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A Toolbox of Commutative Bi-Complex Algebra

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Abstract : *Hyper-complex numbers forming four-dimensional quaternion-scalar space are considered. Corresponding complementary algebra is introduced as an additional non-vector extension over the field of complex numbers. Similarly to conventional complex numbers this commutative 4th-rank algebra possesses division, conjugation, rooting and factorization along with direct analog of Euler's formula. Rotations can be represented consistently within this algebra as well. Some of direct applications include electromagnetic wave theory, beam and accelerator physics.*

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I. Introduction

Conventional quaternion was introduced as a four-vector extension over the field of complex numbers [1,2]. Its vector part is a generalization of the imaginary part and forms three-dimensional quaternion-vector space. Although quaternions do not belong to the standard mathematical apparatus, they found numerous applications in computational mathematics and many branches of physics including special relativity and optics, elementary particle theories and astrophysics, theory of fields and mechanics. For example, in the physics of charged particle beams the quaternions are very efficient to address the problem of spin transport [3,4].

In a number of analytical studies we deal with complex numbers or functions mixed with 2×2 matrixes on some intermediate stage. Matrix representation may have alternatives in some cases. For example, quantum mechanics can be elegantly formulated with geometric algebra. In other situations it is more convenient to proceed with completely scalar expressions rather than vector-like quaternions. Those situations may contain functions of complex variables as well. Corresponding practical cases include, e.g., eigenmode analysis of some boundary problems [5,6], charged particle beam transport and dynamics in accelerators [7]; and vacuum electron devices [8,9] as well. Scalar quaternions are designed originally to encapsulate the matrix transformations into a new space of pseudoscalar numbers extending conventional independent spaces of complex numbers (associated, e.g., with the time and space domains correspondingly).

Lewin [6] was one of the first who applied in fact scalar hyper-complex values to analyze electromagnetic waves propagating in various waveguide structures: dielectric-loaded, with magnetized ferrite, surface anisotropy and corrugations. He introduced phenomenologically an additional imaginary unit (see (1.1)) to distinguish complex numbers reflecting behavioral difference of time (or/and longitudinal) variable - from one hand, and the space (or transversal/angular) variables – on the other hand. Corresponding imaginary units form a commutative group:

$$i^2 = -1,\ j^2 = -1,\ \ ij = ji \neq -1 \ \ or \ \ \sqrt{-1}\ . \tag{1.1}$$

This approach led Lewin to a compact scalar dispersion equation for normal modes with four-component complex numbers. Further development and application of this approach in [10,11] allowed to characterize rigorously a self-consistent system composed by beam interacting with slow-wave structure and solenoidal field. It was shown [10] that conventional matrix approach gives equivalent solution of dispersion system of equations resulted finally in exactly the same increments and threshold currents of the regenerative Beam Break-Up (BBU) instability. However, scalar-quaternion representation is much more compact and produces very transparent physical solution. For example, the collective frequency $\tilde{\nu}$ found from a single hyper-complex dispersion equation has the following meaningful components: $\mathrm{Re}_i\,\mathrm{Re}_j\,\tilde{\nu}$ is the detuning of the collective frequency with respect to modal eigen-frequency, $\mathrm{Im}_i\,\mathrm{Im}_j\,\tilde{\nu}$ is the angular velocity of rotation of the self-consistent degenerated dipole mode coupled with the propagating beam, and $\mathrm{Im}_i\,\mathrm{Re}_j\,\tilde{\nu} \pm \mathrm{Im}_j\,\mathrm{Re}_i\,\tilde{\nu}$ are the "left-hand" and "right-hand" increments of the gyromagnetic BBU effect.

In the preceding work [7] the lack of an additional imaginary unit introduced correctly gave improper mixing between different degrees of freedom and leaded to erroneous result of the threshold current in the presence of transverse motion.

