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Matrix Summability of Conjugate and Derived Fourier series

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Abstract: *The object of this paper is to prove that Hankel matrix is almost conservative and hence almost regular. Further, ordinary conjugate and derived Fourier series are both proved to be almost regular Hankel matrix summable.*

Key Words: *Conjugate Fourier series, Derived Fourier series, Conservative matrix, Almost regularity, Hankel Matrix*

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I. Introduction

In an ordinary sense, an infinite matrix is said to be almost-regular if its *A*-transform of *x* is almost convergent to the limit of *x* for each$x \in c$(King, [4]). The sum of a derived Fourier series was found by Rao [5] after applying almost-regular infinite matrix in accordance with King [4]. Also, the concept of almost-regularmatrices was used by Ahmad and Bateineh [1] to achieve*d*-summability of derived series of a Fourier series, while Alotaibi and Mursaleen [3] used regular matrices for summability of conjugate and derived Fourier series in ordinary sense. In Al-Homidan [2] it is proved that Hankel matrix is regular. Furthermore, if we let, in general, $A = a_{nk}$ be an infinite matrix of real or complex entries and$x = (x_k)$ be the sequence of real or complex terms. Then the sequence$n = (\eta_k)$ defined by

$$\eta_n = \sum_{k=0}^{\infty} a_{nk} x_k \quad (1.1)$$

is called the *A*- transform of *x* whenever the series in (1.1) above converges for each *n* = 0, 1, …. The sequence$x = (x_k)$ is said to be *A*-summable to *y* if the sequence (η_k) converges to y. Let *c* be the linear space of all convergent sequences. The infinite matrix is said to be conservative if $x \in c$also implies that$\eta \in c$, at the same time regularity preserves limit. It is one of the objectives of this study to work with almost regular Hankel matrix which is slightly more general than the conservative or regular matrices. It is also an infinite square matrix with constants on each diagonal orthogonal to the main diagonal. Hankel matrix as a transform of a sequence$x = (x_k)$is the sequence$n = (\eta_k)$defined by

$$\eta_n = \sum_{k=0}^{\infty} h_{n+k} x_k \quad (1.2)$$

provided that the series converges for each *n* = 0, 1, 2, …. Since in Al-Homidan [2] it was proved that Hankel matrix is regular, one of the objectives of this study is to prove that Hankel matrix is conservative and almost regular. Furthermore, almost regularity of Hankel matrix is also used to prove ordinary conjugate and derived Fourier series almost summable. Following Alotaibi and Mursaleen [3] we prove that ordinary conjugate and derived Fourier series are both proved to be almost regular Hankel matrix summable.

II. Background And Notations

To prove next results involving almost regularity conditions, that is to prove that Hankel matrix is almost regular the study has to first of all prove that it is almost conservative and to do this, the results of Lorentz [6] and Al-Homidan [2] were employed. The method of Lorentz was able to establish almost convergence of ordinary sequences, while Al-Homidan characterized regular Hankel matrix.These techniques involved the manipulations of row conditions, column conditions and boundedness of infinite matrices. Finally, we shall quote almost regular Hankelconditions which we prove in order to further prove that the derived Fourier series and conjugate Fourier series are almost regular Hankel summable. This will depend on the theory of functions of bounded variations too.

The results proved in the next section also depend on$\tilde{S}_n(x)$ and $\acute{S}_n(x)$standing for the partial sums of conjugate and derived Fourier series of ordinary Fourier series; and the functions of bounded variations:

$$\psi(x;t) = \psi(f,t) = \begin{cases} f(x+t) - f(x-t); 0 < t \le \pi \\ g(x); \quad t = 0 \end{cases} \quad (2.1)$$

where,

$$g(x) = f(x+0) - f(x-0)$$

with

$$g(x,t) = \frac{\psi(x,t)}{4 \sin 1/2} \qquad (2.2)$$

The following lemmas would be useful in the sequel:

Lemma 2.1 If the space of all continuous functions on the closed interval $[0,\pi]$ denoted by $C[0,\pi]$ is equipped with the sup-norm $\|.\|$ and g_n is contained in $C[0,\pi]$. Then $\int_0^\pi g_n dh_x \to 0$ as $n \longrightarrow \infty$ for all $h_x \in BV[0,\pi]$ if, and only if $\|g_n\|$ is bounded for all n and $g_n \longrightarrow 0$as $n \longrightarrow \infty$.

