{-1}\alpha\beta \qquad (2.1)$$

is not self adjoint. Considering the operator,

$$\Delta^*_{\alpha,\beta} = x^{-\alpha} D x^{\alpha-\beta+1} D x^{\beta} = xD^2 + \left(1+\alpha+\beta\right)D + x^{-1}\alpha\beta\ . \qquad (2.2)$$

$\Delta^*_{\alpha,\beta}$ is called the adjoint operator of $\Delta_{\alpha,\beta}$.

Note that $\Delta^*_{\alpha,\beta} = x^{-\alpha-\beta}\Delta_{\alpha,\beta} x^{\alpha+\beta}$ and $D = \frac{d}{dx}; D^*_{\alpha} = x^{-\alpha} D; P^*_{\alpha,\beta} = x^{\alpha-\beta+1} D x^{\beta}$.

Defining the generalized $D^*_\alpha, P^*_{\alpha,\beta}, P^{-1*}_{\alpha,\beta}$ and $\Delta^*_{\alpha,\beta}$ as the adjoint of the classical operators of $D_\beta, P_{\alpha,\beta}, P^{-1}_{\alpha,\beta}$ and $\Delta_{\alpha,\beta}$ respectively the following:

Property 2.2: The operator $f \to \Delta_{\alpha,\beta} f$ defined on $\left({}_pS^{A,B}_{\alpha,\beta}\right)'$ is also a continuous linear mapping of $\left({}_pS^{A,B}_{\alpha,\beta}\right)'$ into itself [9].

The mapping $\hbar^*_{\alpha,\beta} : \left({}_pS^{A,B}_{\alpha,\beta}\right)' \to \left({}_pS^{A,B}_{\alpha,\beta-1}\right)'$ is an isomorphism $\hbar^{*-1}_{\alpha,\beta}$ is its inverse.

III. Multiplier in spaces

The smooth functions on $0 < x < 1$ which are multipliers in the spaces of the type $\left({}_pS^{A,B}_{\alpha,\beta}\right)'$ is defined considering $\theta \in \boldsymbol{C}(I)$ be a function such that:

Definition: The set of all infinitely smooth functions on (0,1)satisfying

$$\left|x^m D^k\left(x^\beta \theta(x)\psi(x)\right)\right| \le C^{\beta,\delta}_k (A+\delta)_{m,k}\, a_{m,k} \tag{3.1}$$

where A, $C^{\beta,\delta}_k$ are positive constants depending on $\theta(x)\psi(x)$ and $a > 0$ being an arbitrary constant.

Thus $\theta(x)\psi(x)$ is in ${}_pS_{\beta,A}$ and the mapping $h_\beta : {}_pS_{\beta,A} \to {}_pS^A_\beta$, $\psi \to \theta\psi$, is continuous.

Taking a ψ in ${}_pS_{\beta,A}$

$$\left|x^m D^k\left(x^\beta \psi(x)\theta(x)\right)\right| \le C^{\beta,\rho}_k (B+\rho)_{m,k}\, b_{m,k} \tag{3.2}$$

where $B, C^{\beta,\rho}_k$ are positive constants depending on $\theta(x)\psi(x)$ and $b > 0$ being an arbitrary constant.

Considering from [5], $\mathcal{J}_{\alpha,\beta}(z) = z^{(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\sqrt{z}\right)$. And as the transformation is an automorphism onto $H_{\alpha,\beta}$ for $\alpha - \beta$, $\dfrac{d^n}{dz^n}\mathcal{J}_{\alpha,\beta}(z) = (-1)^n \mathcal{J}_{\alpha,\beta+n}(z)$, $n \in \boldsymbol{N}$, then for every $\phi \in H_{\alpha,\beta}$ and $m, k \in \boldsymbol{N}$.

$$y^m D^k\left(y^\beta \psi(y)\right) = (-1)^k \int_0^1 (xy)^m \mathcal{J}_{\alpha,\beta+k+m}(xy)\, x^{(-\beta+k+m)} D^m x^\beta \phi(x)\, dx \tag{3.3}$$

where $\psi(y) = \hbar_{\alpha,\beta}\left\{\phi(x)\right\}(y)$.

The virtue of boundedness of the function $z^m \mathcal{J}_{\alpha,\beta+k+m}(z)$, (3.3) is given by

$$\left|y^m D^k\left(y^\beta \psi(y)\right)\right| \le M \sup_{x\in I}\left|x^{(c+k+m)} D^m x^\beta \phi(x)\right| \tag{3.4}$$

for $m, k \in \boldsymbol{N}$ and $\mu \ge 0$, being $c = \left[-\beta\right]$ and M is a constant.

