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On A New Special Differential Equation and Its Polynomials

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Abstract: *Special differential equations and polynomials are very popular in the field of mathematics and serve as important tools in the solution of some engineering problems. Examples of these equations are Legendre, Hermite, Laguerre, Bessel, Gegenbaur differential etc. In this paper, we established a new special differential equation and its polynomial which we named as Mohammed's equation and polynomial. The Rodrigue formula, generating function and recurrence relations of the polynomial are given. We also presented the orthogonality properties of the polynomials and our results are entering the literature for the first time.*
Keywords: *Generating function, Rodrigue formula, generating function, recurrence relations and orthogonality.*

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I. Introduction

The equation (Mohammed's equation) is a second order ordinary differential equation given by

$$xy'' + (1+x)y' - ny = 0 \tag{1.1}$$

where n is a positive integer and the equation converges in the interval $(-\infty, 0)$.

The solution of (1.1) is given by the linear combination

$$y(x) = c_1 M_n(x) + c_2 A_n(x) \tag{1.2}$$

Where $M_n(x)$ is the Mohammed's polynomial of order n and $A_n(x)$ is Mohammed's function of order n

II. Solution of The Equation

The differential equation has a regular singular point at $x = 0$, so we apply Frobenius method.

On letting $y(x) = \sum_{r=0}^{\infty} a_r x^{r+c}$, differentiating twice, plugging these results into (1.2) and simplifying, we obtain

$$\sum_{r=0}^{\infty} (r+c)^2 a_r x^{r+c-1} - \sum_{n=0}^{\infty} (n-r-c) a_r x^{r+c} = 0 \qquad (1)$$

Equating the coefficient of x^{c-1} in the first summation, we get

$$a_0 c^2 = 0, \quad \text{since} \quad a_0 \neq 0 \quad \text{then} \quad c^2 = 0 \quad \therefore c = 0 \text{ twice}$$

Replacing r by $(r + 1)$ in (1) and equating the common coefficient, the recurrence relation becomes

$$a_{r+1} = \frac{n-r-c}{(r+c+1)^2} a_r \qquad (2)$$

If $c = 0$, (2) becomes $a_{r+1} = \frac{n-r}{(r+1)^2} a_r \qquad (3)$

Putting $r = 0, 1, 2, 3, \ldots, r$ in (3), we obtain

$$a_r = \frac{n(n-1)\ldots(n-r+1)}{1^2.2^2\ldots r^2} a_0 = \frac{n(n-1)\ldots(n-r+1)}{[1.2\ldots r]^2} \frac{(n-r)!}{(n-r)!} a_0 = \frac{n! a_0}{(n-r)![r!]^2}; \qquad r = 0,1\ldots$$

$$y_1(x) = \sum_{r=0}^{\infty} a_r x^{r+c} = a_0 \sum_{r=0}^{\infty} \frac{n! x^{r+c}}{(n-r)![r!]^2} \tag{2.1}$$

Also, Putting r = 0, 1, 2, 3,…,r in (2), we get

$$a_r = \frac{(n-c)(n-c-1)(n-c-r+1)}{(c+1)^2(c+2)^2...(c+r)^2}a_0; \qquad r=1,...$$

$$y(x) = a_0x^c + a_0\sum_{r=1}^{\infty}\frac{(n-c)(n-c-1)(n-c-r+1)x^{r+c}}{(c+1)^2(c+2)^2...(c+r)^2}$$

Now, $$\frac{\partial y}{\partial c} = a_0x^c\ln x + a_0\sum_{r=1}^{\infty}\frac{(n-c)(n-c-1)...(n-c-r+1)x^{r+c}\ln x}{(c+1)^2(c+2)^2...(c+r)^2} - a_0\sum_{r=1}^{\infty}\frac{(n-c)(n-c-1)(n-c-r+1)x^{r+c}}{(c+1)^2(c+2)^2...(c+r)^2}\times$$

$$\left\{\frac{1}{n-c}+\frac{1}{n-c-1}+...+\frac{1}{n-c-r+1}+\frac{2}{c+1}+\frac{2}{c+2}+...+\frac{2}{c+r}\right\}$$

$$y_2(x) = \left.\frac{\partial y}{\partial c}\right|_{c=0} = y_1(x)\ln x - a_0\sum_{r=1}^{\infty}\frac{n(n-1)...(n-r+1)x^r}{1^22^2...r^2}\left\{\frac{1}{n}+\frac{1}{n-1}+...+\frac{1}{n-r+1}+2H_r\right\}$$

$$y_2(x) = y_1(x)\ln x - a_0\sum_{r=1}^{\infty}\frac{n!x^r}{(n-r)!(r!)^2}\left\{\frac{1}{n}+\frac{1}{n-1}+...+\frac{1}{n-r+1}+2H_r\right\} \qquad (2.2)$$

Where $H_r = 1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{r}$ is an harmonic function.

