IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 14, Issue 1 Ver. III (Jan. - Feb. 2018), PP 01-13
www.iosrjournals.org

Numerical Efficient For Second Order Lax-Wendroff Difference Scheme of A Fluid Dynamic Traffic Flow Model

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Abstract: *This paper considers a fluid dynamic traffic flow model known as Lighthill, Whitham and Richards (LWR) model appended with a linear velocity-density function. The model reads as a quasi-linear first order hyperbolic partial differential equation (PDE) and in order to incorporate initial and boundary data treated as an initial boundary value problem (IBVP). We presents the exact solution of the PDE as a Cauchy problem and the derivation of a finite difference scheme of a traffic flow model which is second order Lax-Wendroff difference scheme and establish well-posed-ness and stability condition for the scheme. The traffic density* $\rho(t,x)$ *is computed by solving LWR traffic flow model using the scheme. Computer programs for the implementation of the numerical scheme and perform numerical experiments in order to verify stability condition in terms of time step selection. Some numerical simulation results are presented for various parameters and relative errors and verify convergence of errors.*

Keywords: *LWR Traffic Flow Model, Lax-Wendroff Difference Scheme, Numerical Simulation.*

Date of Submission: 12-02-2018 Date of acceptance: 27-02-2018

I. Introduction

Today, when cars are part of every day life and more families tend to own more than one car, the traffic congestion becomes a very important issue. Traffic problems on highways and in urban areas attract considerable attention. So traffic is very well known term to all and it is closely related to real life. As the number of vehicles is increasing rapidly, in recent years, traffic congestion has become especially an acute problem. Traffic jams are now a major problem in most of the cities. So at the core of traffic congestion, development of traffic management is the need of time. Therefore, an efficient traffic control and management is essential in order to grid of such huge traffic congestion. Modeling and computer simulation play an increasing role in the flow management. Many scientists have been working to develop various mathematical models ([2], [3]) in order to describe traffic flow. In this paper, we consider a fluid dynamic traffic flow model developed first by Lighthill and Whitham (1955) ([1], [9]) and Richard (1956) shortly called LWR model based on Habermann (1977) [4], Klar (1996) [2]. In [5], L. S. Andallah, S. Ali, M. O. Gani, M. K. Pandit and J. Akhter have used linear velocity-density function for the development of traffic flow model and they have presented explicit upwind difference scheme. Based on the study of the general finite difference method for the first order non-linear PDE ([6], [7]), we present a second order Lax-Wendroff difference scheme for our first order traffic flow model appended with a linear velocity-density relation. We establish the well-posed-ness and stability condition of the Lax-Wendroff difference scheme. The numerical scheme is implemented in order to perform the numerical results are compared in terms of accuracy by error estimation with respect to the exact solution of the traffic flow model and also, the features of the rate of convergence are presented graphically. The conditions of stability are also numerically verified. Some numerical simulation results are presented for various parameters.

II. General Mathematical Model of Fluid Dynamic Traffic Flow

In this section, the general mathematical model are shortly presented based on ([3], [5], [9]) and work out the qualitative behavior of the flux. The well-known LWR model is formulated by employing the conservation equation

$$\frac{\partial \rho}{\partial t}+\frac{\partial q(\rho)}{\partial x}=0, \text{where } q(\rho)=\rho v(\rho) \qquad (1)$$

In this paper, we will use a linear velocity-density relationship [5] (linear function) can be of the form $v(\rho) = v_{\max}\left(1 - \dfrac{\rho}{\rho_{\max}}\right)$.

Therefore, $q(\rho) = \rho\, v(\rho) = \rho . v_{\max}\left(1 - \dfrac{\rho}{\rho_{\max}}\right) = v_{\max}\left(\rho - \dfrac{\rho^2}{\rho_{\max}}\right)$.

Now, we put flow (flux)-density function that is $q(\rho)$ into the general non-linear partial differential equation (1), we obtain the specific non-linear partial differential equation of traffic flow model in the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(v_{\max}\left(\rho - \frac{\rho^2}{\rho_{\max}}\right)\right) = 0 \qquad (2)$$

III. Exact Solution of Non-Linear PDE of Traffic Flow Model

The traffic flow model appended with the initial condition reads as initial value problem (IVP) is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(v_{\max}\left(\rho - \frac{\rho^2}{\rho_{\max}}\right)\right) = 0 \qquad (3)$$
$$\text{with } \rho(t_0, x) = \rho_0(x)$$

The non-linear PDE (3) can be solved [6] if we know the traffic density at a given initial time, i.e. if we know the traffic density at a given time t_0 we can predict the traffic density for all future time $t \geq t_0$, in principle. Then we have to solve the IVP (3) can be solved by the method of characteristics. The exact solution [5] of the IVP (3) is given by

$$\rho(t,x) = \rho_0\left(x - v_{\max}\left(1 - \frac{2\rho}{\rho_{\max}}\right)t\right) \qquad (4)$$

which is non-linear implicit form and therefore very complicated to evaluate at each $\rho(t,x)$. However, in reality it is very difficult to approximate the initial density $\rho_0(x)$ of the Cauchy problem (3) as a function of t from given initial data. Therefore, there is a demand of some efficient numerical methods for solving the IVP (3).

IV. Numerical Solution of Second Order Lax-Wendroff Difference Scheme By FTCS Techniques

We consider our specific non-linear PDE of traffic flow model as an initial boundary value problem (IBVP):

$$\left.\begin{array}{l} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial x}(q(\rho)) = 0, t_o \leq t \leq T,\ a \leq x \leq b \\ \text{with i.c. } \rho(t_o, x) = \rho_o(x);\ a \leq x \leq b \\ \text{and b.c. } \rho(t,a) = \rho_a(t);\ t_o \leq t \leq T, \end{array}\right\} \qquad (5)$$

$$\text{where } q(\rho) = v_{\max}\left(\rho - \frac{\rho^2}{\rho_{\max}}\right).$$

In order to develop the 2nd order Lax-Wendroff method, named after P. Lax and B. Wendroff, can be derived in terms of the discretization of $\dfrac{\partial \rho}{\partial t}$ is obtained by first order forward difference in time and the discretization of $\dfrac{\partial q}{\partial x}$ is obtained by second order centered difference in space.