To study the image of ${}_pS_\beta$ by h_β . Let ϕ be any element of ${}_pS_{\beta,A}$ invoking (3.4), then

$$\left|y^m D^k\left(y^\beta \psi(y)\right)\right| \le K_{m,\delta}(A+\delta)^k \left\{p\left((c+k+m)\right)\right\}!^a \le C_{m,\delta}(A+\delta)^k (pk)!^a \tag{3.5}$$

for every $m, k \in \boldsymbol{N}$ and $\delta > 0$.

Hence the mapping $h_\beta : {}_pS_{\beta,A} \to {}_pS^A_\beta$ is linear and continuous.

If $\phi \in {}_pS_{\alpha,B}$, then

$$\left|y^m D^k\left(y^{-\alpha}\psi(y)\right)\right| \le M_{k,\rho}\left\{C_{c+k+m,\rho}(B+\rho)^m mp\right\}!^b \tag{3.6}$$

for $m, k \in \boldsymbol{N}$ and $\rho > 0$.

Therefore the mappings:

$h_\alpha : {}_pS_{\alpha,\mathrm{B}} \to {}_pS^B_\alpha$ is linear and continuous.

If $\phi \in {}_pS^{A,B}_{\alpha,\beta}$, then

$$\left| y^m D^k \left(y^{-(\alpha-\beta)} \psi(y) \right) \right| \le C_{m,\delta} (A+\delta)^k (pk)!^a \times M_{k,\rho} \left\{ C_{c+k+m,\rho} (B+\rho)^m mp \right\}!^b$$
$$\le M_{\delta,\rho} \left(Ae^{p^\alpha} + \eta \right)^k \left(Be^{p^\alpha} + \varepsilon \right)^m (p)!^{a+b} (k)!^a (m)!^b \qquad (3.7)$$

for $m, k \in \mathbf{N}$ and $\eta, \varepsilon > 0$.

Thus it has been established that the mapping $h_{\alpha,\beta} : {}_pS^{A,B}_{\alpha,\beta} \to {}_pS^{Ae^{p^\alpha}, Be^{p^\alpha}}_{\alpha,\beta}$ is linear and continuous.

IV. The finite generalized Hankel-Clifford transformation in the spaces

Theorem 4.1: The mappings

i. $h_\alpha : {}_pS_{\alpha,B} \to {}_pS^B_\alpha$

ii. $h_\beta : {}_pS_{\beta,A} \to {}_pS^A_\beta$

iii. $h_{\alpha,\beta} : {}_pS^{A,B}_{\alpha,\beta} \to {}_pS^{Ae^{p^\alpha}, Be^{p^\alpha}}_{\alpha,\beta}$

are linear and continuous.

Defining the finite generalized Hankel-Clifford transformation $h'_{\alpha,\beta}$ as the adjoint of the classical transformation $h_{\alpha,\beta}$, it can be seen as the orthogonal series expansions of generalized functions and the distributional finite generalized Hankel-Clifford transformation in [5] stated where every member $f \in \left({}_pS^{A,B}_{\alpha,\beta} \right)'$ can be expanded into a generalized series of the form $f = \sum_{n=1}^{\infty} \frac{1}{a^{2-\alpha-\beta} \lambda_n \mathcal{J}^2_{\alpha.\beta-1}(\lambda_n a)} \left(f, \phi_n^* \right) \phi_n$ which converges in $\left({}_pS^{A,B}_{\alpha,\beta} \right)'$. For $f \in \left({}_pS^{A,B}_{\alpha,\beta} \right)^{*\prime}$, then $f = \sum_{n=1}^{\infty} \frac{1}{a^{2-\alpha-\beta} \lambda_n \mathcal{J}^2_{\alpha.\beta-1}(\lambda_n a)} \left(f, \phi_n \right) \phi_n^*$ where the series converges in $\left({}_pS^{A,B}_{\alpha,\beta} \right)^{*\prime}$. The distributional finite generalized Hankel-Cliffordtransformation of the first kind of $f \in \left({}_pS^{A,B}_{\alpha,\beta} \right)'$ is defined in [5] as

