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Linear-quadratic optimal control system and its' application in formulating and minimizing cost criterion via optimization principle

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Abstract: *This paper deals with the application of linear-quadratic optimal control (LQOC) system through an optimization principle formulated in term of a cost criterion and supported by an optimal control system that minimizes the cost criterion. In a linear control theory, if the cost criterion is quadratic and the optimization is over an infinite horizon, then the result of such optimal control is a linear feedback with many properties and satisfies closed loop stability the result which is intimately connected to the system in order to certify the stability properties of control theory.*

Keywords*: Optimization, infinite, linear, stability, cost criterion*

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I. Introduction

Various theoretical results on the solvability of the linear –quadratic optimal control problem in (LQOCP) of an inverse eigenvalue problem (IEP) for Hamilton matrices together with numerical examples are systematically reviewed and discussed in respect of the inverse eigenvalue problems for certain singular and non-singular Hamilton matrices in[1],[2],[3] and as well as [4] and [5]. This paper deal with the application of linear-quadratic optimal control system via optimization principle in formulating and minimizing the cost criterion which is quadratic in nature over an infinite time horizon to give linear feedback satisfying closed loop stability result which is intimately connected to the system in order to certify the stability properties of control theory.

II. Theory of Linear system

We let our linear system be of the form:

$$\frac{dx}{dt}(t) = Ax(t) + Bu(t), x(t_i) = x_i, t > t_i \tag{1}$$

Where $x(t)$ is a vector whose entries functions are of t and a matrix-valued function $A(t)$ is a matrix whose entries are functions given as:

$$\begin{bmatrix} x_i(t) \\ . \\ . \\ x_n(t) \end{bmatrix}, A(t) = \begin{bmatrix} a_{11}(t) & . & . & a_{1n}(t) \\ . & . & . & . \\ . & . & . & . \\ a_{1n}(t) & . & . & a_{mn}(t) \end{bmatrix} \tag{2}$$

The calculus operation of taking the limits, differentiating and others are extended to the vector –valued and matrix valued functions by performing the said operations entry wise.
Thus, by definition:

$$\lim_{t \to t_0} x(t) = \begin{bmatrix} \lim_{t \to t_0} x_i(t) \\ . \\ \lim_{t \to t_0} x_n(t) \end{bmatrix} \qquad (3)$$

The limit exists if and only if $\lim_{t \to t_0} x_i(t)$ exists $\forall i \in \{1.....n\}$.

Then the derivatives of a vector valued or matrix valued function is a function obtained by entry-wise differentiation.

Thus:

$$\frac{dx}{dt}(t) = \begin{bmatrix} x'_i(t) \\ . \\ . \\ x'_n(t) \end{bmatrix} \quad , \quad \frac{dA}{dt}(t) = \begin{bmatrix} a'_{11}(t) & . & . & a'_{1n}(t) \\ . & . & . & . \\ . & . & . & . \\ a'_{1n}(t) & . & . & a'_{mn}(t) \end{bmatrix} \qquad (4)$$

Where $x_i'(t)$ is the derivative of $x_i(t)$

So, $\frac{dx}{dt}$ is defined if and only if each of the component functions $x_i(t)$ is differentiable.

The derivative can also be described in vector notation as:

$$\frac{dx}{dt}(t) = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h} \qquad (5)$$

Here $x(t+h) - x(t)$ is computed by the usual vector addition and the h in the denominator represents scalar multiplication by h^{-1}. The limit is obtained by evaluating the limit of each entry separately. The same is applicable in the case of matrix valued functions.

III. Control Theory

We let our differential equation be of the form:

$$\frac{dx}{dt}(t) = f(x(t), u(t)), x(t_i) = x_i, t \geq t_i \qquad (6)$$

Where $x(t) \in \Re^n, u(t) \in \Re^m$ and $f_1 .... f_n$ represent the components of f. u is the free variable called the input and assumed to be piecewise continuous.

We let the class of $\Re^m$-valued piecewise continuous function be denoted by $\underline{u}$ .Under regularity condition on the function $f : \Re^n \times \Re^m \;\; \exists$ a unique solution to the differential equation (6) for any initial condition $x_i \in \Re^n$ and every piecewise contain input u . x is called the state and equation (6) is called the state equation.

