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A Note On Topology Of Non-Newtonian Real Numbers

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Abstract: *In this work, the authors examine the open and closed sets on the non-newtonian real line, and relationship between them.*

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I. Introduction

A generator is a one-to-one function α , whose domain is $\mathbb{R}$, the set of all real numbers, and whose range is a subset of $\mathbb{R}$. Identity function and exponential function can be given as examples of generators. The range of generator α , called Non-Newtonian real line, is denoted by $\mathbb{R}(N)$.

A $\alpha-$positive number is a number x with $\dot{0} \dot{<} x$, similarly a $\alpha-$negative number is a number x with $\dot{0} \dot{>} x$. $\alpha-$zero and $\alpha-$one numbers are denoted by $\dot{0}=\alpha(0)$ and $\dot{1}=\alpha(1)$, respectively. $\alpha-$integers is obtained sequentially by adding $\dot{1}$ to $\dot{0}$ and by subtracting $\dot{1}$ from $\dot{0}$. $\alpha-$integers are as follows:

$$...,\alpha(-2),\alpha(-1),\alpha(0),\alpha(1),\alpha(2),... \ .$$

Each integer n according to $\alpha-$arithmetic is denoted by $\dot{n}=\alpha(n)$.

Non-Newtonian arithmetic operations on $\mathbb{R}(N)$ are represented as follows ([1],[2],[3],[4],[5]).

$$\alpha-addition \qquad x\dot{+}y=\alpha\left(\alpha^{-1}(x)+\alpha^{-1}(y)\right)$$

$$\alpha-substraction \qquad x\dot{-}y=\alpha\left(\alpha^{-1}(x)-\alpha^{-1}(y)\right)$$

$$\alpha-multiplication \qquad x\dot{\times}y=\alpha\left(\alpha^{-1}(x)\times\alpha^{-1}(y)\right)$$

$$\alpha-division \qquad x\dot{/}y=\alpha\left(\alpha^{-1}(x)/\alpha^{-1}(y)\right)$$

$$\alpha-order \qquad x\dot{<}y\left(x\dot{\le}y\right)\Leftrightarrow\alpha^{-1}(x)<\alpha^{-1}(y)\left(\alpha^{-1}(x)\le\alpha^{-1}(y)\right).$$

The open α -intervals on $\mathbb{R}(N)$ are represented by

$$(a,b)_N=\left\{x\in\mathbb{R}(N):a\dot{<}x\dot{<}b\right\}$$
$$=\left\{x\in\mathbb{R}(N):\alpha^{-1}(a)<\alpha^{-1}(x)<\alpha^{-1}(b)\right\}=\alpha\left(\left(\alpha^{-1}(a),\alpha^{-1}(b)\right)\right).$$

It is said that an open $\alpha-$interval has $\alpha-$lenght $b\dot{-}a$ ([2],[3]). Likewise closed and semi-open intervals can be represented.

All proven properties here are the generalization of basic topological properties known in real analysis. The reades can refer to the textbook [nat.] for these properties.

II. Main Results

Definition 1. A point c is called an interior point of the subset $E \subset \mathbb{R}(N)$ if there exists an open α - interval, contained entirely in the set E, which contains this point:

$$c \in (a,b)_N \subset E \Leftrightarrow \alpha^{-1}(c) \in \alpha^{-1}\left((a,b)_N\right) = \left(\alpha^{-1}(a), \alpha^{-1}(b)\right) \subset \alpha^{-1}(E).$$

According to this definition, c is an interior point of the subset $E \subset \mathbb{R}(N)$ if and only if $\alpha^{-1}(c)$ is an interior point of the subset $\alpha^{-1}(E) \subset \mathbb{R}$.

Definition 2. A subset $E \subset \mathbb{R}(N)$ is said to be α -open if all of its points are interior points.