Hence, the general solution becomes $y(x) = Ay_1(x) + By_2(x)$

2.1 Mohammed's Polynomial $M_n(x)$

If we let $a_0 = n!$ in (2.1), it yields $y_1(x) = \sum_{r=0}^{\infty}\frac{[n!]^2}{(n-r)!}\frac{x^r}{[r!]^2}$

The series exists only when $n-r \geq 0$ or $n \geq r$. Hence Mohammed's polynomial becomes

$$M_n(x) = \sum_{r=0}^{n}\frac{[n!]^2}{(n-r)!}\frac{x^r}{[r!]^2} \qquad (3.1)$$

A few of the polynomials are as follow;

$$M_0(x) = 1$$

$$M_1(x) = x+1$$

$$M_2(x) = x^2+4x+2$$

$$M_3(x) = x^3+9x^2+18x+6$$

$$M_4(x) = x^4+16x^3+72x^2+96x+24$$

All these polynomials satisfy the differential equation (1.1) and can be verified. If we also let $a_0 = n!$ in (2.2), we get

$$A_n(x) = M_n(x)\ln x - \sum_{r=1}^{\infty}\frac{[n!]^2x^r}{(n-r)!(r!)^2}\left\{\frac{1}{n}+\frac{1}{n-1}+...+\frac{1}{n-r+1}+2H_r\right\} \qquad (3.2)$$

This is known as Mohammed's function of order *n*.

2.3 Generating Function For $M_n(x)$

The generating function for $M_n(x)$ is given as

$$\sum_{n=0}^{\infty}\frac{M_n(x)t^n}{n!}=\frac{1}{1-t}e^{\frac{xt}{1-t}} \tag{4.1}$$

Proof: $\sum_{n=0}^{\infty}\frac{M_n(x)t^n}{n!}=\sum_{n=0}^{\infty}\frac{t^n}{n!}\left\{\sum_{r=0}^{n}\frac{[n!]^2}{(n-r)!}\frac{x^r}{[r!]^2}\right\}=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{n!}{(n-r)!r!}\frac{x^r}{r!}t^n$

On letting $n-r=s$, we get

$$\sum_{n=0}^{\infty}\frac{M_n(x)t^n}{n!}=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(r+s)!}{s!r!}\frac{x^r}{r!}t^{r+s}=\sum_{r=0}^{\infty}\frac{(xt)^r}{r!}\sum_{s=0}^{\infty}\frac{(r+s)!}{s!r!}t^s=\sum_{r=0}^{\infty}\frac{(xt)^r}{r!}\left[\frac{1}{1-t}\right]^{r+1}$$

$$=\frac{1}{1-t}\sum_{r=0}^{\infty}\frac{1}{r!}\left[\frac{xt}{1-t}\right]^r=\frac{1}{1-t}e^{\frac{xt}{1-t}}$$

Hence the proof

2.4 Rodrigue Formula For $M_n(x)$

The Rodrigue formula for $M_n(x)$ is given as

$$M_n(x)=e^{-x}D^n\left(x^ne^{-x}\right) \tag{5.1}$$

Proof: We know from Leibnitz theorem that, $D^n(uv)=\sum_{r=0}^{n}{}^nC_rD^{n-r}uD^rv$ and that

$$D^mx^n=\frac{n!}{(n-m)!}x^{n-m},\ n\geq m$$

$$\therefore D^n\left(x^ne^{-x}\right)=\sum_{r=0}^{n}{}^nC_rD^{n-r}\left(x^n\right)D^r\left(e^x\right)=\sum_{r=0}^{n}\frac{n!}{(n-r)!r!}\frac{n!}{r!}x^re^x$$

$$=e^{-x}\sum_{r=0}^{n}\frac{n!}{(n-r)!r!}\frac{n!}{r!}x^r=e^{-x}M_n(x)$$

$$\therefore M_n(x)=e^xD^n\left(x^ne^{-x}\right)$$

This completes the proof. The Rodrigue formula can be used to establish the polynomials $M_n(x)$.