A control system is an equation of the type (6), with input u and state x. Once the input u and the initial state $x(t_i) = x_i$ are specified, then the state x is determined.

A characteristic of underdetermined equations is that one can choose the free variable in a way that some desirable effect is produced on the other dependent variable. The state variables x comprise the 'to-be-controlled 'variables, which depend on the free variables u , i.e. the inputs.

3.1 Linear Control and Controllability

If the function f is linear, that is, if $f(x,u) = Ax + Bu$ for some $A \in \Re^{n \times n}$ and $B \in \Re^{n \times m}$ then the control system is said to be linear. Thus a linear control system is given by:

$$\dot{X}(t) = Ax(t) + Bu(t), t \leq t_i \qquad (7)$$

Equation (7) is said to be controllable if every pair of vectors $x_i, x_f \in \Re^{nn}, \exists, t_f > t_i$ and a control $u \in \left(C[t_i, t_f]\right)^m$ Such that the solution of x in equation (7) with $x(t_i) = x_i$ satisfies

$x(t_f) = x_f$. Controllability means that any state can be driven to any other state using appropriate control.

IV. Linear Quadratic Optimal Control Problem

Here we consider a linear system of the form:

$$\dot{X} = Ax + Bu \qquad X_0 = x_0 \tag{8}$$

Where u is the admissible control unit and be of the form: $u = \phi(t)$

For u to be the admissible control unit, it must satisfy the following conditions:

(i) ϕ must be a continuous function

(ii) The closed loop system must have a unique solution.

(iii) The closed loop system results in $\lim_{t\to\infty} x(t) = 0$.

Besides, the control objective is to find a control strategy that minimizes the cost functional.

$$J(x,\phi) = \int_0^\infty \left[X^T(t)Qx(t) + \phi^T(t)R\phi(t)\right]dt \tag{9}$$

Where

Q is a symmetric positive semi definite matrix.

R is a symmetric positive definite matrix.

Then, this type of control problem is called **Linear -quadratic control** problem

Since Q is positive semi definite, then $x^T(t)Qx(t) \geq 0$; R is positive definite $\phi^T(t)R\phi(t) > 0$ unless $\phi(t) = 0$

The diagonal matrix: $Q = \begin{bmatrix} q_1 & 0 & 0 & - & 0 \\ 0 & q_2 & - & - & 0 \\ 0 & 0 & - & & 0 \\ - & - & - & - & - \\ 0 & - & - & - & q_n \end{bmatrix}$

Which gives the quadratic form: $x^T(t)Qx(t) = \sum_{i=1}^{n} q_i x^2{}_i(t)$ where $q_i(s)$ is the magnitude of $x_i(t)$.

For stability, we let (A, B) be stable by introducing K as feedback control such that the closed loop system.

$$\dot{X} = (A + Bk)x \tag{10}$$

Where $u = Kx$ and clearly admissible..

The solution to the equation (10) gives; $x(t) = e^{(A+Bk)t} x_0$ (11)

Which satisfy condition (ii) above.

We let the cost function be: $J(x_0, K_x) = x_0{}^T \int_0^\infty e^{(A+Bk)^T t}(Q + K^T RK)e^{(A+Bk)t} x_0$ (12)

This optimal control problem can be solved by dynamic programming in which we define the instantaneous cost as;

$$L(x,u) = x^T Qx + u^T Ru \tag{13}$$

For the final state $x_0 = x$ and we define the optimal cost or value function as

$$V(x) = \inf_{\phi \in u} J(x,\phi) \tag{14}$$

Where inf is the greatest lower bound. We let $u(t), 0 \leq t \leq r$ be the control over $[0, r]$

Then, $J = \int_0^r L(x(t), u(t)dt + V(x(r))$ (15)

Where $u(t)$ is an arbitrary and the optimal cost satisfies the equation

$$V(x) = \min_{v(t)0 \le t \le r} \left[\int_0^T L(x(t), u(t)dt + V(x(r)) \right] \qquad (16)$$

Expanding equation (16), as $\frac{\sigma(r)}{r} \to 0, r \to 0$

$$\Rightarrow \int_0^T L(x(r), u(t)dt - rL(x,u) + 0(r) \ \Rightarrow V(x(r) = v(x) + r\frac{\partial v}{rx}(x)(Ax + Bu) + 0(r) \qquad (17)$$