Forward difference in time:

From the Taylor's series expansion we can write

$$\rho(x,\, t+k) = \rho(x,t) + k\frac{\partial \rho}{\partial t} + \frac{k^2}{2!}\frac{\partial^2 \rho}{\partial t^2} + \;..........\;..... \tag{6}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = \frac{\rho(x,\, t+k) - \rho(x,t)}{k} - o(k^2)$$

$$\Rightarrow \frac{\partial \rho}{\partial t} \approx \frac{\rho(x,\, t+k) - \rho(x,t)}{k}$$

$$\therefore \frac{\partial \rho\,(t^n, x_i)}{\partial t} \approx \frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} \tag{7}$$

Central difference in space:
From the Taylor's series expansion we can write

$$q(x+h,\, t) = q(x,t) + h\frac{\partial q}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 q}{\partial x^2} + \;...........$$

$$q(x-h,\, t) = q(x,t) - h\frac{\partial q}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 q}{\partial x^2} + \;...........$$

Subtracting the above two series, we obtain

$$\Rightarrow \frac{\partial q}{\partial x} \approx \frac{q(x+h,\, t) - q(x-h,\, t)}{2h}$$

$$\Rightarrow \frac{\partial q(t^n, x_i)}{\partial x} \approx \frac{q_{i+1}^n - q_{i-1}^n}{2\Delta x}$$

$$\therefore \frac{\partial}{\partial x}\Big(q(\rho)(t^n, x_i)\Big) \approx \frac{q\left(\rho_{i+1}^n\right) - q\left(\rho_{i-1}^n\right)}{2\Delta x} \tag{8}$$

Now in equation (6), where the time derivatives can be replaced space derivatives using $\rho_t + \left(q(\rho)\right)_x = 0$ (9)

This has been done by so called Cauchy-Kawalewski technique which implies $\frac{\partial \rho}{\partial t} = -\frac{\partial q(\rho)}{\partial x}$.

$$\therefore \frac{\partial^2 \rho}{\partial t^2} = \frac{\partial}{\partial t}\left(-\frac{\partial q(\rho)}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial q(\rho)}{\partial t}\right) = \frac{\partial}{\partial x}\left(-q'(\rho)\frac{\partial \rho}{\partial t}\right) = \frac{\partial}{\partial x}\left(-q'(\rho).-\frac{\partial q(\rho)}{\partial x}\right) = \frac{\partial}{\partial x}\left(q'(\rho)\frac{\partial q(\rho)}{\partial x}\right);$$

where $q'(\rho) = \frac{\partial q(\rho)}{\partial \rho}$.

Substitute the preceding expression of time derivatives (9) into the Taylor's series of $\rho(x, t+k)$ in equation (6) to obtain

$$\rho(x,\, t+k) = \rho(x,t) - k\frac{\partial q(\rho)}{\partial x} + \frac{k^2}{2!}\frac{\partial}{\partial x}\left(q'(\rho)\frac{\partial q(\rho)}{\partial x}\right) + o(\Delta t^3) \tag{10}$$

Using equation (8), we get

$$\frac{\partial}{\partial x}\left(q'(\rho)\frac{\partial q(\rho)}{\partial x}\right)\left(t^n, x_i\right) = \frac{q'\left(\rho^n_{i+\frac{1}{2}}\right)\left(q\left(\rho_{i+1}^n\right) - q\left(\rho_i^n\right)\right) - q'\left(\rho^n_{i-\frac{1}{2}}\right)\left(q\left(\rho_i^n\right) - q\left(\rho_{i-1}^n\right)\right)}{(\Delta x)^2} + o(\Delta x^2)$$

From equation (6) we get,

$$\frac{\rho(x, t+k)-\rho(x,t)}{k} = -\frac{\partial q(\rho)}{\partial x} + \frac{k}{2!}\frac{\partial}{\partial x}\left(q'(\rho)\frac{\partial q(\rho)}{\partial x}\right)$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = -\frac{\partial q(\rho)}{\partial x} + \frac{k}{2!}\frac{\partial}{\partial x}\left(q'(\rho)\frac{\partial q(\rho)}{\partial x}\right)$$

$$\Rightarrow \frac{\partial \rho}{\partial t}\left(t^n, x_i\right) = -\frac{\partial q(\rho)}{\partial x}\left(t^n, x_i\right) + \frac{k}{2!}\frac{\partial}{\partial x}\left(q'(\rho)\frac{\partial q(\rho)}{\partial x}\right)\left(t^n, x_i\right)$$

$$\Rightarrow \frac{\rho_i^{n+1}-\rho_i^n}{\Delta t} = -\frac{1}{2\Delta x}\left(q(\rho_{i+1}^n)-q(\rho_{i-1}^n)\right) + \frac{\Delta t}{2!}\frac{q'\left(\rho^n_{i+\frac{1}{2}}\right)\left(q\left(\rho_{i+1}^n\right)-q\left(\rho_i^n\right)\right)-q'\left(\rho^n_{i-\frac{1}{2}}\right)\left(q\left(\rho_i^n\right)-q\left(\rho_{i-1}^n\right)\right)}{(\Delta x)^2}$$

$$\therefore \rho_i^{n+1} = \rho_i^n - \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)\left(q(\rho_{i+1}^n)-q(\rho_{i-1}^n)\right) + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2\left(q'\left(\rho^n_{i+\frac{1}{2}}\right)\left(q\left(\rho_{i+1}^n\right)-q\left(\rho_i^n\right)\right)-q'\left(\rho^n_{i-\frac{1}{2}}\right)\left(q\left(\rho_i^n\right)-q\left(\rho_{i-1}^n\right)\right)\right) \quad (11)$$

where $q'\left(\rho^n_{i\pm\frac{1}{2}}\right) = v_{\max}\left(1-\frac{2.\frac{1}{2}\left(\rho_{i\pm1}^n+\rho_i^n\right)}{\rho_{\max}}\right) = v_{\max}\left(1-\frac{1}{\rho_{\max}}\left(\rho_{i\pm1}^n+\rho_i^n\right)\right)$

and $q\left(\rho_{i+1}^n\right) = v_{\max}\left(\rho_{i+1}^n - \frac{\left(\rho_{i+1}^n\right)^2}{\rho_{\max}}\right)$, $q\left(\rho_i^n\right) = v_{\max}\left(\rho_i^n - \frac{\left(\rho_i^n\right)^2}{\rho_{\max}}\right)$, $q\left(\rho_{i-1}^n\right) = v_{\max}\left(\rho_{i-1}^n - \frac{\left(\rho_{i-1}^n\right)^2}{\rho_{\max}}\right)$.

