IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 14, Issue 3 Ver. I (May - June 2018), PP 30-37
www.iosrjournals.org

On Scalar Quasi - Weak Commutative Algebras

G.Gopalakrishnamoorthy[1], S.Anitha[2] and M.Kamaraj[3]
[1]*Principal, Sri krishnasamy Arts and Science College , Sattur – 626203, Tamilnadu.*
[2]*Lecturer, Raja Doraisingam Government Arts College, Sivagangai – 630 561, Tamil Nadu.*
[3]*Dept. of Mathematics,Government Arts and Science College,Sivakasi - 626124.*

Abstract: *The concepts of scalar commutativity defined in an Algebra A over a commutative ring R and Quasi - weak commutativity defined in a near- ring are mixed together to coin the concept of scalar quasi weak commutativity in an algebra A over a commutative ring R and its various properties are studied.*

Date of Submission: 20-04-2018 Date of acceptance: 07-05-2018

I. Introduction:

Koh,Luh,Putcha [5] called an algebra A over a commutative ring R to be scalar commutative if for every x,y ∈ A there exists an $\alpha \in$ R depending on x and y such that xy = αyx.Coughlin and Rich[1] and Coughlin,Kleinfield [2] have studied scalar commutativity in algebras over a field F.G.Gopalakrishnamoorthy,S.Geetha and M.kamaraj[3] have defined a near - ring N to be Quasi – weak commutative if xyz = yxz for all x,y,z ∈ N.They have obtained many interesting results of Quasi – weak commutativity in Near – rings. In this paper we call an algebra A over a commutative ring R to be scalar Quasi – weak Commutative, if for every x,y,z ∈ A ,there exists a scalar $\alpha \in$ R depending on x,y,z such that xyz = αyxz. We prove many interesting results.

II. Preliminaries:

In this section we give the basic definitions and various well known results which we use in the sequel.

2.1 Definition[5]:
Let A be an algebra over a commutative ring R.If for every x,y ∈ A,there exitsts an element $\alpha \in$ R depending on x,y such that xy = αyx, then A is said to be scalar commutative.If for every x,y ∈ A, there exists an element $\alpha \in$ R depending on x,y such that xy = - αyx, then A is said to anti-scalar commutative.

2.2 Definition[3]:
Let N be a near – ring inwhich xyz = yxz for all x,y,z ∈ N.Then N is called Quasi - weak commutative near-ring.If xyz = - yxz for all x,y,z ∈ N,then N is said to be Quasi - weak anti-commutative.

2.3 Lemma 3.5[4]:
Let N be a distributive near-ring.If xyz = $\pm$ yxz for all x,y,z ∈ N,then N is either quasi weak commutative or quasi weak anti- commutative.

III. Main Results:

3.1 Definition:
Let A be an algebra over a commutative ring R.If for every x,y,z ∈A,there exists $\alpha \in$ R depending on x,y,z such that xyz = αyxz,then A is said to be scalar quasi weak commutative.
If xyz = - αyxz,then A is said to be scalar quasi weak anti- commutative.

3.2 Theorem:
Let A be an algebra (not necessarily associative) over a field F.If A is scalar quasi weak commutative,then A is either quasi weak commutative or quasi weak anti commutative.

Proof:
Suppose xyz = yxz for all x,y,z ∈ A,there is nothing to prove.Suppose not,we shall prove that xyz = - yxz for all x,y,z ∈A.We shall first prove that if x,y,z ∈A such that xyz ≠ yxz,then $x^2z = y^2z = 0$.

Let x,y,z ∈A such that xyz ≠ yxz.Since A is scalar quasi weak commutative,there exists $\alpha = \alpha$(x,y,z) ∈ F such that

$$xyz = \alpha yxz \quad \rightarrow (1)$$

Also there exists $\gamma = \gamma$(x,x+y,z) ∈ F such that

$$x(x+y)z = \gamma(x+y)xz \quad \rightarrow (2)$$

(1) – (2) gives

$xyz - x^2z - xyz = \alpha yxz - \gamma x^2z - \gamma yxz.$

i.e., $\gamma x^2z - x^2z = (\alpha - \gamma)yxz.$

$(\gamma - 1) - x^2z = (\alpha - \gamma)yxz$

i.e., $(1 - \gamma)\, x^2z = (\gamma - \alpha)yxz \quad \rightarrow (3)$

Now

yxz ≠ 0 for if yxz = 0, then from (1) we get xyz = 0 and so xyz = yxz, contradicting our assumption that xyz ≠ yxz.

