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Some Reduction Formulas Using Cauchy's Residue Theorem And A New Representation of Beta Function

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Abstract: *In this paper, we presented some definite integral formula using the well-known Cauchy Residue theorem. These formulas may not be easily derived using the elementary integral calculus method. We also presented a new representation of beta function.*
Keywords: *Definite integral, Residue's theorem*

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I. Introduction

There are many definite integral integrals that the elementary methods cannot be used to evaluate, most especially the improper integrals. This is why Mathematicians have developed various methods of evaluating these integrals; one of these methods is Cauchy's Residue theorem.
Particularly, the results obtained in this paper cannot be obtained using the elementary method; this motivated the authors to employ complex variable method. The important of this work is that, we made use of complex variable method.

Theorem (1.1): Cauchy's Residue theorem
If $f(z)$ *is analytic inside and a simple closed contour c except for a finite number of singular points* $z_1, z_2, \ldots, z_n$ *interior to c, then*

$$\int_c f(z)\,dz = 2\pi i\{R_1 + R_2 + \ldots + R_n\} \tag{1.1}$$

where R_i denotes the residue of $f(z)$ at $z = z_i$ and the integral is taken in the counter-clockwise sense around c
Residue theorem is an important tool in the theory of complex variables and can also used to obtain the inverse Laplace transform.

II. our results

1. If n is a positive integer, then

i. $$\int_0^{\frac{\pi}{2}} \sin^{2n} x dx = \int_0^{\frac{\pi}{2}} \cos^{2n} x dx = \frac{\pi(2n)!}{2^{2n+1}[n!]^2}$$

(2.1)

ii. $$\int_0^{\frac{\pi}{2}} \sin^{2n} x \cos^{2n} x dx = \frac{(2n)!}{2^{4n+1}[n!]^2}$$

(2.2)

iii. $$\int_0^{\pi} \tan^{2n} x dx = (-1)^n \pi$$

(2.3)

2. If m and n are positive integers, then

$$\beta(m, n) = \sum_{r=0}^{n-1} {}^{n-1}C_r \frac{(-1)^r}{(m+r)}$$

(2.4)

To carry out the analysis of our proofs, some lemma and their proofs are necessary.

Lemma 1.1: *If n is a positive integer, then*

$$\lim_{z\to 0}\frac{d^{2n}}{dz^{2n}}\left(z^2-1\right)^{2n}=(-1)^n\,{}^{2n}C_n\,(2n)!=(-1)^n\left[\frac{(2n)!}{n!}\right]^2$$

Proof: If $n = 1$; $\lim_{z\to 0}\frac{d^2}{dz^2}\left(z^2-1\right)^2=\lim_{z\to 0}\frac{d^2}{dz^2}\left(z^4-2z^2+1\right)=-2.2!=(-1)^1\,{}^2C_1 2!$

If $n = 2$, $\lim_{z\to 0}\frac{d^4}{dz^4}\left(z^2-1\right)^4=\lim_{z\to 0}\frac{d^2}{dz^2}\left(z^8-4z^6+6z^4-4z^2+1\right)=+6.4!=(-1)^2\,{}^4C_2 4!$

Continuing in this manner yields the result.

Lemma 1.2: *If n is a positive integer, then*

$$\lim_{z\to 0}\frac{d^{4n}}{dz^{4n}}\left(z^{4n}-1\right)^{2n}=(-1)^n\,{}^{2n}C_n\,(4n)!=(-1)^n\frac{(2n)!(4n)!}{[n!]^2}$$

Proof: Replacing n by $2n$ on the derivative of lemma (1.1), we get

$$\lim_{z\to 0}\frac{d^{2(2n)}}{dz^{2(2n)}}\left(z^2-1\right)^{2n}=(-1)^n\,{}^{2n}C_n\,\big(2(2n)\big)! \qquad \therefore \lim_{z\to 0}\frac{d^{4n}}{dz^{4n}}\left(z^2-1\right)^{2n}=(-1)^n\,{}^{2n}C_n\,(4n)!$$

Hence the result follows

Lemma 1.3: *If n is a positive integer, then*

$$\lim_{z\to 0}\frac{d^{2n-1}}{dz^{2n-1}}\frac{\left(z^2-1\right)^{2n}}{z\,(z+i)^{2n}}=0; \quad i=\sqrt{-1}.$$

Proof: The proof follows the method adopted in proving the lemma (1.1).

