IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 14, Issue 2 Ver. II (Mar. - Apr. 2018), PP 47-50
www.iosrjournals.org

Extension of Some Theorems In General Metric Spaces

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Abstract
We prove a version of Caristi-Kirk - BrowderTheorem and Park's Theorem [3,4] in G-metric space. And then give some corollaries.
Keywords
G-metric spaces,fixed point.
AMS (2010) subject classification: *46Axx, see also 47H10, 55M20*

Date of Submission: 20-03-2018 Date of acceptance: 09-04-2018

I. Introduction and preliminaries

In 2005, a general metric spaces was introduced by Mustafa and Sims as appropriate notion of generalized metricspace called G −metric spaces as follows.

Definition (1.1) [1]

'Let$X$ be a non-empty set and $G: X \times X \times X \to R^+$ be a function for all x, y, z, a in X satisfying the following conditions:

(1) $G(x, y, z) = 0$ if $x = y = z$
(2) $0 < G(x, x, y)$ with $x \neq y$
(3) $G(x, x, y) \leq G(x, y, z)$ with $y \neq z$
(4) $G(x, y, z) = G\big(P(x, y, z)\big), P(x, y, z)$ is permutation of x, y, z
(5)$G(x, y, z) \leq G(x, a, a) + G(a, y, z)$

Then the ordered pair (X, G) is called a generalized metric or G-metric space. X is said to be symmetric if for all x, y in X

$$G(x, y \ \ , y) = G(y, x, x)."$$

Proposition (1. 3) [1]

"Let (X, G) be a G-metric space Then for any $u, v, w, and\ b \in X$, the following are satisfies

(1)if$G(u, v, w) = 0$ Then $u = v = w$
(2)$G(u, v, w) \leq G(u, u, v) + G(u, u, w)$
(3)$G(u, v, v) \leq 2G(v, u, u)$
(4)$G(u, v, w) \leq G(u, b, w) + G(b, v, w)$
(5)$G(u, v, w) \leq 2/3\big(G(u, v, b) + G(u, b, w) + G(b, w, y)\big)$
(6)$G(u, v, w) \leq G(u, b, b) + G(v, b, b) + G(w, b, b)$"

Definition (1.4) [2]

"Let (X, G) be a G-metric space, let(x_n) be a sequence of points of X a point $x \in X$ is said to be the limit of the sequence (x_n) if

$$\lim_{n,m\to\infty} G(x, x_n , x_m) = 0$$

Thus, that if $x_n \to x_0$ in a G-metric space (X, G),then for any $\varepsilon > 0$ there exists $K \in N$ such that $G(x, x_n, x_m) < \varepsilon$ for all $n, m \geq K$."

Definition (1.5) [2]

"Let (X, G) be a G–metric space a sequence (x_n) is called G –Cauchy if given $\varepsilon > 0$, there is $K \in N$ such that

$$G(x_n, x_m, x_l) < \varepsilon, \text{ for all } n, m, l \geq K ,$$

that is , if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$."

Definition (1.6) [2]

"A G-metric space (X, G) is said to be G-complete or (complete G-metric) if every G-cauchy sequence in (X, G) is convergent in (X, G)."

Definition (1.7) [1]

"Let (X, G) and (X', G') be two G-metric spaces, and let $f: (X, G) \to (X', G')$ be a function .then f is said to be G-continuaus at a point $a \in X$ if and only if given $\varepsilon > 0$,there exists $\delta > 0$ such that $x, y \in X$; and

$$G(a, x, y) < \delta \Rightarrow G'\big(f(a), f(x), f(y)\big) < \varepsilon$$

A function f is G-continuous on X if and only if it is G-continuous at all $a \in X$."

Definition (1.8) [2]

"Let (X, G) be a G-metric spase, the mapping $T: X \to X$ then for all $x, y, z \in X$

i-T is called $G -$ contraction mapping if

$$G(T(x), T(y), T(z)) \leq k\, G(x, y, z), \text{forsome } k \in (0,1)$$

ii-.T is called a G - contractive if

$$G(T(x), T(y), T(z)) < G(x, y, z), \text{ for all } x, y, z \text{ in } X \text{ with } x \neq y \neq z$$

iii-T is called G -expansive mapping if

$$G(T(x), T(y), T(z)) \geq a\, G(x, y, z), \text{ for some } a > 1"$$

"The version of Banach's fixed point Theorem in G-metric space is

Theorem (1.10) [2]

"If (X, G) be a complete G-metric space and $T: X \to X$ be a G – contraction mapping, then T has unique fixed point $z in X$, and $\lim_{n\to\infty} T^n(x) = z$, for any intial point $x in\, X$."