IV-A Well-posed-ness and Stability Condition

The implementation of LWDS is not straight forward. Since vehicles are moving in only one direction, so the characteristic speed $\frac{dq}{dt}$ must be positive. One needs to ensure the well-posed-ness condition $q'\left(\rho_i^n\right) = v_{\max}\left(1-\frac{2\rho_i^n}{\rho_{\max}}\right) \geq 0.$

Since the maximum velocity $v_{\max} > 0$,

$$\therefore 1-\frac{2\rho_i^n}{\rho_{\max}} \geq 0;$$

$$\Rightarrow \frac{2\rho_i^n}{\rho_{\max}} \leq 1$$

$$\Rightarrow \rho_{\max} \geq 2\rho_i^n; \text{ which is the condition for well-posed-ness.}$$

$$\therefore\ q'(\rho_i^n) \leq v_{\max} \quad (12)$$

Proposition IV-B: The well-posed-ness and stability condition of the Lax-Wendroff difference scheme (11) is guaranteed by the simultaneous conditions $0 < \left(v_{\max}\frac{\Delta t}{\Delta x}\right) \leq 1/\left(1-\frac{2\max(\rho_i^0)}{\rho_{\max}}\right)$ and $-\Delta x \leq v_{\max}\,\Delta t \max(\rho_i^0) \leq \Delta x.$

Proof: Rewriting the non-linear PDE in (5) as

$$\frac{\partial \rho}{\partial t} + q'(\rho)\frac{\partial \rho}{\partial x} = 0$$

The LWDS (11) takes the form

$$\rho_i^{n+1}=\rho_i^n-\frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)\left(q(\rho_{i+1}^n)-q(\rho_{i-1}^n)\right)+\frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 q'\left(\rho_{i+\frac{1}{2}}^n\right)\left(q\left(\rho_{i+1}^n\right)-q\left(\rho_i^n\right)\right)-\frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 q'\left(\rho_{i-\frac{1}{2}}^n\right)\left(q\left(\rho_i^n\right)-q\left(\rho\right.\right.$$

$$=\rho_i^n-\frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)q'\left(\rho_i^n\right)\left(\rho_{i+1}^n-\rho_{i-1}^n\right)+\frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 q'\left(\rho_{i+\frac{1}{2}}^n\right)q'\left(\rho_i^n\right)\left(\rho_{i+1}^n-\rho_i^n\right)-\frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 q'\left(\rho_{i-\frac{1}{2}}^n\right)q'\left(\rho_i^n\right)\left(\rho_i^n-\right.$$

$$=\left(1-\frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 q'\left(\rho_{i+\frac{1}{2}}^n\right)q'\left(\rho_i^n\right)-\frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 q'\left(\rho_{i-\frac{1}{2}}^n\right)q'\left(\rho_i^n\right)\right)\rho_i^n+\left(-\frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)q'\left(\rho_i^n\right)+\frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 q'\left(\rho_{i+\frac{1}{2}}^n\right)q\right.$$

$$+\left(\frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)q'\left(\rho_i^n\right)+\frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 q'\left(\rho_{i-\frac{1}{2}}^n\right)q'\left(\rho_i^n\right)\right)\rho_{i-1}^n$$

$$=\left(1-r_1-r_2\right)\rho_i^n+\left(-r+r_1\right)\rho_{i+1}^n+\left(r+r_2\right)\rho_{i-1}^n$$

$$=\left(r+r_2\right)\rho_{i-1}^n+\left(1-r_1-r_2\right)\rho_i^n+\left(r_1-r\right)\rho_{i+1}^n \qquad (13)$$

where $r=\frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)q'\left(\rho_i^n\right)$, $r_1=\frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 q'\left(\rho_{i+\frac{1}{2}}^n\right)q'\left(\rho_i^n\right)$ and $r_2=\frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 q'\left(\rho_{i-\frac{1}{2}}^n\right)q'\left(\rho_i^n\right)$

This equation (13) implies that if

$$0\le r+r_2\le 1 \qquad (14)$$

$$0\le 1-r_1-r_2\le 1 \qquad (15)$$

$$0\le r_1-r\le 1 \qquad (16)$$

then the new solution is a convex combination of the two previous solutions. That is the solution at new time-step $(n+1)$ at a spatial node is an average of the solutions at the previous time-step at the spatial nodes $i-1,\ i$ and $i+1.$ This means that the extreme value of the new solution is the average of the extreme values of the previous two solutions at the three consecutive nodes. Therefore, the new solution continuously depends on the initial value $\rho_i^o,\ i=1,2,3,.......,M$.

Since $r_2=\frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 q'\left(\rho_{i-\frac{1}{2}}^n\right)q'\left(\rho_i^n\right)$

$$=\frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2\left(v_{\max}\right)^2\left(1-\frac{\left(\rho_{i-1}^n+\rho_i^n\right)}{\rho_{\max}}\right)\left(1-\frac{2\rho_i^n}{\rho_{\max}}\right)=\frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2\left(v_{\max}\right)^2\left(1-\frac{2\max(\rho_i^o)}{\rho_{\max}}\right)$$

$$\left(1-\frac{2\max(\rho_i^o)}{\rho_{\max}}\right)=\frac{1}{2}\left(\frac{\Delta t}{\Delta x}v_{\max}\right)^2\left(1-\frac{2\max(\rho_i^o)}{\rho_{\max}}\right)^2$$

Similarly, $r_1=\frac{1}{2}\left(\frac{\Delta t}{\Delta x}v_{\max}\right)^2\left(1-\frac{2\max(\rho_i^o)}{\rho_{\max}}\right)^2$. *i.e.* $r_1=r_2$.

Equation (15) implies,

$$0 \le 1 - r_2 \le 1 \Rightarrow -1 \le -2r_2 \le 0 \Rightarrow 1 \ge 2r_2 \ge 0 \Rightarrow 0 \le 2r_2 \le 1 \Rightarrow 0 \le r_2 \le \frac{1}{2}$$

$$\Rightarrow 0 \le \frac{1}{2}\left(\frac{\Delta t}{\Delta x} v_{\max}\right)^2 \left(1 - \frac{2\max(\rho_i^o)}{\rho_{\max}}\right)^2 \le \frac{1}{2} \Rightarrow 0 \le \left(\frac{\Delta t}{\Delta x} v_{\max}\right)\left(1 - \frac{2\max(\rho_i^o)}{\rho_{\max}}\right) \le 1$$

$$\Rightarrow 0 \le \left(\frac{\Delta t}{\Delta x} v_{\max}\right) \le 1 / \left(1 - \frac{2\max(\rho_i^o)}{\rho_{\max}}\right).$$

and equation (14) implies,

$$0 \le r + r_2 \le 1 \Rightarrow 0 \le r + \frac{1}{2} \le 1 \Rightarrow -\frac{1}{2} \le r \le \frac{1}{2} \Rightarrow -\frac{1}{2} \le \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right) q'\left(\rho_i^n\right) \le \frac{1}{2} \Rightarrow -1 \le \left(\frac{\Delta t}{\Delta x}\right) q'\left(\rho_i^n\right) \le 1$$

$$\Rightarrow -\Delta x \le \Delta t\, q'\left(\rho_i^n\right) \le \Delta x \Rightarrow -\Delta x \le \Delta t\, v_{\max}\left(1 - \frac{2\rho_i^n}{\rho_{\max}}\right) \le \Delta x \Rightarrow -\Delta x \le \Delta t\ v_{\max} \max\left(\rho_i^o\right) \le \Delta x.$$