Also $\gamma \neq 1$, for if $\gamma = 1$, then from (3) we get $\alpha = \gamma = 1$.

Then from (1) we get xyz = yxz, again contradicting our assumption that xyz ≠ yxz.

Now from (3) we get

$$x^2z = \frac{\gamma-\alpha}{1-\gamma}\ yxz$$

i.e., $x^2z = \beta$ yxz for some $\beta \in F$ $\rightarrow (4)$

Similarly $y^2z = \delta$ yxz for some $\delta \in F$ $\rightarrow (5)$

Now corresponding to each choice of $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in F$,there is an $\eta \in F$ such that

$(\alpha_1 x + \alpha_2 y)(\alpha_3 x + \alpha_4 y)z = \eta(\alpha_3 x + \alpha_4 y)(\alpha_1 x + \alpha_2 y)z$

$(\alpha_1\alpha_3 x^2 + \alpha_1\alpha_4 xy + \alpha_2\alpha_3 yx + \alpha_2\alpha_4 y^2)z = \eta(\alpha_3\alpha_1 x^2 + \alpha_3\alpha_2 xy + \alpha_4\alpha_1 yx + \alpha_4\alpha_2 y^2)z.$

$\alpha_1\alpha_3 x^2z + \alpha_1\alpha_4 xyz + \alpha_2\alpha_3 yxz + \alpha_2\alpha_4 y^2z = \eta(\alpha_3\alpha_1 x^2z + \alpha_3\alpha_2 xy z + \alpha_4\alpha_1 yx z + \alpha_4\alpha_2 y^2z) \quad \rightarrow (6)$

$\alpha_1\alpha_3\beta yxz + \alpha_1\alpha_4 xyz + \alpha_2\alpha_3 yxz + \alpha_2\alpha_4\delta yxz = \eta(\alpha_3\alpha_1\beta yx z + \alpha_3\alpha_2 xy z + \alpha_4\alpha_1 yx z + \alpha_4\alpha_2\delta yxz)$

(using (4) and (5))

$\alpha_1\alpha_3\beta\alpha^{-1}xyz + \alpha_1\alpha_4 xyz + \alpha_2\alpha_3\alpha^{-1} xyz + \alpha_2\alpha_4\delta\alpha^{-1} xyz$

$= \eta(\alpha_3\alpha_1\beta yx z + \alpha_3\alpha_2\alpha yxz + \alpha_4\alpha_1 yx z + \alpha_4\alpha_2\delta yxz)$

i.e., $(\alpha_1\alpha_3\beta\alpha^{-1} + \alpha_1\alpha_4 + \alpha_2\alpha_3\alpha^{-1} + \alpha_2\alpha_4\delta\alpha^{-1})$ xyz

$= \eta(\alpha_3\alpha_1\beta + \alpha_3\alpha_2\alpha + \alpha_4\alpha_1 + \alpha_4\alpha_2\delta)yxz \quad \rightarrow (7)$

If in (7) we choose $\alpha_2 = 0$, $\alpha_3 = \alpha_1 = 1$, $\alpha_4 = -\beta$, the right hand side of (7) is zero where as the left hand side of (7) is

$(\beta\alpha^{-1} - \beta)$ xyz = 0

i.e., $\beta(\alpha^{-1} - 1)$ xyz = 0.

Since xyz ≠ 0 and $\alpha \neq 1$, we get $\beta = 0$.

Hence from (4) we get $x^2z = 0$.

Also if in (7) ,we choose $\alpha_3 = 0$, $\alpha_4 = \alpha_2 = 1$, $\alpha_1 = -\delta$, the right hand side of (7) is zero where as the left hand side of (7) is

$(-\delta + \delta\alpha^{-1})$ xyz = 0.

i.e., $\delta(\alpha^{-1} - 1)$ xyz = 0.

Since xyz ≠ 0 and $\alpha \neq 1$, we get $\delta = 0$.

Hence from (5) we get $y^2z = 0$.