2.5 Recurrence Relations For $M_n(x)$

On differentiating both sides of the generating function with respect to *t*, equating the coefficients of t^n and simplifying, we obtain

$$M_{n+1}(x)=(x+2n+1)M_n(x)-n^2M_{n-1}(x)$$

(6.1)

Differentiating also with respect to *x* and equating the coefficients of t^n yields

$$M_n'(x)=nM_{n-1}(x)+nM_{n-1}'(x)$$

(6.2)

These are the recurrence relations for $M_n(x)$

III. Orthogonality Properties

The orthogonality properties for $M_n\left(x\right)$ is given as

$$\int_{-\infty}^{0} e^x M_n\left(x\right) M_m\left(x\right) dx = \left[n!\right]^2 \delta_{mn}$$

(7.1)

Where δ_{mn} is the kronelka delta

Proof: Since $M_n(x)$ and $M_m(x)$ satisfy (1.1), we must have that

$$xM_n''\left(x\right) + \left(1+x\right)M_n'\left(x\right) - nM_n\left(x\right) = 0 \qquad \text{(i)}$$

$$xM_m''\left(x\right) + \left(1+x\right)M_m'\left(x\right) - mM_m\left(x\right) = 0 \qquad \text{(ii)}$$

Multiplying (i) by $M_m(x)$ and (ii) by $M_n(x)$ and simplifying yields

$$x\frac{dM}{dx} + \left(1+x\right)M = \left(m-n\right)M_m\left(x\right)M_n\left(x\right); \qquad where\ \ M = M_m'M_n - M_n'M_m$$

Using integrating factor methods, the solution yields

$$M\exp\left(\int\left(1+\frac{1}{x}\right)dx\right) = \int \exp\left(\int\left(1+\frac{1}{x}\right)dx\right)\frac{\left(m-n\right)M_m\left(x\right)M_n\left(x\right)}{x}dx$$

$$\left[Mxe^x\right]_{-\infty}^{0} = \left(m-n\right)\int_{-\infty}^{0} e^x M_m\left(x\right)M_n\left(x\right)dx = 0$$

If $m \neq n$, then $\int_{-\infty}^{0} e^x M_m\left(x\right)M_n\left(x\right)dx = 0$

Now, If $m = n$;

Squaring both sides of the generating function, we have

$$\sum_{n=0}^{\infty}\frac{\left[M_n\left(x\right)\right]^2 t^{2n}}{\left[n!\right]^2} = \frac{1}{\left(1-t\right)^2}e^{\frac{2xt}{1-t}} \qquad \therefore \sum_{n=0}^{\infty}\frac{e^x\left[M_n\left(x\right)\right]^2 t^{2n}}{\left[n!\right]^2} = \frac{e^x}{\left(1-t\right)^2}e^{\frac{2xt}{1-t}} = \frac{1}{\left(1-t\right)^2}e^{\frac{x(1+t)}{1-t}}$$

$$\therefore \sum_{n=0}^{\infty}\frac{t^{2n}}{\left[n!\right]^2}\int_{-\infty}^{0} e^x\left[M_n\left(x\right)\right]^2 dx = \frac{1}{\left(1-t\right)^2}\int_{-\infty}^{0} e^{\frac{x(1+t)}{1-t}} = \frac{1}{1-t^2} = \sum_{n=0}^{\infty} t^{2n}$$

Equating the coefficients of t^{2n} from both sides, we get

$$\frac{1}{\left[n!\right]^2}\int_{-\infty}^{0} e^x\left[M_n\left(x\right)\right]^2 dx = 1 \qquad \therefore \int_{-\infty}^{0} e^x\left[M_n\left(x\right)\right]^2 dx = \left[n!\right]^2$$

This completes the proof

IV. Further Research

In our next research work, we (or any interested researcher in the field) intend to present the following

1. Application of Mohammed's polynomial
2. Integral representation of the polynomial
3. Series of the type $f\left(x\right) = \sum_{n=0}^{\infty} c_n M_n\left(x\right)$
4. Confluent hypergeometric representation of (1.1) and lots more

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