According to this definition, an α -open set G is the set that the reverse image $\alpha^{-1}(G)$ is an open in $\mathbb{R}$. Thus, we can say that G is an α -open set in $\mathbb{R}(N)$ if and only if there exists an open α - interval $(a,b)_N$ in $\mathbb{R}(N)$ such that $c \in (a,b)_N \subset G$ for all $c \in G$. Indeed, for any $c \in G$ we have

$$\alpha^{-1}(c) \in (a,b) \subset \alpha^{-1}(G) \Leftrightarrow c \in \alpha\left((a,b)\right) = \alpha\left(\left(\alpha^{-1}(\alpha(a)), \alpha^{-1}(\alpha(b))\right)\right) = \left(\alpha(a), \alpha(b)\right)_N \subset G.$$

Examples. 1) Every open α - interval $(a,b)_N$ is an α -open set in $\mathbb{R}(N)$. Indeed, if $c \in (a,b)_N$, then we have

$$\begin{aligned}
&\alpha^{-1}(c) \in \left(\alpha^{-1}(a), \alpha^{-1}(b)\right) \Rightarrow \exists r > 0 \ni \left(\alpha^{-1}(c) - r, \alpha^{-1}(c) + r\right) \subset \left(\alpha^{-1}(a), \alpha^{-1}(b)\right) \\
&\Rightarrow \exists r > 0 \ni \alpha\left(\alpha^{-1}(c) - \alpha^{-1}(\alpha(r)), \alpha^{-1}(c) + \alpha^{-1}(\alpha(r))\right) \subset \alpha\left(\alpha^{-1}(a), \alpha^{-1}(b)\right) \\
&\Rightarrow \exists r > 0 \ni \left(\alpha\left(\alpha^{-1}(c) - \alpha^{-1}(\alpha(r))\right), \alpha\left(\alpha^{-1}(c) + \alpha^{-1}(\alpha(r))\right)\right) \subset \alpha\left(\alpha^{-1}(a), \alpha^{-1}(b)\right) \\
&\Rightarrow \exists r > 0 \ni \left(c \dot{-} \alpha(r), c \dot{+} \alpha(r)\right)_N \subset (a,b)_N.
\end{aligned}$$

2) The set $\mathbb{R}(N)$ of all non-Newtonian real numbers and the void set ϕ are open.

Theorem 1. The composition of an arbitrary family of α -open sets in $\mathbb{R}(N)$ is an α -open.

Proof. Let $G = \bigcup_i G_i$, where all of the sets G_i are open in $\mathbb{R}(N)$. If $c \in G$, the $c \in G_{i_0}$ for some i_0. Since G_{i_0} is an open set in $\mathbb{R}(N)$, there exists an open α -interval $(a,b)_N$ such that $c \in (a,b)_N \subset G_{i_0} \subset G$. This completes the proof.

Theorem 2. The intersection of a finite number of α -open sets in $\mathbb{R}(N)$ is an α -open set.

Proof. Let G_i be α -open set in $\mathbb{R}(N)$ for all $i = 1, 2, ..., n$ and $G = \bigcap_{i=1}^{n} G_i$. If it is given any element $c \in G$, then there exists an α -interval $(a_i, b_i)_N$ such that

$$c \in (a_i, b_i)_N = \alpha\left(\left(\alpha^{-1}(a_i), \alpha^{-1}(b_i)\right)\right) \subset G_i \Rightarrow \alpha^{-1}(c) \in \left(\alpha^{-1}(a_i), \alpha^{-1}(b_i)\right) \subset \alpha^{-1}(G_i)$$

for all $i = 1, 2, ..., n$.

Now we set the numbers λ and μ by

$$\lambda = \max\left\{\alpha^{-1}(a_1), ..., \alpha^{-1}(a_n)\right\} \text{ and } \mu = \min\left\{\alpha^{-1}(b_1), ..., \alpha^{-1}(b_n)\right\}.$$

Then we have

$$\alpha^{-1}(c) \in (\lambda, \mu) \subset \bigcap_{i=1}^{n} \alpha^{-1}(G_i) \Rightarrow c \in \alpha\left((\lambda, \mu)\right) = \left(\alpha(\lambda), \alpha(\mu)\right)_N \subset \alpha\left(\bigcap_{i=1}^{n} \alpha^{-1}(G_i)\right) \subset G.$$

This shows that G is an α -open set.

Theorem 3. If the set G is an open in $\mathbb{R}(N)$, then its complement $G^c = \mathbb{R}(N) - G$ is closed.

Proof. Let $c \in G$ be an arbitrary point. Then there exists an open α -interval $(a,b)_N$ in $\mathbb{R}(N)$ such that $c \in (a,b)_N \subset G$. According to this, any point of G can not be a limit point of G^c, hence G^c contains all of of its limit points and G^c is closed.

Theorem 4. If the set F is closed in $\mathbb{R}(N)$, then its complement F^c is open.