Where $\frac{\partial v}{\partial x}$ is the gradient of v with respect to $x(1 \times n, row - vector)$

Substituting into equation (16) we get the Hamilton-Jacobi Bellman (HJB) equation for

V Satisfies by $V(x)$.

$$\min_{u \in R^n} \left\{ \frac{\partial v}{\partial x}(x)(Ax + Bu) + L(x,u) \right\} = 0 \qquad (18)$$

Given that $R > 0$, minimizing the element u in the equation (18), we get the following square quadratic form of equation:

$$u^T Ru + 2\alpha^T u + \beta = (u + R^{-1}\alpha) + \beta - \alpha^T R^{-1}\alpha$$

$$\Rightarrow \min_u (u^T Ru + 2\alpha^T u + \beta) = \beta - \alpha^T R^{-1}\alpha \qquad (19)$$

Minimizing u by $u = -R^{-1}\alpha$, and substituting it into equation (18) yields;

$$u = -\frac{1}{2}R^{-1}B^T \frac{\partial v}{\partial x}(x) \quad \Rightarrow \frac{\partial v^T}{\partial x}(x)(Ax) + x^T Qx - \frac{1}{4}\frac{\partial v}{\partial x}(x)BR^{-1}B^T \frac{\partial v^T}{\partial x}(x) = 0 \qquad (20)$$

Solving equation (20), we apply trial solution $V(x) = x^T px$

For $p \ge 0$,

$$\frac{\partial}{\partial x_k}(x^T px) = \frac{\partial}{\partial x_k}\sum_{ij=1}^{n} x_i p_{ij} x_j$$
$$= \sum_j p_{kj} x_j + \sum_i x_i p_i k \qquad (21)$$
$$= (Px)_k + (p^T x)_k$$
$$= 2(Px)_k$$

Substituting into equation (20) yields:

$$x^T (A^T P + PA - PBR^{-1}B^T P + Q)x = 0 \qquad (22)$$

Since this is true for all x. P Must satisfy the matrix quadratic equation of the form:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \qquad (23)$$

This is called the **Algebraic Riccati Equation**

In term of P_i, the minimizing u would then be given as:

$$u = -R^{-1}B^T Px \qquad (24)$$

V. Theorem on Riccati

Assume (A, B) is stabilizable and $(\sqrt{Q}, A)$ is detectable. Then, there exists a unique solution P in the class of positive semi definite matrices to the algebraic Riccati equation (23) above and the closed loop system matrix $A - BR^{-1}B^T P$ is stable.

Proof

If (A, B) is stabilizable and $(\sqrt{Q}, A)$ is detectable, then equation (23) will be an admissible as it is stabilizing, we then verify that it is optimal by completing the square.

(i).For any admissible u

$$J(x_0,u) = \int_0^\infty \left[x^T(t)Qx(t) + u^T(t)Ru(t)\right]dt$$

$$= \int_0^\infty \left[x^T(t)PBR^{-1}B^T Px(t) + u^T(t)Ru(t) - x^T(t)(A^T P + PA)x(t)\right]dt$$

$$= \int_0^\infty \left[u(t) + R^{-1}B^T Px(t)\right]^T R\left[(u(t) + R^{-1}B^T Px(t)\right]dt$$

$$= -\int_0^\infty \left[u^T(t)B^T Px(t) + x^T(t)PBu(t) + x^T(t)APx(t) + x^T(t)PAx(t)\right]dt$$

$$\int_0^\infty \left[(u(t) + R^{-1}B^T Px(t)\right]^T R\left[(u(t) + R^{-1}B^T Px(t)\right] - \dot{x}^T(t)Px(t) - x^T(t)P\dot{x}(t)]dt$$

$$= x_0{}^T Px_0 + \int_0^\infty \left[(u(t) + R^{-1}B^T Px(t)\right]^T R\left[(u(t) + R^{-1}B^T Px(t))\right]dt$$

Since $x_0 Px_o$ is constant and $u = -R^{-1}B^T Px$ is admissible with $R > 0$, then, the optimal control will be: $u(t) = -R^{-1}B^T Px(t)$ while the optimal cost is $V(x) = x^T Px$