IV-C Numerical Simulation and Results Discussion

We implement numerical finite difference scheme that is second order Lax-Wendroff difference scheme by computer programming and perform numerical simulation as described below.

IV-C (I) Comparative Profile of Density in different time step

In this section we present traffic density of exact solution and numerical simulation results based on second order Lax-Wendroff difference scheme (LWDS) for some specific cases of flow parameters like $\rho_{\max}, v_{\max}$ etc. Now we consider the initial density using sine function, $\rho(0,x) = \rho_o(x) = 15 * \sin\left(\frac{x}{4}\right) + 16$ and perform traffic density in different time step for exact solution and LWDS in the spatial domain [0, 10] in km. Here we use exact solution result as prescribe left boundary value and right boundary value as corresponding two sided boundary values for LWDS. For the above initial and boundary conditions with $v_{\max} = 0.167\,(0.1\,\text{km/sec}) = 60.12\,\text{km/hour},$ satisfying the physical constraints condition (14) $\rho_{\max} = 10 \max_i \rho_o(x_i) = 550/\text{km}$ in the spatial domain [0km, 10km], we perform the numerical experiment for 6 minutes in $\Delta t = 0.1$ time steps for a highway of 10 km in 101 spatial grid points with step size $\Delta x = 100\ \text{meters} = 0.25$ which guarantees the stability condition

$$0 < \left(v_{\max} \frac{\Delta t}{\Delta x}\right) \le 1 / \left(1 - \frac{2\max(\rho_i^0)}{\rho_{\max}}\right) \text{ and } -\Delta x \le v_{\max}\, \Delta t \max(\rho_i^0) \le \Delta x \text{ respectively.}$$

Figure-1 shows density profile of exact solution in different time step when $v_{\max} = 60\,\text{km/hour}.$

Figure-2 shows comparison of density in initial time step between exact solution and LWDS. The figure shows that at initial stage the density of car in a 10 km highway is overlapping. **Figure-3** shows comparison of density in different time step and **Figure-4** shows comparison of density in 600th, 1200th and 1800th time step. From the above figure we see that the density profile of LWDS is close nearer to exact solution. In **figure-5(a)** solid line represents the exact solution and the red line represents LWDS of density profile in last time step. Here we see that density profile in right boundary red line has enough jigjag, it depends on the discretization parameters $\Delta t = 0.1$ and $\Delta x = 0.2$ respectively. In discretization parameters $\Delta t = 0.6$ and $\Delta x = 0.4$ **figure-5(b)** last time step jigjag is no more than figure 5(a). When discretization parameters $\Delta t = 0.9$ and $\Delta x = 0.2$ in **figure-5(c)** right boundary has few jijjag. In **figure-5(d)** right boundary has only one jigjag. Finally, when discretization parameters $\Delta t = 0.01$ and $\Delta x = 0.04$ **figure-5(e)** shows the density profile has no jigjag. From above discretization parameters satisfying the stability conditions of LWDS.

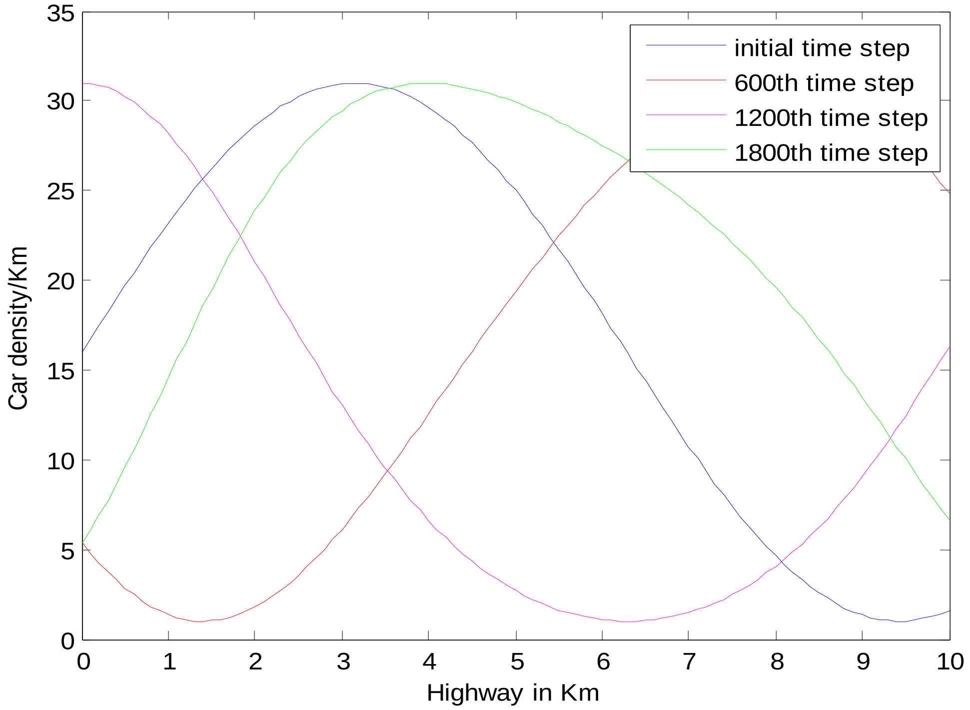

Figure-1: Density profile of exact solution in different time step when $v_{\max} = 60\,\text{km/hour}$