Then (6) becomes

$\alpha_1\alpha_4 xyz + \alpha_2\alpha_3 yxz = \eta(\alpha_3\alpha_2 xyz + \alpha_4\alpha_1 yxz)$

$\alpha_1\alpha_4 xyz + \alpha_2\alpha_3\alpha^{-1} xyz = \eta(\alpha_3\alpha_2 xyz + \alpha_4\alpha_1\alpha^{-1} xyz)$

i.e.,$(\alpha_1\alpha_4 + \alpha_2\alpha_3\alpha^{-1})$ xyz = $\eta(\alpha_3\alpha_2 + \alpha_4\alpha_1\alpha^{-1})$ xyz.

This is true for any choice of $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in$ F.Choose $\alpha_1 = \alpha_3 = \alpha_4 = 1$, $\alpha_2 = \alpha^{-1}$,we get

$(1 - (\alpha^{-1})^2)$xyz = 0.

Since xyz ≠ 0 , $1 - (\alpha^{-1})^2 = 0$.Hence $(\alpha^{-1})^2 = 1$.

i.e., $\alpha = \pm 1$.Since $\alpha \neq 1$,we get $\alpha = -1$.

i.e., xyz = - yxz for x,y,z ∈ A.

Thus A is either quasi weak commutative or quasi weak anti commutative.

3.3 Lemma:

Let A be an algebra (not necessarily associative) over a commutative ring R.

Suppose A is scalar quasi- weak commutative .Then for all x,y,z ∈ A,$\alpha \in$ R, αxyz = 0 iff α yxz = 0.Also xyz = 0 iff yxz = 0**.**

Proof:

Let x,y,z ∈ A, $\alpha \in$ R such that αxyz = 0.

Since A is scalar quasi weak commutative,there exists $\beta = \beta\ (\alpha y,x,z) \in$ R such that

$(\alpha y)xz = \beta x(\alpha y)z = \beta\alpha xyz = 0$

(ie) $\alpha yxz = 0$.

Similarly

If $\alpha yxz = 0$, then there exists $\gamma \epsilon$R such that $\gamma = \gamma(x,\alpha y,z)\ \epsilon$ R, such that

$x(\alpha y)z = \gamma(\alpha y)xz = \gamma\alpha yxz = 0$

(ie) $\alpha xyz = 0$

Thus $\alpha xyz = 0$ iff $\alpha yxz = 0$.

Assume $xyz = 0$.

Since A is scalar quasi weak commutative there exists $\delta = \delta(y,x,z)\ \epsilon$ R such that

$yxz = \delta xyz = 0$

Similarly if $yxz = 0$, then there exists $\eta = \eta(x,y,z)\ \epsilon$ R such that

$xyz = \eta yxz = 0$

Thus $xyz = 0$ iff $yxz = 0$.

3.4 Lemma:

Let A be an algebra over a commutative ring R. Suppose A is scalar quasi weak commutative. Let x,y,z,u ϵ A, $\alpha,\beta\ \epsilon$ R such that

$xu = ux$, $yxz = \alpha xyz$ and $(y+u)xz = \beta x(y+u)z$. Then $(xu-\alpha xu-\beta xu+\alpha\beta xu)z = 0$.

Proof:

Given

$(y+u)xz = \beta x(y+u)z$ → (1)

$yxz = \alpha xyz$ → (2)

and $xu = ux$ → (3)

From (1) we get

$yxz + uxz = \beta xyz + \beta xuz$

$\alpha xyz + uxz = \beta xyz + \beta xuz$ (using (2))

i.e., $\alpha xyz + xuz = \beta xyz + \beta xuz$ (using (3))

i.e., $x(\alpha y + u - \beta y-\beta u)z = 0$.

By Lemma 3.3 we get

$(\alpha y + u - \beta y- \beta u)xz = 0$.

$\alpha yxz + uxz - \beta yxz - \beta uxz = 0$.

$\alpha yxz + uxz - \alpha\beta xyz - \beta uxz = 0$. (using (4)) →(4)

From (1) we get

$yxz + uxz = \beta xyz + \beta xuz$

$yxz + uxz = \beta xyz + \beta uxz$ (using (3))

i.e., $yxz - \beta xyz = \beta uxz - uxz$.