$$\left(\hbar'_{\alpha,\beta} f \right)(n) = F_{\alpha,\beta}(n) = \left(f(x), \phi_n^*(x) \right) = \left(f(x), x^{-(\alpha+\beta)} \mathcal{J}_{\alpha,\beta}(\lambda_n x) \right) \qquad (4.1)$$

for each value of $n = 1, 2, 3, \ldots$. Its corresponding inversion formula is given as

$$\left(\hbar'^{-1}_{\alpha,\beta} F_{\alpha,\beta} \right)(x) = f(x) = \sum_{n=1}^{\infty} \frac{F_{\alpha,\beta}(n) \mathcal{J}_{\alpha,\beta}(\lambda_n x)}{a^{2-\alpha-\beta} \lambda_n \mathcal{J}^2_{\alpha.\beta-1}(\lambda_n a)}. \qquad (4.2)$$

This formula may be rewritten analogous to [5] as

$$\hbar'_{\alpha,\beta} \left(\Delta^k_{\alpha,\beta} f \right) = \left(-\lambda_n \right)^k \hbar'_{\alpha,\beta} f \qquad (4.3)$$

for every $f \in \left({}_pS^{A,B}_{\alpha,\beta} \right)'$ and $k = 0, 1, 2, \ldots$. To introduce other variant of the distributional finite generalized Hankel-Clifford transformation of the first kind in the space $\left({}_pS^{A,B}_{\alpha,\beta} \right)'$ by means of

$$\left(\hbar^{*\prime}_{\alpha,\beta} f \right)(n) = F^*_{\alpha,\beta}(n) = \left(f, \phi_n \right) = \left(f, x^{-(\alpha+\beta)} \mathcal{J}_{\alpha,\beta}(\lambda_n x) \right) \qquad (4.4)$$

where $f \in \left({}_pS^{A,B}_{\alpha,\beta} \right)^{*\prime}$ for each value of *n*= 1,2,…

The inversion formula is given through

$$\left(\hbar^{*'^{-1}}_{\alpha,\beta} F^*_{\alpha,\beta}\right)(x) = f(x) = \sum_{n=1}^{\infty} \frac{F^*_{\alpha,\beta}(n)}{a^{2-\alpha-\beta} \lambda_n \mathcal{J}^2_{\alpha,\beta-1}(\lambda_n a)} x^{-(\alpha+\beta)} \mathcal{J}_{\alpha,\beta}(\lambda_n x) . \qquad (4.5)$$

Since $\left({}_pS^{A,B}_{\alpha,\beta}\right)^* \subset \left({}_pS^{A,B}_{\alpha,\beta}\right)'$, it out to be an extension to distributions and an immediate consequence of the inclusion ${}_pS^{A,B}_{\alpha,\beta} \subset \left({}_pS^{A,B}_{\alpha,\beta}\right)^{*'}$, agrees with the distributional finite generalized Hankel-Clifford transformation (4.1), so that theorem 4.1 appears now as distributional sense.

Theorem 4.2: The operators

i. $\hbar^{*'}_{\alpha,\beta} : \left({}_pS^{A,B}_{\alpha,\beta}\right)^{*'} \to \left({}_pS^{Ae^{p^{\alpha}},Be^{p^{\alpha}}}_{\alpha,\beta}\right)^{*'}$

ii. $h^{*'}_{\alpha} : \left({}_pS_{\alpha,B}\right)^{*'} \to \left({}_pS^{B}_{\alpha}\right)^{*'}$

iii. $h^{*'}_{\beta} : \left({}_pS_{\beta,A}\right)^{*'} \to \left({}_pS^{A}_{\beta}\right)^{*'}$

are linear and continuous.

The proof is analogous as in [2].

V. Applicationsusing Kepinski-Myller-Lebedev partial differential

Equation using operator $\Delta^*_{\alpha,\beta}$ in heat equation:

To illustrate the use of the distributional finite generalized Hankel-Clifford transformation in heat equation, the following generalized Kepinski-Myller-Lebedev partial differential equation in a finite interval is solved.

$$x\frac{\partial^2 v}{\partial x^2} + (1-\alpha-\beta)\frac{\partial v}{\partial x} + x^{-1}\alpha\beta v - P\frac{\partial v}{\partial t} = 0, 0 < x < a, t > 0 \qquad (5.1)$$

satisfying boundary conditions

i) As $t \to 0+; v(x,t_0) \approx \phi_0(x) \in \left({}_pS^{A,B}_{\alpha,\beta}\right)'$

ii) As $t \to \infty$; $v(x,t)$ converges uniformly to zero on $0 < x < a$

iii) As $x \to a-$; $v(x,t)$ converges to zero on $t_0 \le t < \infty$ for each $t_0 < 0$

iv) As $x \to 0+$; $v(x,t) = O\left(x^{(\alpha-\beta)}\right)$ on $t_0 \le t < \infty$.