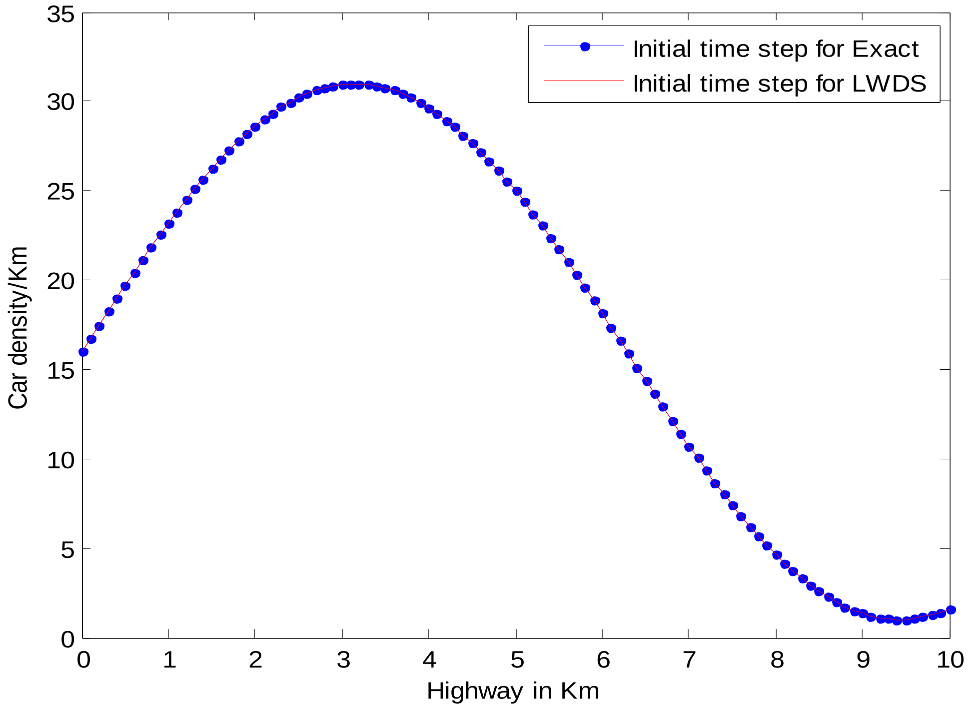

Figure-2: Comparison density profile of exact solution and LWDS initial time step

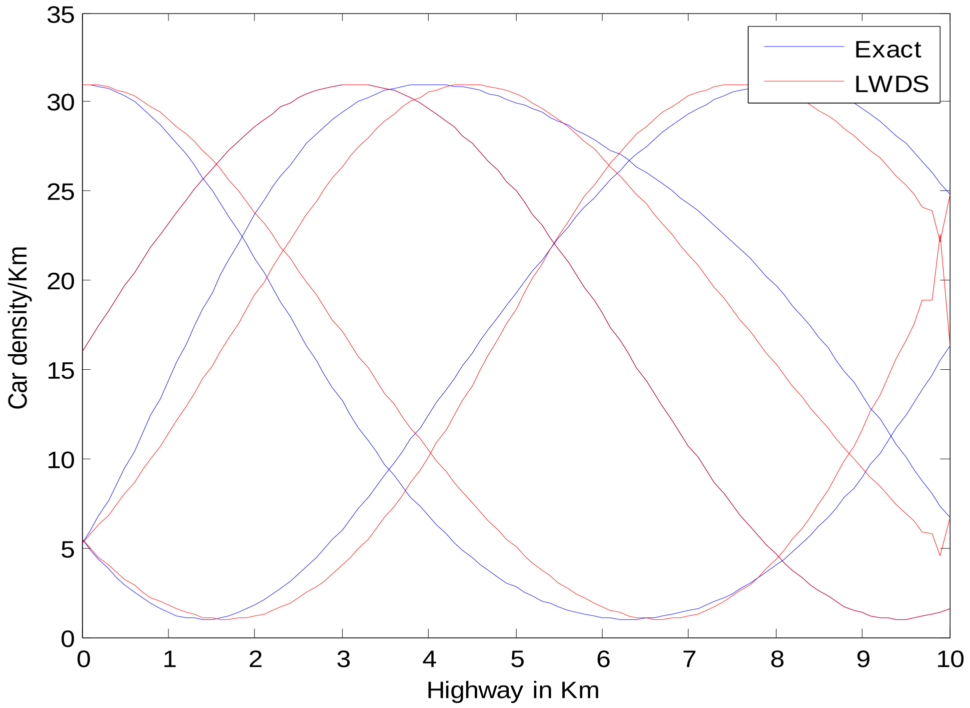

Figure-3: Comparison density profile of exact solution and LWDS in different time step

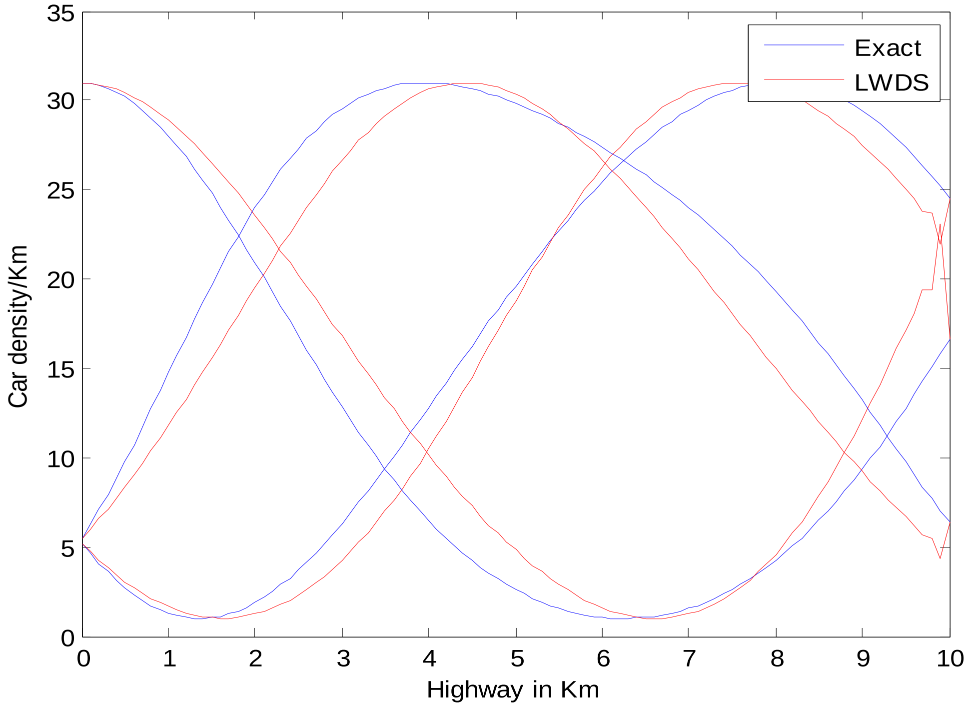

Figure-4: Comparison density profile of exact solution and LWDS in 600th, 1200th, 1800th time step