Multiply by α,

$\alpha yxz - \alpha\beta xyz = \alpha\beta uxz - \alpha uxz$ →(5)

From (4) and (5) we get

$\alpha\beta uxz - \alpha uxz + uxz - \beta uxz = 0$

i.e., $(\alpha\beta ux - \alpha ux + ux - \beta ux)z = 0$

i.e.,$(ux - \alpha ux - \beta ux + \alpha\beta ux)\ z = 0$

i.e.,$(xu - \alpha xu - \beta xu + \alpha\beta xu)z = 0$ ($\because$ xu = ux)

Hence the Lemma.

3.5 Corollary:

Taking u = x, we get

$(x^2 - \alpha x^2 - \beta x^2 + \alpha\beta\ x^2)z = 0$.

$(x - \alpha x)\ (x - \beta x)\ z = 0$.

3.6 Theorem:

Let A be an algebra over a commutative ring R.Suppose A has no zero divisors.If A is scalar quasi weak commutative ,then A is quasi weak commutative.

Proof:

Let x, y , z $\in$ A.

Since A is scalar quasi weak commutative,there exists scalars $\alpha = \alpha(y,x,z) \in$ R and $\beta = \beta(y+x,x,z) \in$ R such that

$(y+x)xz = \beta x(y+x)$ → (1)

and

$yxz = \alpha xyz$ → (2)

From (1) we get

$yxz + x^2z = \beta xyz + \beta x^2z.$

$\alpha xyz + x^2z - \beta xyz - \beta x^2z = 0.$ (using (2))

$x(\alpha y + x - \beta y - \beta x)z = 0.$

By Lemma 3.3,

$(\alpha y + x - \beta y - \beta x)xz = 0.$

$\alpha yxz + x^2z - \beta yxz - \beta x^2z = 0$

$\alpha yxz + x^2z - \alpha\beta xyz - \beta x^2z = 0$ (using (2)) → (3)

From (1) we get

$yxz + x^2z = \beta xyz + \beta x^2z.$

$yxz - \beta xyz = \beta x^2z - x^2z.$

$\alpha yxz - \alpha\beta xyz = \alpha\beta x^2z - \alpha x^2z$ → (4)

From (3) and (4) we get

$\alpha\beta x^2z - \alpha x^2z + x^2z - \beta x^2z = 0.$

$(x^2 - \alpha x^2 - \beta x^2 + \alpha\beta x^2)z = 0.$

$(x - \alpha x)(x - \beta x)z = 0.$

Since A has no zero dvisors.

$x = \alpha x$ or $x = \beta x.$

If $x = \alpha x$, then from (2) we get

$yxz = xyz.$

If $x = \beta x$,then from (1) we get

$(y+x)xz = x(y+x)z$

$yxz + x^2z = xyz + x^2z$

$yxz = xyz$

Thus A is quasi weak commutative.

3.7 Definition:

Let R be any ring and x,y,z ∈ R.We define xyz – yxz as the quasi weak commutator of x,y,z.

(ie) $xyz - yxz = (xy - yx)z$

$= [x, y]z$ is called the quasi weak commutator of x, y, z.

3.8 Theorem:

Let A be an algebra over a commutative ring R. Let A be a scalar quasi-weak commutative. If A has an identity, then the square of every quasi-weak commutator is zero.

(ie)., $(xyz - yxz)^2 = 0$ for all $x,y,z \in A$.

Proof:

Let $x,y,z \in A$. Since A is scalar quasi-weak commutative, there exists scalars $\alpha = \alpha(y,x,z) \in R$ and $\beta = \beta(x,y+1,z) \in R$ such that

$yxz = \alpha xyz$ → (1)

and

$x(y+1) = \beta(y+1)xz$ → (2)

From (2) we get

$xyz + xz - \beta yxz - \beta xz = 0$

$xyz + xz - \alpha\beta xyz - \beta xz = 0$

$x(y+1 - \alpha\beta y - \beta)z = 0$

By Lemma 3.3,

$(y+1 - \alpha\beta y - \beta)xz = 0$

$yxz + xz - \alpha\beta yxz - \beta xz = 0$

$\alpha xyz + xz - \alpha\beta yxz - \beta xz = 0$ (using(1)) → (3)