Lemma 2.2: The trigonometric Fourier series of a 2π periodic function f having bounded variation converges to $\frac{[f(x+0)-f(x-0)]}{2}$ of $g(x)/\pi$ for every x, and this convergence is uniform on every closed interval on which f is continuous.

Lemma 2.3 (Theorem 2, Alotaibi and Mursaleen [3]): Let $f(x)$ be a function integrable in the sense of Lebesgue in [0, 2] with period 2π. Let $a = (a_{nk}) = \{h_{n+k}\}$ be a regular Hankel matrix. Then for each $\beta_x(0+) \in BV[0,2\pi]$ the Hankel matrix transform of the sequence $(\acute{S}_n(x))$ is $\beta_x(0+)$ if, and only if $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} \sin(k+1/2)t = 0$.

Lemma 2.4 (Theorem 3, Alotaibi and Mursaleen [3]): Let $f(x)$ be a function integrable in the sense of Lebesgue in [0, 2] with period 2π. Let $a = (a_{nk}) = \{h_{n+k}\}$ be a regular Hankel matrix. Let $a = (a_{nk}) = \{h_{n+k}\}$ be a regular Hankel matrix. Then the Hankel matrix transform of the sequence $(\tilde{S}_n(x))$ converges to the function $g(x)\pi^{-1}$ if, and only if $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk}\ coskt = 0$.

III. Main Results

In this section we prove that Hankel matrix $H = \{h_{n+k}\}$ is almost conservative and, hence almost regular. Further, we prove that ordinary conjugate and derived Fourier series are almost regular Hankel matrix summable.

Theorem 3.1:A Hankel matrix $H = \{h_{n+k}\}$ is almost conservative if, and only if

(i) $\sup_m \left\{\sum_{k=0}^{\infty} \frac{1}{m} \left|\sum_{j+k=n}^{n+P-1} h_{j+k,k}\right| : m \in \mathbb{Z}^+\right\} < \infty$ for n = 0, 1, 2, … ,

There exists $\gamma_k \in \mathbb{C}$, with k = 0, 1, 2, … such that

(ii) $\lim_{m\longrightarrow\infty} \frac{1}{m} \sum_{j+k=n}^{n+P-1} h_{j+k,k} = \gamma_k$uniformly in n, and there exists $\gamma \in \mathbb{C}$ such that

(iii) $\lim_{m\longrightarrow\infty} \frac{1}{m} \sum_{j+k=n}^{n+P-1} \sum_{k=0}^{\infty} h_{j+k,k} = \gamma$, uniformly in n.

Proof: Assume (i), (ii) and (iii) hold. We must show that H is almost conservative matrix. To do this, fix $n \in \mathbb{Z}$ and take $x \in$ c. Then

$$\iota_{mn}(x) = \frac{1}{m} \sum_{j+k=\sigma(n)}^{n+P-1} \sum_{k=0}^{\infty} h_{j+k,k}\, x_k$$

$= \frac{1}{m} \sum_{k=0}^{\infty} \sum_{j+k=\sigma(n)}^{n+P-1} h_{j+k,k}\, x_k$. So,

$|\iota_{mn}(x)| \le \frac{1}{m} \sum_{k=0}^{\infty} \left|\sum_{j+k=\sigma(n)}^{n+P-1} h_{j+k,k}\right| \|x_k\| \le$

$\|\iota\| \|x\| \le$

$k_n \|x\|$where,$k_n \le \|\iota\|$ (By (i), and k_n is clearly independent of m).

Hence, $\iota_{mn} \in c'$, p = 1, 2, 3, …and the sequence $\{\|\iota_{mn}\|\}$ is bounded for each $n \in \mathbb{Z}^+$. (ii) and (iii) imply that $\lim_m \iota_{mn}(e)$ and $\lim_m \iota_{mn}(e_k)$ exist for each n, k = 0, 1, 2, … . Since the sequence $(e, e_0, e_1, \ldots)$ is a fundamental set in c it follows from that $\lim_{m\to\infty} \iota_{mn}(x) = \iota(x)$ exists and $\iota_n \in c'$.Therefore,ι_n has the form

$\iota_n(x) = \lim x\,[\gamma - \sum_{k=0}^{\infty} \gamma_k] + \sum_{k=0}^{\infty} x_k \gamma_k$,the$F$-limit of $\iota(x)$; where $\gamma = \iota_n(e)$ and $\gamma_k = \iota_n(e_k)$, k = 0, 1, … by (i) and (ii) respectively. Hence, $\lim_{m\to\infty} \iota_{mn}(x) = \iota(x)$ exists for each $x \in c$ and n = 0, 1, 2, …. With

(iv) $\iota_n(x) = \lim x\,[\gamma - \sum_{k=0}^{\infty} \gamma_k] + \sum_{k=0}^{\infty} x_k \gamma_k$.