II. Main Results

We begin with following

Theorem (2.1): Let Mbe a subset of a complete G-metric space and $T : X \to X$ be a mapping such that $\emptyset : X \to R^+ G(x, x, Tx) \leq \emptyset(x) - \emptyset(T(x))$, for all $x \in X$.

where$\emptyset$ is lower semi continuous function

Proof:

For $x_0 \in X\ and\ n, m \in N\ with\ n < m$, we have$\emptyset: X \to R$, then , by similar argument of proof of Theorem (2.1) in [2]

$$G(T^n(x_0), T^n(x_0), T^{m+1}(x_0)) \leq \sum_{i=n}^{m} G\left(T^i(x_0), T^i(x_0), T^{i+1}(x_0)\right)$$
$$\leq \emptyset(T^n(x_0)) - T^{m+1}(x_0)$$

In particular,

$$\sum_{i=0}^{\infty} G\left(T^i(x_0), T^i(x_0), T^{i+1}(x_0)\right) < \infty$$

Therefore, $(T^n(x_0))$is Cauchy sequence. Since T is continuous, then $(T^n(x_0))$ converges to a fixed point of T.

Definition(2.2):

A real valued function $\emptyset$ on X has a $G -$point $p \in X$ if

$$\emptyset(p) - \emptyset(x) < G(p, p, x), \quad \text{forother point } x \in X, x \neq p.$$

Proposition (2.3) :

Every lower semi continuous function $\emptyset: X \to R^+$on a complete X has a $G -$point p in X.

Proof:

By putting $T = I$ and $T(x) = p$ in theorem (2.1).

Theorem (2.4)

Let M be a subset of a complete G-metric space X and $f, g : M \to X$ be maps such that

(i) f is surjective

(ii) There exist a lower semi continuous function$\emptyset : X \to R^+$ satisfying

$$G(f(x), f(x), g(x)) \leq \emptyset(f(x)) - \emptyset(g(x)) \qquad \ldots \qquad (2.1)$$

for each $x \epsilon M$. Then f and g have a coincidence point.

Proof:

By proposition (2.3), then $\emptyset$ has a G-point $p \epsilon X$, means that

$$\emptyset(p) - \emptyset(x) < G(p, x, x)$$

Now, let $x \epsilon f^{-1} p$, suppose $fx = gx$ since $p = fx$ and $gx \epsilon X$, we have

$$\emptyset(f(x)) - \emptyset(g(x)) < G(f(x), f(x), g(x))$$

which contradicts (ii).

By putting $X = M$and $= I$, Theorem (2.1) reduces to the version of Caristi-Kirk Theorem in G-metric space: Consequently, we obtain the following:

Corollary (1)

(i) If $M = X$ and $f = I_x$,then the above theorem reduces to the version Caristi-Kirk -Browder theorem in this case ,if g is G-continuous then for any $x \in X$ the sequence $\{g_x^n\}$ G-converges to a fixed point of g

(ii) If $M = X$ and $g = 1_x$ .then f has a fixed point

Corollary (2)

Let X be a G-metric space and $f: X \to X$ be onto mapping such that for all x, y in X if there is a constant $a > 1$ such that

$$G\big(f(x), f(x), f(y)\big) \geq a\, G(x, x, y) \; .... \quad (2.2)$$

then f has a unique fixed point

Proof:

From (2.2) f is clearly injective. Since f is also surjective, $g = f^{-1}$ exists and is surjective for any x, y in X we obtain, from (2.3)

$$G(x, x, y) \geq a\, G(gx, gx, gy)$$

and g is G-continuous. One could use Theorem (1.10) at this point to prove that g has a unique fixed point. Adding $(a-1)G(x, x, y)$ to each side of the above inequality to get

$$a\, G(x, x, y) - a\, G(gx, gx, gy) \geq (a-1)G(x, x, y)$$

Now, put $y = gx$ to get

$G(x, x, gx) \leq \emptyset(x) - \emptyset(gx)$,

where, define $\emptyset$ as

$$\emptyset(x) = \frac{a\, G(x, x, gx)}{(a-1)}$$

since g is G-continuous, $\emptyset$ is lower semi continuous , and g has a fixed point by Corollary (1-i). For any $x \in X$,the sequence $\{g^n x\}$ G-converges to a fixed point of g , that is ,of f. From (2.2) the fixed point is unique.