III. Proof of our Results

3.1 Proof of (2.1)

We know that if $f(x)=f(2a-x)$, then $\int_0^{2a} f(x)\,dx=2\int_0^a f(x)\,dx$

$$\therefore \int_0^{\frac{\pi}{2}}\sin^{2n}x\,dx=\frac{1}{4}\int_0^{2\pi}\sin^{2n}x\,dx$$

On letting $z=e^{ix}$ and simplifying, we obtain

$$\therefore \int_0^{\frac{\pi}{2}}\sin^{2n}x\,dx=\frac{1}{2^{2n+2}i^{2n+1}}\int_c\frac{\left(z^2-1\right)^{2n}}{z^{2n+1}}dz=\frac{(-1)^n}{2^{2n+2}i}\int_c\frac{\left(z^2-1\right)^{2n}}{z^{2n+1}}dz=\frac{(-1)^n}{2^{2n+2}i}\left[2\pi\mathrm{i}\,\mathrm{Rez}\{f(z)\}\right]$$

Now, $f(z)=\frac{\left(z^2-1\right)^{2n}}{z^{2n+1}}$ has singular point $z = 0$ at a pole of order $(2n + 1)$ in the unit disk c. Thus, the residue of f(z) at this pole becomes

$$\mathrm{Rez}\{f(z),\ z=0\}=\frac{1}{(2n)!}\lim_{z\to 0}\frac{d^{2n}}{dz^{2n}}\frac{\left(z^2-1\right)^{2n}}{z^{2n+1}}=\frac{1}{(2n)!}(-1)^n\,{}^{2n}C_n\,(2n)!=(-1)^n\frac{(2n)!}{[n!]^2}$$

where we have applied lemma (1.1). Therefore, by residue theorem, we get

$$\therefore \int_0^{\frac{\pi}{2}}\sin^{2n}x\,dx=\frac{(-1)^n}{2^{2n+2}i}\left[2\pi i(-1)^n\frac{(2n)!}{[n!]^2}\right]=\frac{\pi\,(2n)!}{2^{2n+1}[n!]^2}$$

Similarly, $\int_0^{\frac{\pi}{2}}\cos^{2n}x\,dx=\frac{\pi\,(2n)!}{2^{2n+1}[n!]^2}$

Verification of the Result

If $n = 1$, $\therefore \int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \dfrac{\pi\,(2)!}{2^{2+1}[1!]^2} = \dfrac{\pi}{4}$

By direct method, $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \dfrac{1}{2}\int_0^{\frac{\pi}{2}}(1-\cos 2x)\,dx = \dfrac{1}{2}\left[x - \dfrac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}} = \dfrac{1}{2}\left[\dfrac{\pi}{2}\right] = \dfrac{\pi}{4}$

3.2 Proof of (2.2)

Proceeding as before, we have that

$$\int_0^{\frac{\pi}{2}} \sin^{2n} x\cos^{2n} x\,dx = \frac{1}{4}\int_0^{2\pi} \sin^{2n} x\cos^{2n} x\,dx$$

On letting $z = e^{ix}$ and simplifying, we obtain

$$\int_0^{\frac{\pi}{2}} \sin^{2n} x\cos^{2n} x\,dx = \frac{(-1)^n}{2^{4n+2}i}\int_c \frac{\left(z^4-1\right)^{2n}}{z^{4n+1}}\,dz$$