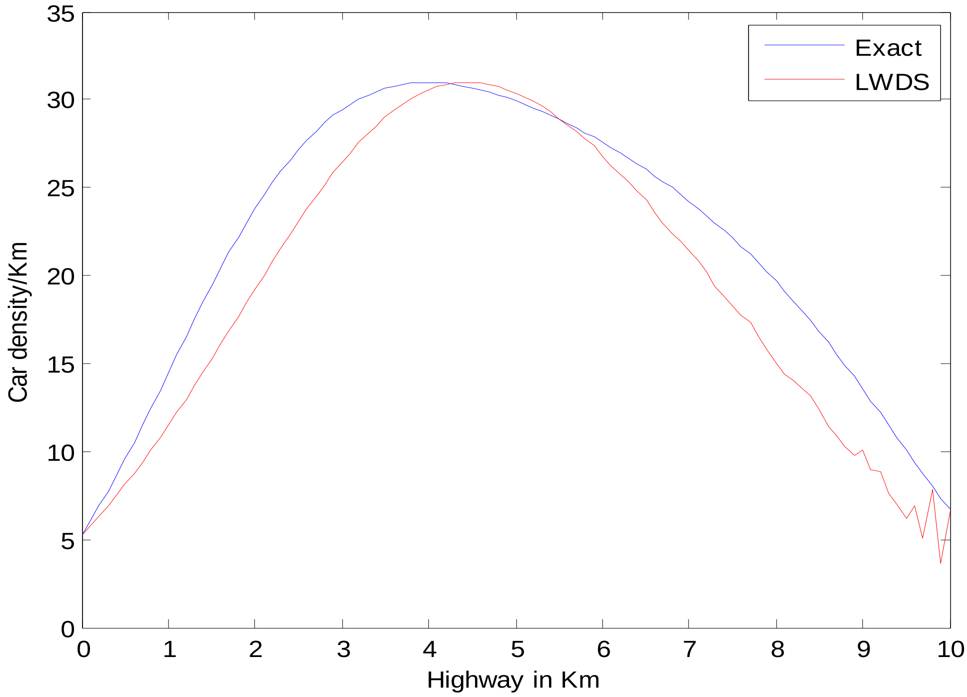

Figure-5(a): Comparison density profile of exact solution and LWDS in last time step when $\Delta t=0.1$ and $\Delta x=0.2$

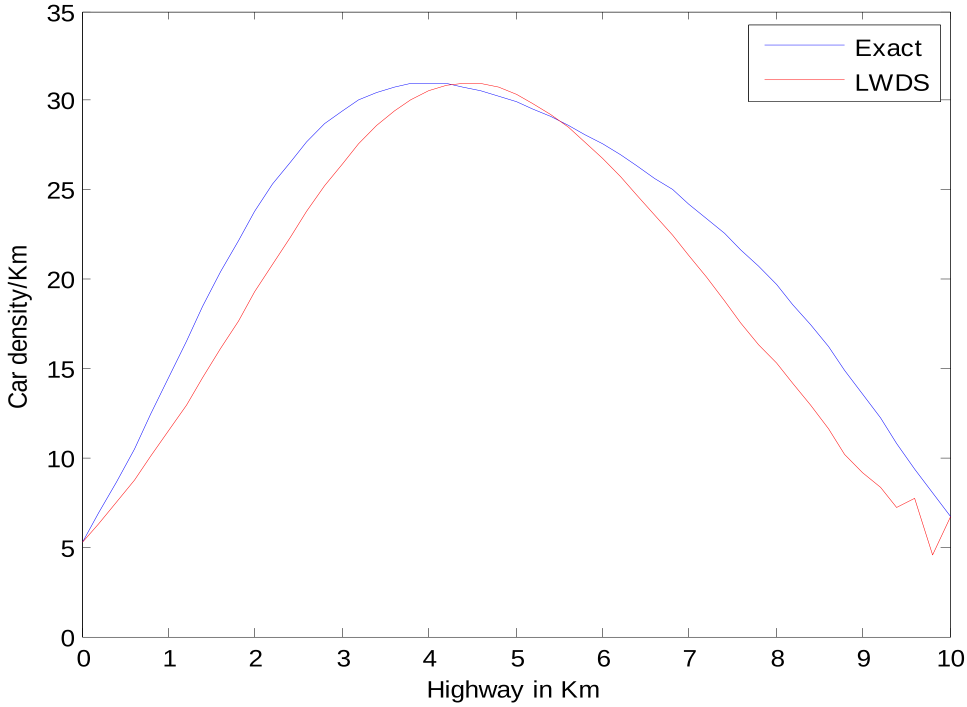

Figure-5(b): Comparison density profile of exact solution and LWDS in last time step when $\Delta t=0.6$ and $\Delta x=0.4$

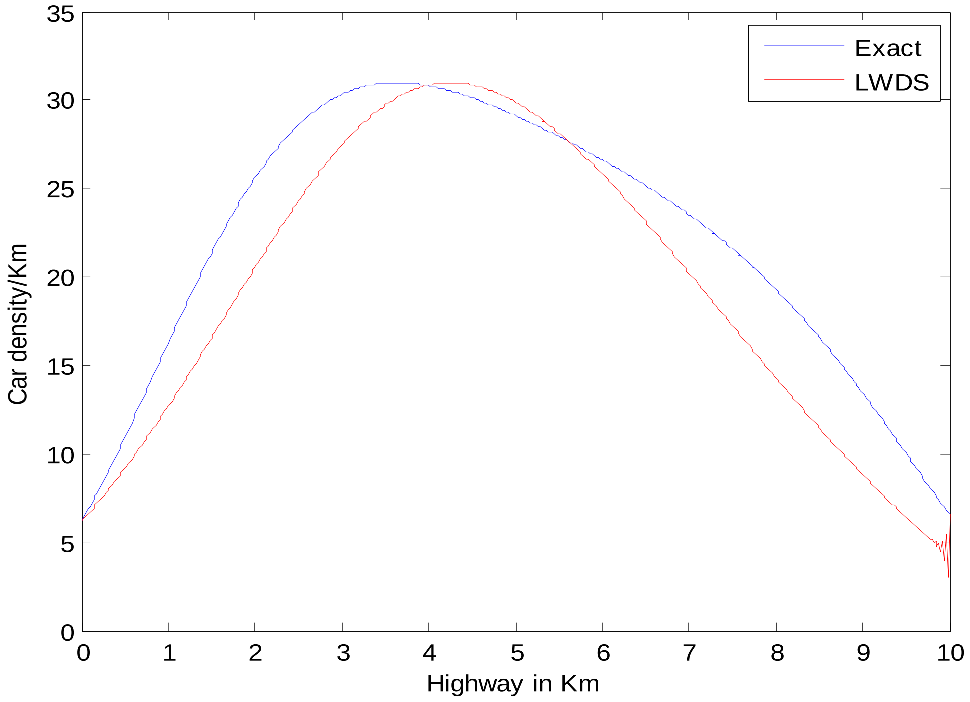

Figure-5(c): Comparison density profile of exact solution and LWDS in last time step when $\Delta t=0.9$ and $\Delta x=0.2$