Originally the commutative algebra of corresponding hyper-complex numbers was introduced in [10] for accelerator and beam physics applications. It was defined as a closed generalization over two different fields of complex numbers i and j which forms a commutative 4th-rank linear algebra with division and main attributes of conventional complex numbers. In this paper we present basic properties and the simplest analytical continuation. Such terms as “four-component number”, “hyper-complex number” and “scalar quaternion” we use here as being equivalent.

II. Basics of commutative algebra of four-component complex numbers

We start from writing the four-component complex number that looks like a conventional quaternion:

$$\tilde{a} = \alpha_0 + i\alpha_1 + j\alpha_2 + ij\alpha_3, \qquad (2.1)$$

where the components $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are real numbers; i, j are the independent imaginary units, and ij is the composite imaginary unit.

We consider here commutative algebra of hyper-complex numbers (1.1, 2.1) as being additive, associative, distributive, and closed with respect to addition, multiplication, division.

Particularly, a product of any two complex numbers from different i - and j - spaces forms a scalar quaternion:

$$(a+ib)\cdot(c+jd) = \alpha_0 + i\alpha_1 + j\alpha_2 + ij\alpha_3, \text{ where } \alpha_0 = ac, \alpha_1 = bc, \alpha_2 = ad, \alpha_3 = bd. \qquad (2.2)$$

So we treat here independent spaces of regular complex numbers as projections of hyper-complex space. Therefore we can smoothly redefine operators of real and imaginary parts:

$$\mathrm{Re}_i\,\tilde{a} = \alpha_0 + j\alpha_2, \ \mathrm{Im}_i\,\tilde{a} = \alpha_1 + j\alpha_3, \qquad (2.3)$$

where the indexes denote that corresponding operation to be made only in the space-projection of complex numbers associated with i or j correspondingly.

Let us assume the Pauli matrixes $\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\hat{\sigma}_2 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}$, $\hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as operators assigned, for example, to the j - space only. Then, considering $\tilde{a}$ as a column matrix $\begin{pmatrix} \mathrm{Re}_j\,\tilde{a} \\ \mathrm{Im}_j\,\tilde{a} \end{pmatrix}$, we can use the following substitutions in transition to our pseudo-scalar space:

$$\hat{\sigma}_1\tilde{a} \to j\tilde{a}^{*j}, \ \ \hat{\sigma}_2\tilde{a} \to -\tilde{a}, \ \text{ and } \ \hat{\sigma}_3\tilde{a} \to \tilde{a}^{*j}, \qquad (2.4)$$

i.e. the matrix operators can be represented formally as $\hat{\sigma}_1 \to j()^{*j}$, $\hat{\sigma}_2 \to -1$, and $\hat{\sigma}_3 \to ()^{*j}$.

Similarly to the algebra of Pauli spin matrixes we have from (2.4):

$$\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3\hat{\sigma}_3\tilde{a} = \hat{\sigma}_1\hat{\sigma}_2\tilde{a} = -j\hat{\sigma}_3\tilde{a}; \quad \hat{\sigma}_2\hat{\sigma}_3\tilde{a} = -j\hat{\sigma}_1\tilde{a}; \quad \hat{\sigma}_3\hat{\sigma}_1\tilde{a} = -j\hat{\sigma}_2\tilde{a}.$$

However, unlike the geometric algebra the operators (2.4) are commutative in the pseudo-scalar space. So, an arbitrary 2×2 matrix operator $\hat{U}$ belonging to the j - space can be represented, for example, as:

$$\hat{U} \equiv \rho\hat{E} - j(\lambda\hat{\sigma}_1 + \mu\hat{\sigma}_2 + \nu\hat{\sigma}_3) \to \rho + j\mu + (\lambda - j\nu)()^{*J},$$

where $\hat{E}$ is the 2×2 unit matrix, $\rho^2 + \lambda^2 + \mu^2 + \nu^2 = \det\hat{U}$, and ρ, λ, μ, ν are the real numbers describing the operator $\hat{U}$ associated with the j - space.