Also from (2) we get

$xyz + xz = \beta yxz + \beta xz$

Multiplying by α

$\alpha xyz + \alpha xz = \alpha\beta yxz + \alpha\beta xz$

$\alpha xyz - \alpha\beta yxz = \alpha\beta xz - \alpha xz$ → (4)

From (3) and (4) we get

$xz - \beta xz + \alpha\beta xz - \alpha xz = 0$

(ie) $x(z - \alpha z) = x(\beta z - \alpha\beta z)$

Multiplying by (y+1) on the left we get

$(y+1)\, x\, (z - \alpha z) = (y+1)\, x\, (\beta z - \alpha\beta z)$

$(y+1)(xz - \alpha xz) = \beta(y+1)xz - \alpha\beta(y+1)xz$

(ie) $(y+1)(x - \alpha x)z = \beta x(y+1)z - \alpha x(y+1)z$ (using (2))

$[yx - \alpha yx + x - \alpha x]\, z = [xy + x - \alpha xy - \alpha x]z$

(ie) $[yx - xy + x - x]z = [\alpha yx + \alpha x - \alpha xy - \alpha x]z$

(ie) $yxz - xyz = \alpha yxz - \alpha xyz$

$\alpha xyz - xyz = \alpha.\, \alpha xyz - \alpha xyz$

(ie) $xyz - 2\alpha xyz + \alpha^2 xyz = 0$

(ie) $x(y - 2\alpha y + \alpha^2 y)z = 0$ → (5)

Now

$(xyz - yxz)^2 = (xyz - \alpha xyz)^2$

$= (xyz - \alpha xyz)(xyz - \alpha xyz)$

$= xyzxyz - \alpha xyzxyz - \alpha xyz + \alpha^2 xyzxyz$

$= x\,(y - 2\alpha y + \alpha^2 y)zxyz$

$= 0\ xyz$ (using(5))

Thus $(xyz - yxz)^2 = 0.$

(ie) Square of every quasi-weak commutator is zero.

3.9 Definition:

Let R be a P.I.D (Principal ideal domain) and A be an algebra over R. Let $a\epsilon$R. Then the order of a, denoted as O(a) is defined to be the generator of the ideal I = $\{\alpha\epsilon R\ /\ \alpha a=0\}$. O(a) is unique upto associates and O(a)=1 if and only if a=0.

3.10 Lemma:

Let A be an algebra with identity over P.I.D. If A is scalar quasi-weak commutative, yϵR with O(y) = 0, then y is in the center of A.

Proof:

Let yϵR with O(y) = 0.

For every xϵA, there exists scalars $\alpha = \alpha(x,y,1)\epsilon R$ and $\beta = \beta(y,x+1,1)\epsilon R$ such that

$x\, y.1 = \alpha\, yx.1$

$xy = \alpha\, yx$ → (1)

and $y(x+1) = \beta(x+1)y.1$

(ie) $y(x+1) = \beta(x+1)y$ → (2)

From (2) we get

$yx + x = \beta xy + \beta y$ (using(1))

$yx + x - \beta xy - \beta y = 0$

$y\,(x + 1 - \alpha\beta x - \beta.1).1 = 0$

Using Lemma 3.3 we get

$(x + 1 - \alpha\beta x - \beta.1)y.1 = 0$

$xy + y - \alpha\beta xy - \beta y = 0$ → (3)

Also from (2) we get

$yx + y - \beta xy - \beta y = 0$

Multiply by α

$\alpha yx + \alpha y - \alpha\beta xy - \alpha\beta y = 0$

$xy + \alpha y - \alpha\beta xy - \alpha\beta y = 0$ (using(1)) → (4)

From (3) and (4) we get

$y - \beta y - \alpha y + \alpha\beta y = 0$

$(y - \beta y) - \alpha(y - \beta) = 0$

$(1 - \alpha)(1 - \beta)\, y = 0$

Since O(y) = 0 we get $\alpha = 1$ or $\beta = 1$

If $\alpha = 1$, from (1) we get xy = yx

If $\beta = 1$ from (2) we get $y(x+1) = (x+1)y$

$yx+y = xy + y$

$yx = xy$

(ie) y commutes with x.
As $x\epsilon A$ is arbitrary, y is in the center of A.