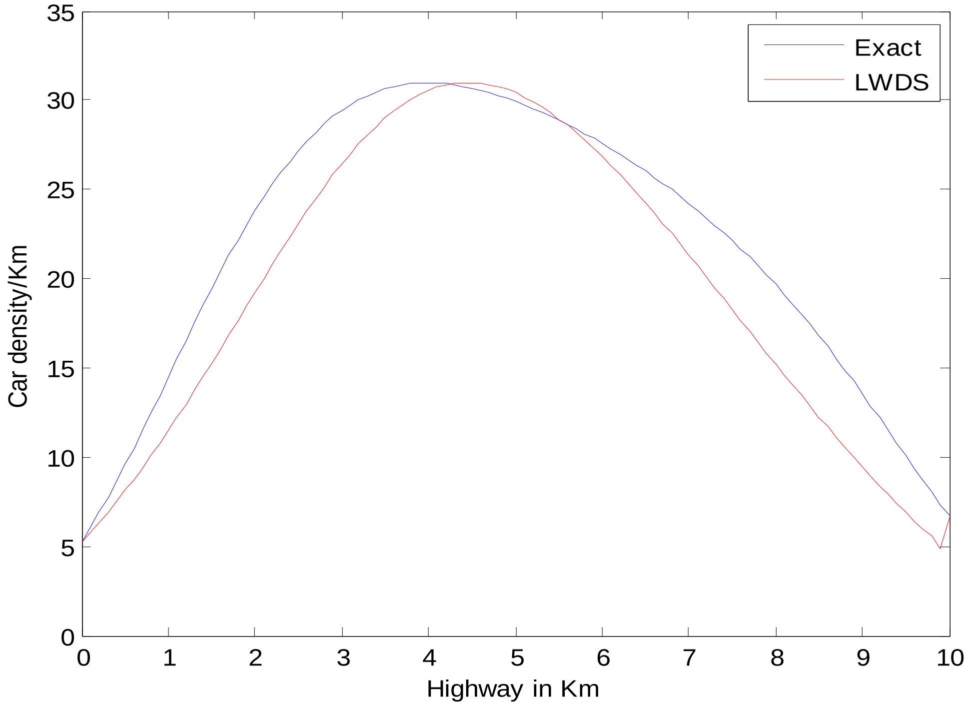

Figure-5(d): Comparison density profile of exact solution and LWDS in last time step when $\Delta t=0.05$ and $\Delta x=0.04$

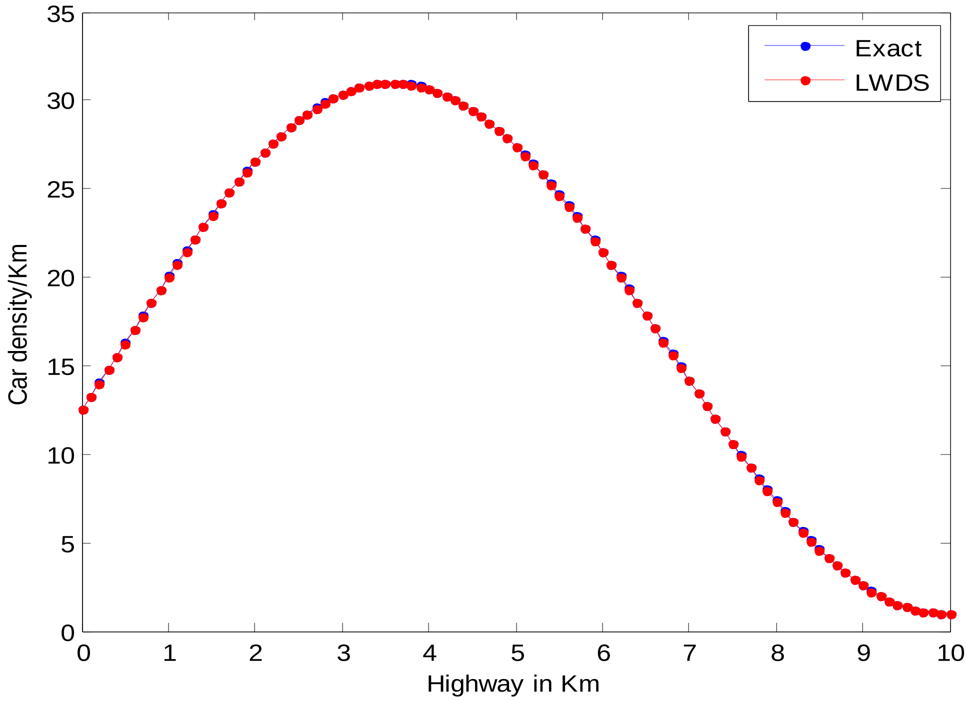

Figure-5(e): Comparison density profile of exact solution and LWDS in last time step when $\Delta t=0.01$ and $\Delta x=0.04$

IV-C (II) Error Estimation

In order to perform error estimation for density (ρ), we consider exact solution (4) with initial condition i.e. linear function $\rho_0(x)=\frac{1}{2}x,$ we have

$$\rho(t,x)=\rho_0(x_0)=\frac{1}{2}\left(x-v_{\max}\left(1-\frac{2\rho}{\rho_{\max}}\right)t\right)$$

$$\Rightarrow \rho(t,x)=\frac{(x-v_{\max}t)\big/2}{(1-v_{\max}t)\big/\rho_{\max}}.$$

We compute the relative error in L_1-norm defined by $\|e\|_1=\frac{\|\rho_e-\rho_n\|_1}{\|\rho_e\|_1}$ for all time ρ_e is the exact solution and ρ_n is the numerical solution computed by finite difference scheme.