To generalize of the 2×2 matrix operator and rotation representation in the i, j - hyperspace we can, e.g., replace formally the complex unit j by i in $\hat{U}$ and $\hat{\sigma}_2$ (i.e. $\hat{\sigma}_2 \to ij$) representations as follows:

$$\hat{U} = \rho\hat{E} - i(\lambda\hat{\sigma}_1 + \mu\hat{\sigma}_2 + \nu\hat{\sigma}_3) \to \rho + j\mu - (ij\lambda + i\nu)()^{*J}, \qquad (2.5)$$

where $\hat{U}$ belongs to both i - and j - spaces simultaneously.

If $\hat{U}$ is the unimodular matrix and $\rho^2 + \lambda^2 + \mu^2 + \nu^2 = 1$, Eqn. (2.5) represents rotations in 4D i, j - space.

Before considering the full length in the 4D space let us define partial determinant in each of these spaces-projections:

$$\det_i \tilde{a} = (\mathrm{Re}_i \tilde{a})^2 + (\mathrm{Im}_i \tilde{a})^2 = \tilde{a} \cdot \tilde{a}^{i*} \equiv |\tilde{a}|_i^2 = \alpha_0^2 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2 + 2j(\alpha_0\alpha_2 + \alpha_1\alpha_3). \qquad (2.6)$$

The commutation law (1.1) and definitions (2.1, 2.3, 2.6) give the following obvious identities:

$$\begin{aligned}
&\tilde{a} \cdot \tilde{b} = \tilde{b} \cdot \tilde{a}, \\
&\mathrm{Re}_i\, \mathrm{Re}_j\, \tilde{a} = \mathrm{Re}_j\, \mathrm{Re}_i\, \tilde{a} = \alpha_0 \equiv \mathrm{Re}_{ij}\, \tilde{a} = \mathrm{Re}_{ji}\, \tilde{a} \equiv \mathrm{Re}\,\tilde{a}, \\
&\mathrm{Im}_i\, \mathrm{Re}_j\, \tilde{a} = \mathrm{Re}_j\, \mathrm{Im}_i\, \tilde{a} = \alpha_1, \\
&\mathrm{Im}_i\, \mathrm{Im}_j\, \tilde{a} = \mathrm{Im}_j\, \mathrm{Im}_i\, \tilde{a} = \alpha_3 \equiv \mathrm{Im}_{ij}\, \tilde{a} = \mathrm{Im}_{ji}\, \tilde{a} \equiv \mathrm{Im}\,\tilde{a}, \\
&(\tilde{a}^{*i})^{*j} \equiv \tilde{a}^{*i*j} = \tilde{a}^{*j*i} \equiv (\tilde{a}^{*j})^{*i} = \alpha_0 - i\alpha_1 - j\alpha_2 + ij\alpha_3, \\
&\tilde{a} + \tilde{a}^{*i} = 2\,\mathrm{Re}_i\, \tilde{a}, \quad \tilde{a} - \tilde{a}^{*i} = 2i\, \mathrm{Im}_i\, \mathrm{Re}_j\, \tilde{a} + 2j\, \mathrm{Re}_i\, \mathrm{Im}_j\, \mathrm{Re}_i\, \tilde{a}, \\
&(\tilde{a} + \tilde{a}^{*i*j}) + C.C._i = (\tilde{a} + \tilde{a}^{*i*j}) + C.C._j = 4\,\mathrm{Re}_i\, \mathrm{Re}_j\, \tilde{a} \equiv 4\,\mathrm{Re}\,\tilde{a}, \\
&\det_i \det_j \tilde{a} \equiv \left| |\tilde{a}|_j^2 \right|_i^2 = \left| |\tilde{a}|_i^2 \right|_j^2 \equiv \det_j \det_i \tilde{a}.
\end{aligned} \qquad (2.7)$$

The rules above describe a plain scalar unification of two superposed fields of regular complex numbers. They can be used to simplify some typical problems by reducing to convenient algebraic form (e.g. in waveguide [6] and wake function [5] theories, polarimetry, and analytical description of periodic magnetostatic fields [12]). However, to form a complete hyper-complex field this linear algebra needs to be closed with respect to operations of multiplication and division, raising to powers and rooting.