Proof. Let $c \in F^c$ be an arbitrary point. Then c is not a limit point of F and thus there exists an open α -interval $(a,b)_N$ in $\mathbb{R}(N)$ such that such that $c \in (a,b)_N$ and $(a,b)_N \cap F = \phi$. Hence $c \in (a,b)_N \subset F^c$. Finally, each point of F^c is its interior points and this completes the proof.

Examples. 1) If G is an α -open set in $\mathbb{R}(N)$ and $[a,b]_N$ is a closed α -interval containing G in $\mathbb{R}(N)$, then the set $[a,b]_N - G$ is an α -closed set in $[a,b]_N$.

Solution. Let us accept $G \subset [a,b]_N$. Then we can write

$$G \subset \alpha\left(\left[\alpha^{-1}(a), \alpha^{-1}(b)\right]\right) \Rightarrow \alpha^{-1}(G) \subset \left[\alpha^{-1}(a), \alpha^{-1}(b)\right]$$

and obtain that

$$\left[\alpha^{-1}(a), \alpha^{-1}(b)\right] - \alpha^{-1}(G) = \left[\alpha^{-1}(a), \alpha^{-1}(b)\right] \cap \left(\alpha^{-1}(G)\right)^c$$

is an closed set in $\mathbb{R}$. Hence the set $[a,b]_N - G$ is an α -closed set in $[a,b]_N$.

2) Similar to the previous one, we say that If F is an α -open set in $\mathbb{R}(N)$ and $(a,b)_N$ is a open α -interval containing G in $\mathbb{R}(N)$, then the set $(a,b)_N - F$ is an α -open set in $[a,b]_N$.

On the other hand, if F is a α -closed subset in $\mathbb{R}(N)$ ve $F \subset [a,b]_N$, then the set $[a,b]_N - F$ is always not α -open. For example, let $F = [\alpha(0), \alpha(1)]_N$ and $[a,b]_N = [\alpha(0), \alpha(2)]_N$. Then we have the set

$$\begin{aligned}
[a,b]_N - F &= \alpha\left(\left[\alpha^{-1}\left(\dot{0}\right), \alpha^{-1}\left(\dot{2}\right)\right]\right) - \alpha\left(\left[\alpha^{-1}\left(\dot{0}\right), \alpha^{-1}(1)\right]\right) \\
&= \alpha\left(\left[\alpha^{-1}\left(\dot{0}\right), \alpha^{-1}\left(\dot{2}\right)\right] - \left[\alpha^{-1}\left(\dot{0}\right), \alpha^{-1}(1)\right]\right) \\
&= \alpha\left(\left(\alpha^{-1}\left(\dot{1}\right), \alpha^{-1}\left(\dot{2}\right)\right]\right) = \alpha\left((1,2]\right) = \left(\dot{1}, \dot{2}\right]_N,
\end{aligned}$$

what is neither open nor closed.

Definition 3. Let E be a non-void bounded subset in $\mathbb{R}(N)$ and let $a = \inf_N E,\ b = \sup_N E$. The closed α -interval $S = [a,b]_N$ is called the smallest closed interval containing E. Here is obviously $\inf_N E = \alpha\left(\inf \alpha^{-1}(E)\right)$ and $\sup_N E = \alpha\left(\sup \alpha^{-1}(E)\right)$.

Theorem 5. If $S = [a,b]_N$ is the smallest closed α -interval containing the bounded closed subset F in $\mathbb{R}(N)$, then the set $S - F$ is α -open in $\mathbb{R}(N)$.

Proof. Since

$$S - F = [a,b]_N - F = \alpha\left(\left[\alpha^{-1}(a), \alpha^{-1}(b)\right] - \alpha^{-1}(F)\right) = \alpha\left(\left(\alpha^{-1}(a), \alpha^{-1}(b)\right) \cap \left(\alpha^{-1}(F)\right)^c\right)$$

and the set $\left(\alpha^{-1}(a), \alpha^{-1}(b)\right) \cap \left(\alpha^{-1}(F)\right)^{c}$ is an open set in $\mathbb{R}$.

III. Conclusion

The authors make up the substructure for identification and examination the Lebesgue measure on non-newtonian real line in this work, as in references [1] and [3]. After this step, one can define and examine the Lebesgue measure on non-newtonian real line.

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