$f(z) = \dfrac{\left(z^4-1\right)^{2n}}{z^{4n+1}}$ encloses singular point $z = 0$ in the unit disk c at a pole of order $(4n + 1)$

$$\therefore \operatorname{Rez}\{f(z),\ z=0\} = \frac{1}{(4n)!}\lim_{z\to 0}\frac{d^{2n}}{dz^{2n}}\frac{\left(z^4-1\right)^{2n}}{z^{4n+1}} = \frac{1}{(4n)!}(-1)^n\frac{(2n)!(4n)!}{[n!]^2} = (-1)^n\frac{(2n)!}{[n!]^2}$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^{2n} x\cos^{2n} x\,dx = \frac{(-1)^n}{2^{4n+2}i}\int_c \frac{\left(z^4-1\right)^{2n}}{z^{4n+1}}\,dz = \frac{(-1)^n}{2^{4n+2}i}\left\{2\pi i\times \operatorname{Rez}\left(f(z),0\right)\right\}$$

$$= \frac{(-1)^n}{2^{4n+2}i}\left[2\pi i(-1)^n\frac{(2n)!}{[n!]^2}\right] = \frac{\pi\,(2n)!}{2^{4n+1}[n!]^2}$$

Verification of the Result

If $n = 1$, $\int_0^{\frac{\pi}{2}} \sin^2 x\cos^2 x\,dx = \dfrac{\pi\,(2)!}{2^{4+1}[1!]^2} = \dfrac{\pi}{16}$

Using the elementary method

$$\int_0^{\frac{\pi}{2}} \sin^2 x\cos^2 x\,dx = \frac{1}{8}\int_0^{\frac{\pi}{2}}(1-\cos 4x)\,dx = \frac{1}{8}\left[x-\frac{\sin 4x}{4}\right]_0^{\frac{\pi}{2}} = \frac{1}{8}\left[\frac{\pi}{2}\right] = \frac{\pi}{16}$$

Hence the result is verified.

3.3 Proof of 2.3

$$\int_0^{\pi}\tan^{2n} x\,dx = \frac{1}{2}\int_0^{2\pi}\tan^{2n} x\,dx = \frac{1}{2}\int_c\left[\frac{z-z^{-1}}{2i}\right]^{2n}\left[\frac{2}{z+z^{-1}}\right]^{2n}\frac{dz}{iz} = \frac{(-1)^n}{2i}\int_c\frac{\left(z^2-1\right)^{2n}}{z\left(z^2+1\right)^{2n}}\,dz$$

Now, $f(z) = \dfrac{\left(z^2-1\right)^{2n}}{z\left(z^2+1\right)^{2n}}$ encloses singular point $z = 0$ at a simple pole and $z = \pm i$ at a pole of order $(2n)$

$$\therefore \operatorname{Rez}\{f(z),\ z=0\} = \lim_{z\to 0}\frac{\left(z^2-1\right)^{2n}}{\left(z^2-1\right)^{2n}} = 1$$

$$\operatorname{Rez}\{f(z),\ z=i\}=\frac{1}{(2n-1)!}\lim_{z\to i}\frac{d^{2n-1}}{dz^{2n-1}}\frac{(z^2-1)^{2n}}{z(z+i)^{2n}}=0$$

$$\operatorname{Rez}\{f(z),\ z=-i\}=\frac{1}{(2n-1)!}\lim_{z\to i}\frac{d^{2n-1}}{dz^{2n-1}}\frac{(z^2-1)^{2n}}{z(z-i)^{2n}}=0$$

$$\therefore \int_0^{\pi}\tan^{2n}x\,dx=\frac{1}{2}\int_0^{2\pi}\tan^{2n}x\,dx=\frac{(-1)^n}{2i}(2\pi i)(1+0+0)=(-1)^n\pi$$

3.4 Proof of (2.4)

We know that $\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx=\int_0^1 x^{m-1}\sum_{r=0}^{n-1}{}^{n-1}C_r(-1)^r x^r dx=\sum_{r=0}^{n-1}{}^{n-1}C_r(-1)^r\int_0^1 x^{m+r-1}dx$

$$=\sum_{r=0}^{n-1}{}^{n-1}C_r(-1)^r\left[\frac{x^{m+r}}{m+r}\right]_0^1=\sum_{r=0}^{n-1}{}^{n-1}C_r\frac{(-1)^r}{(m+r)}$$

IV. Conclusion

The important thing we did in this research work is that we have used complex variable method to obtain some reduction formulas. Also, it is not necessary to understand the concept of gamma function before evaluating the beta function of positive integers.

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