To make this next step we postulate additional to (1.1) rules:

$$ij = ji \neq \pm 1, \quad i \cdot ij = -j, \quad ji \cdot j = -i, \quad i^2 j^2 = (ji)^2 = 1. \qquad (2.8)$$

Other properties of scalar quaternion numbers and corresponding functional analytical continuations can be derived from (1.1, 2.8) similarly to known properties of conventional complex numbers. Fore example:

$$1/ij = ij; \quad \sqrt{1} = \pm 1, \pm ij; \quad ij = \exp(\pm (i + j)\pi / 2). \qquad (2.9)$$

Evidently, a square root to be a four-valued in this 4-th rank algebra.

Another example rules the multiplication of hyper-complex numbers $\tilde{a}$ and $\tilde{b} = \beta_0 + i\beta_1 + j\beta_2 + ij\beta_3$:

$$\begin{aligned}
\tilde{a} \cdot \tilde{b} = {} & \alpha_0\beta_0 + \alpha_3\beta_3 - \alpha_1\beta_1 - \alpha_2\beta_2 + i(\alpha_1\beta_0 + \alpha_0\beta_1 - \alpha_3\beta_3 - \alpha_2\beta_3) + \\
& j(\alpha_2\beta_0 + \alpha_0\beta_2 - \alpha_3\beta_1 - \alpha_1\beta_3) + ij(\alpha_3\beta_0 + \alpha_2\beta_1 + \alpha_1\beta_2 + \alpha_0\beta_3)
\end{aligned}$$

III. Conjugation and absolute value, hyper-poles and division

One can define a full conjugate as an extension of partial conjugates:

$$\tilde{a}^* = \tilde{a}^{*i}\tilde{a}^{*j}\tilde{a}^{*i*j}. \qquad (3.1)$$

A few additional identities and inequalities for conjugation may be useful:

$$\begin{aligned}
&\tilde{a} + \tilde{a}^{*i}\tilde{a}^{*j} + \tilde{a}^{*i*j} = 4\,\mathrm{Re}_i\, \mathrm{Re}_j\, \tilde{a}, \\
&\tilde{a}^{*i*j}\tilde{a} = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + 2ij(\alpha_3\alpha_3 - \alpha_1\alpha_2), \\
&\tilde{a} + \tilde{a}^{*i*j} \neq 2\,\mathrm{Re}\,\tilde{a}, \quad \tilde{a} + \tilde{a}^* \neq 2\,\mathrm{Re}\,\tilde{a}.
\end{aligned} \qquad (3.2)$$

A natural way to define the full determinant is to use the partial determinants (2.6):

$$\det \tilde{a} = \det_i \det_j \tilde{a} \equiv \left| |\tilde{a}|_j^2 \right|_i^2 \equiv \tilde{a}\tilde{a}^{*i}\tilde{a}^{*j}\tilde{a}^{*i*j} = \tilde{a} \cdot \tilde{a}^* = (\alpha_0^2 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2)^2 + 4(\alpha_0\alpha_2 + \alpha_1\alpha_3)^2. \qquad (3.3)$$

One can see from (3.3), that for some non-zero components α_n the determinant turns to zero. We will call such numbers as polar numbers, hyper-poles, or hyper-zeros. We have such poles, e.g., when $|\alpha_0| = |\alpha_3| \neq 0$ at $\alpha_1 = \alpha_2 = 0$ or when $|\alpha_1| = |\alpha_2| \neq 0$ at $\alpha_0 = \alpha_3 = 0$.