And, since $\iota_{mn} \in c'$ for each m and n, it has the form

(v) $\iota_{mn}(x) = \lim x\,[\iota_{mn}(e) - \sum_{k=0}^{\infty} \iota_{mn}(e_k)] + \sum_{k=0}^{\infty} x_k \iota_{mn}(e_k)$.

Clearly, (iv) is independent of n. Let this expression be $L(x)$. In order to see that $\lim_{m\to\infty} \iota_{mn}(x) = L(x)$ uniformly in n, set $f_{mn}(x) = \iota_{mn}(x) - L(x)$. Then $f_{mn} \in c'$, $\|f_{mn}\| \le 2\|H\|\ \forall m,n, \lim_{m\to\infty} f_{mn}(e) = 0$ uniformly in n and $\lim_{m\to\infty} f_{mn}(e_k) = 0$ uniformly in n for each k.

Let K be arbitrary positive integer. Then$x = (\lim x)e + \sum_{k=1}^{K} (x_k - limx)e^{(k)} + \sum_{k=K+1}^{\infty} (x_k - limx)e^{(k)}$; and we have

$f_{mn}(x) = (\lim x) f_{mn}(e) + \sum_{k=1}^{K} (x_k - limx) f_{mn}(e^{(k)}) + f_{mn}(\sum_{k=K+1}^{\infty} (x_k - limx)e^{(k)})$.

Now,

$\left|f_{mn}\left[\sum_{k=K+1}^{\infty}(x_k - limx)e^{(k)}\right] \leq 2\|H\|(sup_{k\geq K+1}|x_k - limx|)\right|$, for all *m*, *n*.
By first choosing a fixed k larger enough, each of the three terms of $f_{mn}(x)$ can be made to be uniformly small in absolute value for all sufficiently large m, so that$\lim_{m\to\infty} f_{mn}(x) = 0$uniformly in *n*. This shows that $lim_m \iota_{mn}(x) = L(x)$ uniformly in *n*. Hence $Hx \in f$ for all $x \in c$ and the Hankel matrix, H is almost conservative.

Conversely, suppose H is almost conservative. Now fix $n \in \mathbb{Z}^+$ and let $\iota_{mn}(x) = \sum_{j=\sigma(n)}^{\sigma(n)+P-1} \eta_j(x)$, where $\eta_j(x) = \sum_{k=0}^{\infty} h_{j+k,k} x_k$, so that $\eta_j = \sum_{k=0}^{\infty} h_{j+k,k}$. This means $\eta_j(x) = \frac{1}{p}\sum_{j=\sigma(n)}^{\sigma(n)+P-1}\sum_{k=0}^{\infty} h_{j+k,k} x_k$.

It is clear that, $\eta_j \in c'$, *j* = 0, 1, …. Hence, $\iota_{mn} \in c'$, with $\iota_{mn} = \sum_{j=n}^{n+P-1} \eta_j$ for *m* = 1, 2, …. Since H is almost conservative, $\lim_{p\to\infty} \iota_{mn}(x) = \iota(x)$ uniformly in *n*. This implies that the sequence $(\|\iota_{mn}\|)$ is bounded.

Now, for each $r \in \mathbb{Z}^+$ define the sequence $y = y(m,n)$ by

$$y_k = \begin{cases} \sum_{j=n}^{n+P-1} a_{j+k,k}, & 0 \leq k \leq r \\ 0, & r > k \end{cases}$$

Then,$y \in c, \|y\| = 1$, and also $|\iota_{mn}(y)| = \frac{1}{p}\sum_{k=0}^{r}\left|\sum_{j=n}^{n+P-1} a_{j+k,k}\right|$. Therefore, $\frac{1}{p}\sum_{k=0}^{r}\left|\sum_{j=n}^{\sigma n+P-1} a_{j+k,k}\right| \leq \|\iota_{mn}\|$, so that (i) follows.