We prescribe the corresponding two sided boundary values for LWDS by the equations

$$\rho_a(t)=\rho(t,x_a)=\frac{(x_a-v_{\max}t)\big/2}{(1-v_{\max}t)\big/\rho_{\max}} \text{ and } \rho_b(t)=\rho(t,x_b)=\frac{(x_b-v_{\max}t)\big/2}{(1-v_{\max}t)\big/\rho_{\max}}$$

Figure-6 below shows the relative error for density (ρ) of Lax-Wendroff difference scheme, which remains 0.0000008 which is quite acceptable. **Figure-7** presents that the density (ρ) error is decreasing with respect to the smaller descretization parameters Δt and Δx which shows the convergence of Lax-Wendroff difference scheme. We observe that as we increase number of grid points the error is decreasing and also shows the rate of convergence of the numerical solutions.

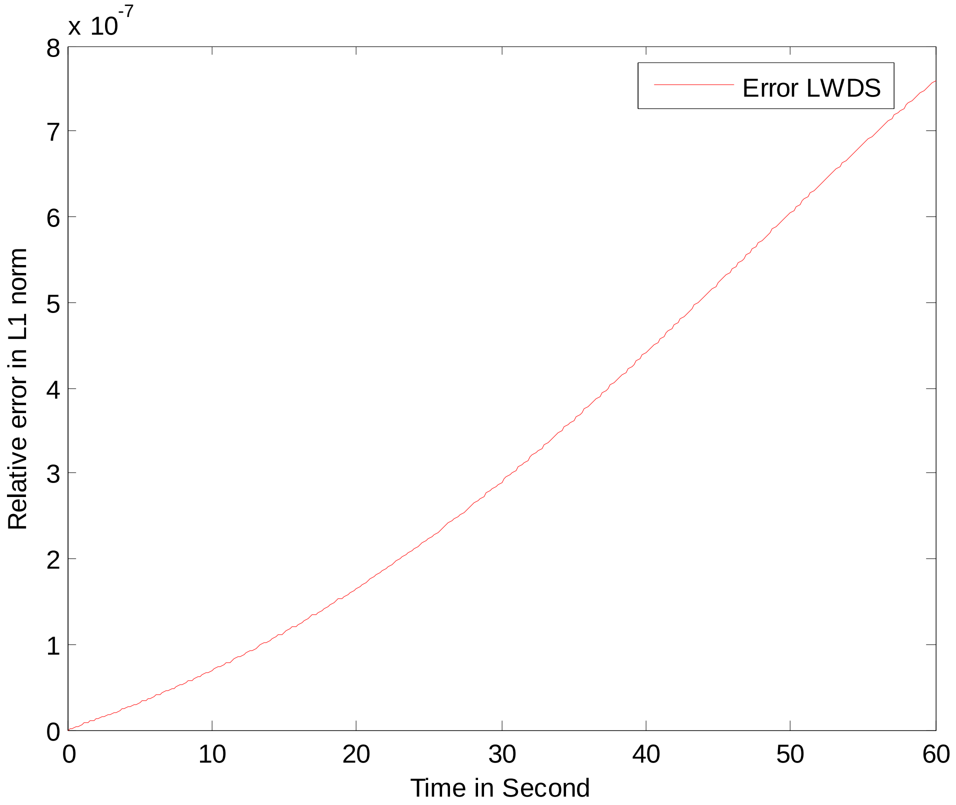

Figure-6: Relative errors of Lax-Wendroff difference scheme

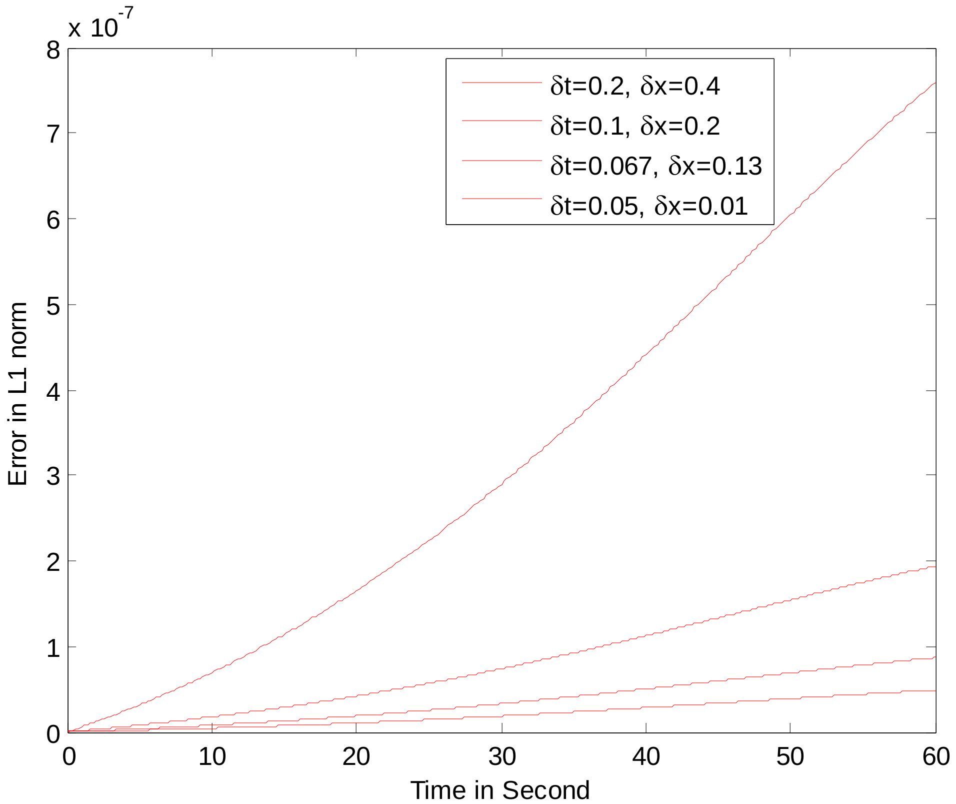

Figure-7: Convergence of errors of Lax-Wendroff difference scheme

V. Conclusion

From the numerical results we observe that Lax-Wendroff difference scheme for the considered traffic flow model is adequate for traffic flow simulation. The time-step in the established stability condition and well-posed-ness condition is not stiff and this resulted computational efficiency of the scheme. The computational results showed the accuracy up to six decimal places and a good rate of convergence. The scheme can be extended for multi-lane traffic flow simulations which we left for future work.

VI. References

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Md. Shajib Ali "Numerical Efficient For Second Order Lax-Wendroff Difference Scheme Of A Fluid Dynamic Traffic Flow Model" IOSR Journal of Mathematics (IOSR-JM) 14.1 (2018): 01-13.