Unlike partial determinants the full determinant is a real non-negative number. Therefore we define an absolute value (or scalar quaternion norm) using a 4th order arithmetic root:

$$|\tilde{a}| \equiv N(\tilde{a}) = \sqrt[4]{\det \tilde{a}} \equiv \sqrt[4]{\tilde{a}\tilde{a}^{*i}\tilde{a}^{*j}\tilde{a}^{*i*j}} = \sqrt[4]{\left| |\tilde{a}|_j^2 \right|_i^2}. \qquad (3.4)$$

Note polar numbers $i \pm j,\ 1 \pm ij$ have zero norm (i.e. absolute value or hyper-length). As we shall see below, the numbers $2\pi(i \pm j)$ and $\pi(i \pm j)$ are the hyper-periods for the hyperbolic functions $\cosh(x), \sinh(x)$ and $\tanh(x), \mathrm{cotanh}(x)$ as well as the numbers $2\pi(1 \pm ij)$ and $\pi(1 \pm ij)$ are the hyper-periods of the trigonometric functions $\cos(x), \sin(x)$ and $\tan(x), \mathrm{cotan}(x)$ respectively.

The full determinant and conjugation defined above can be used directly to find the inversed number provided the determinant is not zero:

$$\tilde{a}^{-1} \equiv \frac{1}{\tilde{a}} = \frac{\tilde{a}^*}{\det \tilde{a}}. \tag{3.5}$$

One can obtain (3.5) using partial transformations in the spaces-projections applying successively corresponding rules for regular complex numbers:

$$\frac{1}{\tilde{a}} = \frac{\tilde{a}^{*j}}{|\tilde{a}|_j^2} \equiv \frac{\tilde{a}^{*i}}{|\tilde{a}|_i^2} \equiv \frac{\tilde{a}^{*i}}{\tilde{a} \cdot \tilde{a}^{*i}} = \frac{\tilde{a}^{*i} \cdot \left(\tilde{a}\tilde{a}^{*i}\right)^{*j}}{\tilde{a}\tilde{a}^{*i} \cdot \left(\tilde{a}\tilde{a}^{*i}\right)^{*j}} = \frac{\tilde{a}^{*i}\tilde{a}^{*j}\tilde{a}^{*i*j}}{\tilde{a}\tilde{a}^{*i}\tilde{a}^{*j}\tilde{a}^{*i*j}} = \frac{\tilde{a}^*}{|\tilde{a}|^4}..$$

Inverted hyper-zeros can be interpreted as hyper-infinities within this ij-algebra.

IV. Euler's formula, factorization and rooting

Before defining rooting for general scalar quaternion let us consider two partial cases.
The first case is a scalar quaternion (2.2) represented by a product of two complex numbers $a + ib$ and $c + jd$ belonging to i - and j - spaces correspondingly. It means that $\alpha_0\alpha_3 = \alpha_1\alpha_2$, i.e. corresponding 2×2 matrix composed by its components $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ is degenerated.

Interesting to note, this trivial case corresponds to a 2×2 matrix operator (or rotation) applied to a "flat" vector (a conventional complex number) belonging to the i - space ($\mathrm{Im}_j = 0$). Indeed, from (3.3, 2.2) we have

$|\tilde{a}| = \sqrt{\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2} = \sqrt{a^2c^2 + b^2c^2 + a^2d^2 + b^2d^2}$ and from (2.5) $\hat{U} \to \rho - i\nu + j\mu - ij\lambda$ assuming the quaternion $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$ is proportional to $\{\rho, -\nu, \mu, -\lambda\}$.

Thus in this particular case the root of the n-th -order yields:

$$\sqrt[n]{\tilde{a}} = \sqrt[n]{(a + ib)\cdot(c + jd)} = \sqrt[n]{|\tilde{a}|}\exp\left[\left(i\arctan b/a + j\arctan d/c + 2\pi(ki + lj)\right)/n\right], \tag{4.1}$$

where $k, l = \{0,1..., n-1\}$ are the integers.

So the period of the exponential function is $2\pi(ki + lj)$ in our hyperspace. It gives in general n^2 values for $\sqrt[n]{\tilde{a}}$.