Let the sequences e and e_k, k = 0,1,…, be defined by e = (1, 1, 1, …) and e_k = (0, …, 0, …) where the last 1 is in the k^{th} position. Since e and e_k are convergent sequences, k = 0, 1, …,$\lim_{m\to\infty}\iota_{mn}(e)$ and $\lim_{m\to\infty}\iota_{mn}(e_k)$ exists uniformly in *n*. Hence (ii) and (iii) hold. This completes the proof.

Theorem 3.2: The Hankel matrix $H = h_{n+k,k}$ is almost regular if, and only if

(i) $\|H\|$=$sup_n\left\{\sum_{k=0}^{\infty}\frac{1}{m}\left|\sum_{j+k=n}^{n+m-1} h_{j+k,k}\right| : m \in \mathbb{Z}^+\right\}$

(ii) $lim_{m\to\infty}\frac{1}{m}\sum_{j+k=n}^{n+m-1} h_{j+k,k} = 0$, uniformly in *n*, *k* = 0, 1, …, and

(iii) $lim_{m\to\infty}\frac{1}{m}\sum_{j+k=n}^{n+p-1}\sum_{k=0}^{\infty} h_{j+k,k} = 1$, uniformly in *n*.

Proof: Almost regular implies conservative. To see this, let H be almost conservative so that (i) holds by theorem 1. For if $x \in c$, then $f - limit$ of *Hx* is *L*(*x*) which reduces to lim*x*, since $\gamma = 1$ and $\gamma_k = 0$ for each *k*. Hence by (ii) and (iii) regularity *H* is almost regular matrix.
Conversely, let H be almost regular, then $f - \lim He = 1, f - \lim He^{(k)}$ =0 and $\|H\| < \infty$
as in the proof of theorem 3.1. This completes the proof of theorem 3.2.

We have just proved that a Hankel matrix is almost regular if, and only is (i) to (iii) holds.

Theorem 3.3: Let a Hankel matrix $H = (h_{m+n})$ be almost regular infinite matrix of real numbers. Then for every $x \in [= -\pi, \pi]$ for which $g_x(t)$ is of bounded variation on the closed interval $[0, \pi]$. Then *H*-transform of the sequence of partial sums of an ordinary derived Fourier series $\{s'_n(x)\}$ converges to $g_x(0)$, that is,

(a)$lim_{p\to\infty}\frac{1}{p}\sum_{j=0}^{p-1}\gamma'_{m+j}(x) = g_x(0)$, uniformly in m if, and only if

(b)$lim_{p\to\infty}\frac{1}{p}\sum_{j=0}^{p-1}\sum_{n=1}^{\infty} h_{m+j+n,n}\sin\left(n+\frac{1}{2}\right)t = 0 \;\forall\, t \in [0,\pi]$, where$\gamma'_m(x) = \sum_{n=1}^{\infty} h_{m+n}\, s'_n(x)$ and to be more precise, $s'_n(x)$is the partial sums of the derived Fourier series of a function.

Proof:$s'_n(x)$ can be written as

$$s'_n(x) = \frac{1}{\pi}\int_0^{\pi}\psi(x;t)\left(\sum_{k=1}^{n} ksinkt\right)dt$$

$$= -\frac{1}{\pi}\int_0^{\pi}\psi(x;t)\frac{d}{dt}\left(\frac{sin(n+1/2)t}{2\sin(t/2)}\right)dt$$

But, $g(x;t) = \frac{\psi(x;t)}{4sint/2}$.

$\Rightarrow s'_n(x) = I_n + \frac{2}{\pi}\int_0^{\pi}\sin(n+1/2)\, tdg(x;t)$, where

$I_n = \frac{1}{\pi}\int_0^{\pi} g(x;t)\frac{sin(n+1/2)t}{\tan(t/2)}dt$.

Now, $\gamma'_m(x) = \sum_{n=1}^{\infty} h_{m+n}\, s'_n(x)$

$$= \sum_{n=1}^{\infty} h_{m+n} I_n + \frac{2}{\pi}\int_0^{\pi}\sum_{n=1}^{\infty} h_{m+n}\sin(n+1/2)\, tdg(x;t)$$