3.11 Lemma:

Let A be an algebra with unity over a principal ideal domain R. If A is scalar quasi-weak commutative, $x\epsilon A$ such that O(xz) = 0, then xyz = yxz for all $y,z\epsilon A$.

Proof:

Let $x\ \epsilon A$ with O(xz) = 0
For $y,z\ \epsilon A$, there exists scalars $\alpha = \alpha(y,x,z)\epsilon R$ and $\beta = \beta(x,y+1,z)\epsilon R$ such that

$$yxz = \alpha xyz \quad \rightarrow (1)$$

$$(y+1)xz = \beta x(y+1)z \quad \rightarrow (2)$$

From (2) we get

$$yxz + xz = \beta xz + \beta xz \quad \rightarrow (3)$$

$\alpha xyz + xz - \beta xyz - \beta xz = 0$ (using(1))
$x(\alpha y + x - \beta y - \beta .1)z = 0$
By Lemma 3.3, we get
$(\alpha y + x - \beta y - \beta .1)\ xz = 0$
(ie) $\alpha yxz + x^2z - \beta yxz - \beta xz = 0$ → (4)
Multiplying (3) by α, we get
$\alpha yxz + \alpha xz - \alpha\beta xz - \alpha\beta xz = 0$ → (5)
(4) – (5) gives
$x^2z - \beta xz - \alpha xz + \alpha\beta xz = 0$
$(1 - \alpha - \beta + \alpha\beta)xz = 0$
$(1 - \alpha)(1 - \beta)xz = 0$ → (6)
Since O(xz) = 0, we get $1-\alpha = 0$ or $1-\beta = 0$.
That is $\alpha = 1$ or $\beta = 1$.
If $\alpha = 1$, then from (1) we get yxz = xyz.
If $\beta = 1$, then from (3) we get
yxz + xz = xyz + xz
(ie) yxz = xyz.

3.12 Lemma:

Let A be an algebra with identity over a P.I.D R.Suppose that A is scalar quasi-weak commutative.Assume further that there exists a prime p ∈ R and a positive integer m ∈ Z^+such that p^m A = 0.Then A is quasi-weak commutative.

Proof:

Let x,y ∈ A such that O(yx) = p^k for some k ∈ Z^+.We prove by induction on k that xyu = yxu for all ∈ R.
If k = 0,then O(yx) = p^0 = 1 and so yx = 0.
So yxu = 0.By Lemma 3.3 xyu =0.
Hence xyu = yxu for all u ∈ A.So,assume that k>0 and that the statement is true for all l< k.
We first prove that for any u ∈ A,

$$xyu - yxu \neq 0 \text{ implies}$$
$$y(xu)w - (xu)yw = 0 \text{ for all } w \in A.$$

So,let xyu – yxu ≠ 0.Since A is scalar quasi-weak commutative,there exists scalars
$\alpha = \alpha$ (x,y,u) and
$\beta = \beta$ (x,y+1,u) such that
$xyu = \alpha\ yxu$ → (1)
and
$x(y+1)u = \beta\ (y+1)xu$ → (2)
From (2) we get
$xyu + xu = \beta\ yxu + \beta\ xu$
$\alpha\ yxu + xu = \beta\ yxu + \beta\ xu$ (using (1))
$(\alpha - \beta)\ yxu = (\beta - 1)\ xu$ → (3)
If $(\alpha - \beta)$ yxu = 0,then $(\beta - 1)$ xu = 0 and so $\beta xu = xu$ → (4)
So from (2) we get
x (y+1) u = (y+1) β xu
= (y+1) xu
xyu + xu = yx + xu
i.e., xyu – yxu = 0,contradicting our assumption that xyu ≠ yxu.