Another case of interest is a hyper-complex number having only two components: $\tilde{A} = a + ijd$. From (2.8) and Taylor's expansion we can write the basic formula of exponential representation for this simplest hyper-complex number:

$$\exp(ij\varphi) = \cosh\varphi + ij\sinh\varphi. \tag{4.2}$$

Then the following representation takes place at $|d/a| \neq 1$:

$$\tilde{A} = a + ijd = |a^2 - d^2|\exp\left(ij\,\mathrm{arctanh}\frac{d}{a}\right). \tag{4.3}$$

Note $\mathrm{arctanh}\, d/a$ is a real number at $|d/a| < 1$, otherwise it turns into a complex number in either i - or j - space. However, at $|d/a| > 1$ an additional, "symmetric", representation is possible in the i, j -space:

$$\tilde{A} = ij(d + ija) = ij|a^2 - d^2|\exp\left(ij\,\mathrm{arctanh}\frac{a}{d}\right) = |a^2 - d^2|\exp\left((i + j)\frac{\pi}{2} + ij\,\mathrm{arctanh}\frac{a}{d}\right). \tag{4.4}$$

We used the following formal substitution made in (4.4):

$$\mathrm{arctanh}\,x \xrightarrow[|x|>1]{} -(i + j)\frac{\pi}{2} + \mathrm{arctanh}\frac{1}{x}. \tag{4.5}$$

It gives a hyper-extension of corresponding trigonometric formula $\arctan x = \frac{\pi}{2}\operatorname{sgn} x - \arctan\frac{1}{x}$. Thus the inverse hyperbolic tangent is extended into the space of scalar quaternions.

Using (4.3) we can write the rooting formula for the simple two-component scalar quaternion $\tilde{B} = \tilde{A}^2 = b + ijc$:

$$\sqrt[n]{\tilde{B}} = \sqrt[n]{\left|\tilde{B}\right|}\exp\left(\frac{2\pi(ki+lj)}{n} + \frac{1}{n}ij\operatorname{arctanh}\left(\frac{c}{b}\right)\right), \tag{4.6}$$

where $|c/b| \neq 1$, and, again, $k, l = \{0,1..., n-1\}$.

Now suppose $\tilde{B} = \tilde{A}^2$ just to compare $\tilde{A} \equiv a + ijd$ and $\sqrt{\tilde{B}}$. Substituting in (4.6) $b = a^2 + d^2$ and $c = 2ad$ we have:

$$\sqrt{\tilde{B}} = \pm\left|a^2 - d^2\right| \cdot \left(\cosh\frac{\varphi}{2} + ij\sinh\frac{\varphi}{2}\right), \quad \text{where } \varphi = \operatorname{arctanh}\left(\frac{2ad}{a^2 + d^2}\right). \tag{4.7}$$

Simple transformations of hyperbolic functions in (4.7) give the following:

$$\sqrt{(a+ijd)^2} = \begin{pmatrix} \pm a \pm ijd \\ \pm d \pm ija \end{pmatrix}, \tag{4.8}$$

where the sign alternations are not correlated leading to eight values of the square root $\sqrt{\tilde{B}}$. However, only four of them are linearly independent (see (2.9)), whereas the other four can be produced with multiplication by ij.

In general case at $\left|\tilde{A}\right| \neq 0$ we can generalize the Euler's formula as follows:

$$\tilde{A} \equiv a + ib + jc + ijd = \exp(\alpha_0 + i\alpha_1 + j\alpha_2 + ij\alpha_3) \equiv \exp(\tilde{a}), \tag{4.9}$$

where the relationship between $\tilde{A}$ and $\tilde{a}$ can be found from the system:

$\alpha_0 = \ln\left|\tilde{A}\right|$, and

$$\begin{cases} b_N = \sin\alpha_1\cos\alpha_2\cosh\alpha_3 - \cos\alpha_1\sin\alpha_2\sinh\alpha_3 \\ c_N = \cos\alpha_1\sin\alpha_2\cosh\alpha_3 - \sin\alpha_1\cos\alpha_2\sinh\alpha_3 \\ d_N = \sin\alpha_1\sin\alpha_2\cosh\alpha_3 \end{cases} \tag{4.10}$$

where $b_N = b\big/\left|\tilde{A}\right|$, $c_N = c\big/\left|\tilde{A}\right|$, and $d_N = d\big/\left|\tilde{A}\right|$ are the normalized components.