Let us denote $u(y,t) = \hbar^*_{\alpha,\beta}\{v(x,t); x \to y\}$ and $\phi_0(y) = \hbar^*_{\alpha,\beta}\{\phi_0(x)\}(y)$. According to (4.2), (5.1) becomes

$$\Delta_{\alpha,\beta} v(x,t) - P\frac{\partial v(x,t)}{\partial t} = 0 . \qquad (5.2)$$

By applying $\hbar^*_{\alpha,\beta}$ -transform to (5.2) and making use of (4.3), then

$$\frac{\partial u(y,t)}{\partial t} - P(-y)u(y,t) = 0 \qquad (5.3)$$

and

$$u(y,t_0) \approx \phi_0(y)$$

whose solution is

$$u(y,t) = e^{-Py(t-t_0)}\phi_0(y) \qquad (5.4)$$

where P is a square matrix of polynomials whose solution is because of the boundary conditions (i) and (ii) and y represents the positive zero of the equation $\mathcal{J}_{\alpha,\beta}(ya) = 0$. Invoking the inversion formula (4.5) to provide the required solution

$$v(x,t) = \left(\hbar^{*\prime -1}_{\alpha,\beta} F^{*}_{\alpha,\beta}\right)(x) = \phi(x) = \frac{F^{*}_{\alpha,\beta} e^{-Py(t-t_0)} \mathcal{J}_{\alpha,\beta}(yt)}{a^{2-\alpha-\beta}\, y\, \mathcal{J}^{2}_{\alpha,\beta-1}(ya)}. \quad (5.5)$$

VI. Existence Of generalized solutions

Now considering the initial value problem

$$\frac{\partial u(x,t)}{\partial t} = P\left(B^{*}_{\alpha}\right) u(x,t) \quad (6.1)$$

$u(x,0) = u_0(x)$ with $u_0 \in \varphi'$.

The finite generalized Hankel-Clifford transformation of the first kind $h^{*\prime}_{\alpha}$, leads to the new equivalent problem

$$\frac{\partial v(y,t)}{\partial t} = P(-y)\, v(y,t) \quad (6.2)$$

$v(y,0) = v_0(y)$

where $v(y,t) = h^{*\prime}_{\alpha}\left\{u(x,t), x \to y\right\}$ and $v_0(y) = h^{*\prime}_{\alpha}\left\{u_0(x)\right\}(t)$.

A formal solution of (6.2) is the generalized function $v(y,t) = v_0(y)\, e^{-yt}$.

The distribution $u(x,t) = \hbar'_{\alpha,\beta}\left\{v_0(y)\, e^{-yt}; y \to x\right\} \in \phi'$ is a solution of (6.1).

Accordingly one has:

a) $$\frac{\partial}{\partial t}\left\langle \hbar'_{\alpha,\beta}\left\{v_0(y)\, e^{-yt}\right\}, \phi\right\rangle = \frac{\partial}{\partial t}\left\langle u_0, \hbar_{\alpha,\beta}\left\{e^{-yt}\hbar_{\alpha,\beta}\left\{\phi\right\}\right\}\right\rangle$$
$$= \left\langle u_0, \hbar_{\alpha,\beta}\left\{e^{-yt}\hbar_{\alpha,\beta}\left\{P\left(\Delta_{\alpha,\beta}\right)\right\}\right\}\right\rangle$$
$$= \left\langle P\left(\Delta^{*}_{\alpha,\beta}\right)\hbar'_{\alpha,\beta} e^{-yt}\hbar'_{\alpha,\beta} u_0, \phi\right\rangle.$$

for every $\phi \in \varphi$ and

b) $$\left\langle \hbar'_{\alpha,\beta}\left\{e^{-yt}\hbar'_{\alpha,\beta}\left\{u_0\right\}\right\}, \phi\right\rangle = \left\langle u_0, \hbar_{\alpha,\beta}\left\{e^{-yt}\hbar_{\alpha,\beta}\left\{\phi\right\}\right\}\right\rangle \to \left\langle u_0, \phi\right\rangle$$

for every $\phi \in \varphi$.

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