Corollary (4)

Let X be a G-metric space and $f: X \to X$ be onto mapping such that for all x, y in X if there exist $a, b, c \geq 0$ with $a + b + c > 1$ and $a < 1$ such that

$G\big(f(x), f(x), f(y)\big) \geq a\, G\big(x, x, f(x)\big) + b\, G\big(y, y, f(y)\big) + c\, G(x, x, y)$….(2.3)

with $x \neq y$,then f has a fixed point

Proof:

Since (2.4) is symmetric in x and y assume that $a = b < 1$. Adding, a $G(fx, fx, fy)$ to both sides of (2.3) we have

$$(1 + a)\, G(fx, fx, fy) \geq a[G(x, x, fx) + G(fx, fx, fy) + G(fy, y, y)] + c\, G(x, x, y) \geq (a + c)G(x, x, y)$$

or

$$G(fx, fx, fy) \geq \frac{a + c}{1 + a} G(x, x, y)$$

since $a + c = 0$ implies $a = b > 1$, $(a + c)/(1 + a) > 0$

and f is injective. Since f is also surjective $g = f^{-1}$ exists. Also, since

$$G(x, x, y) \geq \tfrac{a+c}{1+a} G(gx, gx, gy), \text{ for all } x, y \in X,$$

and hence g is G-continuous. (2.3) will be in the form

$$G(x, x, y) \geq a\, G(gx, gx, x) + b\, G(gy, gy, y) + c\, G(gx, gx, gy)$$

set $y = gx$ and then add $(b + c + a - 1)G(x, x, gx)$

to each side to get

$$(b + c)[\, G(x, x, gx) - G(gx, gx, g^2 x)] \geq (b + c + a - 1)G(x, x, gx)$$

or

$$G(x, x, gx) \leq \emptyset(x) - \emptyset(gx)$$

where defined

$$\emptyset(x) = \frac{(b + c)G(x, x, gx)}{(a + b + c - 1)}$$

Since g is G-continuous, $\emptyset$ is (lsc) and g has a fixed point by Corollary (1-i). Moreover for $\forall x \in X$ the sequence $\{g^n x\}$ G-converges to a fixed point of g, that is ,of f

Corollary (5)

Let X be a G-metric space and $f: X \to X$ be onto and G-continuous mapping such that for all $x \in X$ and if $\exists a > 1$ satisfying

$$G\big(f(x), f(x), f^2(x)\big) \geq a\, G\big(x, x, f(x)\big) \qquad .... (2.4)$$

then f has a fixed point

Proof:

Adding $-G\big(x, x, f(x)\big)$ to condition (2.4) yields

$$G(x, x, fx) \leq [G(fx, fx, f^2 x) - G(x, x, fx)]/(a-1)$$

Define, $\emptyset: X \to R^+$ by $\emptyset(x) = \frac{G(x,x,f(x))}{a-1}$
since$f$ is G-continuous, $\emptyset$ is (lsc) and by corollary (1-ii) , f has fixed point

Corollary (6)

Let X be a G-metric space and $f: X \to X$ be onto and G-continuous mapping such that for all x, y in X and if there exists a real constant $a > 1$ such that

$$G\big(f(x), f(x), f(y)\big) \geq a\, min\big[G\big(x, x, f(x)\big), G(y, y, fy), G(x, x, y)\big] \ldots. (2.5)$$

then$f$ has a fixed point

Proof:

Set $y = f(x)$ in condition (2.5)

$$G\big(f(x), f(x), f(y)\big) \geq a\, min[G(x, x, fx), G(y, y, fy), G(x, x, y)]$$

yield to

$$G\big(f(x), f(x), f^2(x)\big) \geq a\ G\ \big(x, x, f(x)\big) \ldots.. (2.4)$$

then$f$ has a fixed point .

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Salwa Salman AbedEbtihal Nabil Jaluobe "Extension of Some Theorems In General Metric Spaces. " IOSR Journal of Mathematics (IOSR-JM) 14.2 (2018): 47-50.