Similar to 3D rotation represented by a conventional quaternion [1], (4.9-4.10) represent rotation $\alpha_1, \alpha_2, \alpha_3$ in pseudo-scalar hyperspace. The degenerate case (2.2, 4.1) can be interpreted as analog of Cardano' case (when $\alpha_3 = 0$ in (4.9)).

Note, unlike conventional complex numbers and the cases (2.2), (2.5), (4.1), the normalized real components b_N, c_N, d_N can vary from $-\infty$ to $+\infty$ in general case.

One can reduce (4.10) to an algebraic system of two unknowns $\tan\alpha_1$ and $\tan\alpha_2$:

$$\begin{cases} \tan^2\alpha_1 - \tan^2\alpha_2 = \left(b_N^2 - c_N^2\right)\left(1 + \tan^2\alpha_1\right)\left(1 + \tan^2\alpha_2\right) \\ b_N\tan^2\alpha_1\tan\alpha_2 + c_N\tan\alpha_1\tan^2\alpha_2 = d_N\left(\tan^2\alpha_1 - \tan^2\alpha_2\right) \end{cases} \tag{4.11}$$

and

$$\alpha_3 = \ln\left(\sin(\alpha_1 + \alpha_2)/(c_N + b_N)\right). \tag{4.12}$$

The system (4.11) can be solved explicitly, but the expressions we obtained with symbolic methods are too lengthy for analysis in this paper.

To provide $\sin(\alpha_1+\alpha_2)/(c_N+b_N)>0$ in (4.12) one can always choose suitable solutions of (4.11) in the form of $\alpha_{1,2}+\pi m$ due to periodicity of tangent. At $b=-c$ formally we have a singularity in (4.12). However, this singularity is avoidable by performing representation (4.9-4.12) for conjugate complex number (in i - or j - space) and applying conjugation again (in the same space) to the result in the right part.

Another way to factorize a scalar quaternion for rooting is to generalize (2.2) as a product of the three principal multiplicands:

$$\alpha_0+i\alpha_1+j\alpha_2+ij\alpha_3=(a+ib)\cdot(c+jd)\cdot(e+ijf). \qquad (4.13)$$

Let us assume for simplicity $\alpha_0=1=a=c=e$. Then (4.13) reduces to the following conventional algebraic system:

$$\begin{cases}\alpha_3=bd(1-bdf)+f\\ \alpha_2=d(1-bdf)-bf\\ \alpha_1=b(1-bdf)-df\end{cases} \qquad (4.14)$$

The solutions $\{b,d,f\}$ of the system (4.14) have much more compact explicit form compared to (4.11). One can show easily that the solutions are always real. Similar to (4.12) for one of the solutions one can find a singularity (e.g. at $\alpha_2+\alpha_1\alpha_3=0$) that is removable.

Thus scalar non-zero ($|\tilde{A}|\neq 0$) quaternion allows to perform factorization and rooting using (4.19), (4.13) or (4.1), (4.6).

V. Discussion

Next considerations of scalar quaternions may include hyper-complex functions, differentiation and integration, conformal mapping and analytical extensions of complex functions with complex variables extended into the algebraic hyperspace introduced here. This pseudo-scalar space allows to represent 2×2 matrixes with the hyper-complex numbers and to combine the properties of conventional complex numbers. One can anticipate further developments and new implications of the ij-algebra, especially in beam, laser, plasma, particle physics and cosmology as well. The possibility of multidimensional hyper-complex numbers of 6th rank and higher remains open.

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