5.1 Linear System via the Riccati Equation

Here we let $A \in \Re^{n\times n}, B \in \Re^{n\times m}, Q \in \Re^{n\times n} : Q = Q^T \geq 0$ and $R \in \Re^{m\times m} : R = R^T > 0$

We wish to find the linear quadratic- optimal control for the functional;

$$Ix_i(u) = \int_{t_i}^{t_f} \frac{1}{2}\left[x(t)^T Qx(t) + u(t)^T Ru(t)\right]dt \quad (25)$$

Subject to differential equation

$$\dot{X}(t) = Ax(t) + Bu(t), t \in \left[t_i, t_f\right], x(t_i) = x_i \quad (26)$$

Then the Hamiltonian equation is given by:

$$H(p,x,u,t) = \frac{1}{2}\left[x^T Qx + u^T Ru\right] + p^T\left[Ax + Bu\right] \quad (27)$$

From the theorem, it then follows that any optimal input $u_\bullet$ and the corresponding state $x_\bullet$ Satisfies:

$$\frac{\partial H}{\partial u}(p_\bullet(t), x_\bullet(t), u_\bullet(t), t) = 0 \Rightarrow u_\bullet(t)^T R + p_\bullet(t)^T B = 0 \quad (28)$$

Thus, $u_\bullet(t) = -R^{-1}B^T p_\bullet(t)$ and the adjoint equation is given as:

$$\left[\frac{\partial H}{\partial x}(p_\bullet(t), k_\bullet(t), u_\bullet(t), t)\right]^T = -\dot{p}_\bullet(t), t \in \left[t_i, t_f\right], p_\bullet(t_f) = 0 \quad (29)$$

$$\Rightarrow \left(x_\bullet(t)^T Q + p_\bullet(t)^T A\right)^T = -\dot{p}_\bullet(t), t \in \left[t_i, t_f\right], p_\bullet(t_f) = 0 \quad (30)$$

We then have;

$$\dot{p}_\bullet(t) = A^T p_\bullet(t) - Qx_\bullet(t), t \in \left[t_i, t_f\right], p_\bullet(t_f) = 0 \quad (31)$$

Consequently;

$$\frac{d}{dt}\begin{bmatrix} x_\bullet(t) \\ p_\bullet(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} x_\bullet(t) \\ p_\bullet(t) \end{bmatrix}, t \in \left[t_i, t_f\right], x_\bullet(t_i) = x_i, p_\bullet(t_f) = 0 \quad (32)$$

Equation (32) is a linear, time variant differential equation in $(x_\bullet, p_\bullet)$

VI. Application of Optimal control to the double integrator

We consider the system of the form: $\frac{d^2t}{dx^2} = u$

With the cost criterion $J = \int_0^\infty \left[y^2(t) + ru^2(t) \right] dt \qquad r > 0$

A state space representation of this system can be given as:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

The cost criterion can be written as: $J = \int_0^\infty \left[x^T(t) Q x(t) + ru^2(t) \right] dt$

Where $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $C^T C = Q$ .it is easy to verify that (C, A) is detectable, and let (A, B) is stabilizable.

We proceed to solve the algebraic Riccati equation

We let $S = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}$, where we have explicitly used the fact that S is symmetric.

The elements of S satisfy the following equations:

$$-\frac{1}{r} s_2{}^2 + 1 = 0; \quad s_1 - \frac{1}{r} s_2 s_3 = 0 \, and \quad 2s_2 - \frac{1}{r} s_3{}^2 = 0$$

$$s_2 = \pm\sqrt{r} \, , s_3 = \pm(2rs_2)^2 \quad \Rightarrow s_2 = \sqrt{r}$$

For S to be positive semidefinite, all its diagonal entries must be nonnegative.