So $(\alpha - \beta)$ yxu $\neq$ 0.Inparticular $\alpha - \beta \neq 0$.
Let $(\alpha - \beta) = p^t \delta$ →(5)
for some t ∈ Z^+ and δ ∈ R with (δ , p) = 1. If t ≥k, then since
O(yx) = p^k , we would get ($\alpha - \beta$) yxu = 0, a contradiction.Hence t < k.
Since p^k yxu = 0,by Lemma 3.3, p^kxyu = 0.
From (3) we get

$$p^{k-t}(\beta\text{-}1)xu = p^{k-t}(\alpha - \beta)\,yxu$$
$$= p^{k-t}p^t\,\delta\,yxu$$
$$= p^k\,\delta\,yxu = 0.$$

Let O(xu) = p^i.If i < k,then by induction hypothesis xyu = yxu contradicting our assumption.
So i ≥ k .
Hence $p^k \mid p^i \mid p^{k-t}(- 1)$.
Thus $p^t \mid \beta - 1$ and let $\beta - 1 = p^t \gamma$ → (6) for some γ ∈ R.
From (3) we get

$$(\alpha\ -\ )\,yxu = (\beta - 1)\,xu$$
$$p^t\,\delta\,yxu = p^t\,\gamma\,xu \qquad \text{(using (4) and (6))}$$
$$\therefore\ p^t(\delta\,y - \gamma.1)\,(xu) = 0.$$

Hence by induction hypothesis

$$(\delta y - \gamma.1)\,(xu)\,w = xu\,(\delta y - \gamma.1)w \text{ for all } w \in A.$$
$$\delta\,yxuw - \gamma xuw = \delta\,xuyw - \gamma xuw$$
$$\therefore \delta\,\{y(xu)w - (xu)yw\} = 0 \qquad \rightarrow (7)$$

Since (δ,p) = 1,there exists μ , δ ∈ R such that $\mu\, p^m + \gamma\delta = 1$.

$$\therefore \mu\, p^m\{ y(xu)w - (xu)yw\} + \gamma\delta\,\{ y(xu)w - (xu)yw\} = \{ y(xu)w - (xu)yw \}$$
$$0+0 = \{ y(xu)w - (xu)yw \} \quad (\because\ p^m A = 0)$$

i.e., y(xu)w – (xu)yw = 0
i.e., xyu –y(xu) ≠ 0 implies y(xu)w – (xu)yw = 0 for all w∈A. → (8)

Now we proceed to show that xyu = yxu for all u ∈A.Suppose not there exists u ∈A such that
xyu – y (xu) ≠ 0 → (9)
Then also we have xy (u+1) – yx (u+ 1) ≠ 0 → (10)
From (8) and (9) we get
y (xu)w – (xu) yw = 0 for all w ∈A. → (11)
From (8) and (10) we get
y (x(u + 1)) w – (x(u+1)yw = 0 for all w∈A. → (12)
From (12) we get
y (xu + x) w – (xu + x) yw = 0
y(xu)w + yxw- (xu)yw - xyw = 0
i.e., yxw – xyw = 0 for all w∈A. (using (11))
a contradiction.This contradiction proves that xyu = yxu for all x,y,u ∈ A.
Thus A is quasi-weak commutative.

3.13 Theorem:

Let A be an algebra with identity over a principal ideal domain R.If A is scalar quasi-weak commutative,then A is quasi-weak commutative.

Proof:

Suppose A is not quasi-weak commutative ,there exists x∈A such that xyz ≠ yxz
for all y , z ∈ A.
Also (x+1) yz ≠ y(x+1)z.
By Lemma 3.10 O(x) ≠ 0 and O(x+1) ≠ 0.
Hence O(1) ≠ 0.Let O(1) = d ≠ 0.Then d is not a unit and hence
d = $p_1^{i_1}\, p_2^{i_2} \ldots\ldots\ldots\ldots p_k^{i_k}$ for some primes $p_1, p_2 \ldots\ldots\ldots\ldots p_k$ ∈ A and some positive integers $i_1, i_2 \ldots\ldots\ldots\ldots\ldots i_k$.
Let A_j = { a ∈ A | $p_j^{i_j}$ a = 0}.Then each A_j is a non-zero sub – algebra of A and
A = $A_1 \oplus A_2 \oplus_{\ldots\ldots\ldots} \oplus A_k$. Being sub algebras of A, each A_j is scalar quasi-weak commutative.
Being homomorphic image of A, all the A_j 's have identity element 1.
By Lemma 3.12 each A_j is quasi-weak commutative and hence A is quasi-weak commutative, a contradiction. This contradiction proves that A is quasi-weak commutative.

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G.Gopalakrishnamoorthy "On Scalar Quasi - Weak Commutative Algebras. "IOSR Journal of Mathematics (IOSR-JM) 14.3 (2018): 30-37.