Hence, $s_3 = \sqrt{2r}^{\frac{3}{4}}, \quad s_1 = \frac{1}{r} s_2 s_3 = \sqrt{2r^{\frac{1}{4}}}$

So, $S = \begin{bmatrix} \sqrt{2r}^{\frac{1}{4}} & \sqrt{r} \\ \sqrt{r} & \sqrt{2r}^{\frac{3}{4}} \end{bmatrix}$

Hence, S is positive definite since $S_{11} > 0$ and $\det S > 0$ (which are the necessary and sufficient conditions for a $2n \times 2n$ matrix to be > 0)

The optimal closed loop system is given by:

$$\dot{x} = \left(A - BR^{-1}B^T S\right) x = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{r} \begin{bmatrix} \sqrt{r} & \sqrt{2r}^{\frac{2}{3}} \end{bmatrix} \right) x \Rightarrow \begin{bmatrix} 0 & 1 \\ -r^{-\frac{1}{2}} & -\sqrt{2r}^{-\frac{1}{4}} \end{bmatrix} x$$

The poles of the closed loop system are given by the roots of the polynomial $s^2 + \sqrt{2r}^{-\frac{1}{4}} s + r^{-\frac{1}{2}}$. This is the form of the standard second order system characteristic polynomial $s^2 + 2\varsigma\omega s + \omega_0{}^2$ with $\omega_0 = r^{-\frac{1}{4}}$ and $\varsigma = \frac{1}{\sqrt{2}}$ .The damping ratio of $\frac{1}{\sqrt{2}}$ of the optimal closed loop system is often referred to as the best compromise between small overshoot and good speed response, and it is independent of r .Now for a fixed damping ratio, the larger the natural frequency ω_0 ,the faster the speed of response(where the peak time is inversely proportional to ω_0 .Thus we can see that if r decreases, the speed of response becomes faster. Since a small r implies small control penalty and hence allowing large control inputs, this behavior give a good interpretation of the role of the quadratic weights in the cost criterion

6.1 Application of optimal control in a servomotor

We let the servomotor system given by the transfer function of the form:

$$y(s) = \frac{1}{s(s+1)} u(s)$$

A state space representation of this system yields:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

We let the cost criterion be : $J = \int_0^\infty \left[y^2(t) + ru^2(t) \right] dt \qquad = \int_0^\infty \left[x^T(t) Q x(t) + ru^2(t) \right] dt$

Where $Q = C^T C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $r > 0$ .it is easy to verify that (C, A) is detectable, and let (A, B) is stabilizable.

We proceed to solve the algebraic Riccati equation

Let $S = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}$, where we have explicitly used the fact that S is symmetric.

The elements of S satisfy the following equations:

$$-\frac{1}{r} s_2{}^2 + 1 = 0; \; s_1 - s_2 - \frac{1}{r} s_2 s_3 = 0 \; and \; 2(s_2 - s_3) - \frac{1}{r} s_3{}^2 = 0$$

Solving: $s_2 = \sqrt{r}$, $s_3 = r\sqrt{1 + 2r^{-\frac{1}{2}}} - r$, $s_1 = \sqrt{r + 2r^{\frac{1}{2}}}$

So, $S = \begin{bmatrix} \sqrt{r + 2r^{\frac{1}{2}}} & \sqrt{r} \\ \sqrt{r} & r\sqrt{1 + 2r^{-\frac{1}{2}}} - r \end{bmatrix}$

The optimal closed loop system matrix is given by:

$$A - BR^{-1}B^T S = \begin{bmatrix} 0 & 1 \\ -\frac{1}{r} s_2 & -1 - \frac{1}{r} s_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -r^{-\frac{1}{2}} & -\sqrt{1 + 2r^{-\frac{1}{2}}} \end{bmatrix}$$

Then the characteristics polynomial of the closed loop system is given by:

$s^2 + \sqrt{1 + 2r^{-\frac{1}{2}}} s + r^{-\frac{1}{2}}$ With the pole located at $\dfrac{-\sqrt{1 + 2r^{-\frac{1}{2}}} \pm \sqrt{1 - 2r^{-\frac{1}{2}}}}{2}$

VII. Conclusion

Based on the reviewed of lots of theoretical results on the solvability of the linear-quadratic inverse eigenvalue problem for Hamilton matrices together with numerical examples, we have successfully investigate and established the application of linear-quadratic optimal control system via optimization principle in formulating and minimizing the cost criterion which is quadratic in nature over an infinite time horizon to give linear feedback satisfying closed loop stability results connected to the system in order to certify the stability properties of control theory

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