From (a), we infer:

$$\frac{1}{p}\sum_{j=0}^{p-1}\gamma'_{m+j}(x) = \frac{1}{p}\sum_{j=0}^{p-1}\sum_{n=1}^{\infty} h_{m+j+n}\, s'_n(x)$$

$$=\frac{1}{p}\sum_{j=0}^{p-1}\sum_{n=1}^{\infty} h_{m+j+n,n}\left[I_n+\frac{2}{\pi}\int_0^{\pi}\sin(n+1/2)\,t dg(x;t)\right]$$

$$=\frac{1}{p}\sum_{j=0}^{p-1}\sum_{n=1}^{\infty} h_{m+j+n,n}I_n+\frac{2}{\pi}\int_0^{\pi}\left[\frac{1}{p}\sum_{j=0}^{p-1}\sum_{n=1}^{\infty} h_{m+j+n,n}\sin\left(n+\frac{1}{2}\right)t\right] \times dg(x;t)$$

$$=\frac{1}{p}\sum_{j=0}^{p-1}\sum_{n=1}^{\infty} h_{m+j+n,n}I_n+\frac{2}{\pi}\int_0^{\pi}L_m(t)dg(x;t)$$

$= A \quad + \quad B$, say, and where

$L_m(x)=\frac{1}{p}\sum_{j=0}^{p-1}\sum_{n=1}^{\infty} h_{m+j+n,n}\sin\left(n+\frac{1}{2}\right)t.$

Since, $dg(x;t)$ is of bounded variations on $[0,\pi]$ then by Jordan's convergence theorem for Fourier series (Lemma 2.1) it follows that $\lim_{p\to\infty}\frac{2}{\pi}\int_0^{\pi}L_m(t)dg(x;t)=0$ holds if, and only if $|L_m(x)|\le M,\ \forall\, m$ and $t\in[0,\ \pi]$. Furthermore (b) holds or $B\longrightarrow 0$ uniformly in m as $p\to\infty$.

■

Next we wish to prove a result for ordinary conjugate Fourier series using almost regular Hankel matrix:

Theorem 3.4: Let $g(x)$ be a function integrable in the sense of Lebesgue in $[0,2\pi]$ and periodic with period 2π. Let $H=(h_{m+n})$ be an almost regular Hankel matrix of real numbers. Then the H-transform of the sequence of partial sums of an ordinary conjugate Fourier series $\{n\tilde{s}_n(x)\}$converges to$\frac{g(x)}{\pi}$, that is,

(i)$\lim_{p\to\infty}\frac{1}{p}\sum_{j=0}^{p-1}\tilde{\gamma}_{m+j}(x)=g(x;t)$, uniformly in m, if, and only if

(ii) $\lim_{p\to\infty}\frac{1}{p}\sum_{j=0}^{p-1}\sum_{n=1}^{\infty}h_{m+j+n,n}cosnt=0$for every $t\in t[0,\pi]$, where $\tilde{\gamma}_m(x)= \sum_{n=1}^{\infty}nh_{m+n}\,\tilde{s}_n(x)$.

Proof: $\tilde{s}_n(x)$ can be written as

$$\tilde{s}_n(x)=\frac{1}{\pi}\int_0^{\pi}\psi(x;t)\sin(nt)\,dt$$

$$=\frac{g(x)}{n\pi}+\frac{1}{n\pi}\int_0^{\pi}\cos(nt)\,d\psi(x;t)$$

$$\sum_{n=1}^{\infty}nh_{m+n}\,\tilde{s}_n(x)=\frac{g(x)}{\pi}\sum_{n=1}^{\infty}h_{m+n}+\frac{1}{\pi}\int_0^{\pi}\sum_{n=1}^{\infty}h_{m+n}\cos(nt)\,d\psi(x;t)$$

$$\Rightarrow\quad \tilde{\gamma}_m(x)=\frac{g(x)}{\pi}\sum_{n=1}^{\infty}h_{m+n}+\frac{1}{\pi}\int_0^{\pi}K_n(t)d\psi(x;t)\,, \qquad (3.1)$$

where, $K_n(t)=\sum_{n=1}^{\infty}h_{m+n}\cos(nt)$.

Taking limits as $p\longrightarrow\infty$ on both sides of (3.1) and using (Lemma 2.2) for Fourier series and at the same time applying the almost regularity conditions of H we get the result because,

$$\lim_{p\to\infty}\frac{1}{p}\sum_{n=1}^{\infty}\tilde{\gamma}_m(x)=\frac{g(x)}{\pi}\lim_{p\to\infty}\frac{1}{p}\sum_{n=1}^{\infty}h_{m+n}+\lim_{p\to\infty}\int_0^{\pi}K_n(t)d\psi(x;t)$$

and the result becomes obvious as in the preceding theorem. ■

IV. Conclusion

In conclusion, this paper has shown almost regular Hankel matrix is another summability method applied in matrix summability of ordinary conjugate and derived Fourier series apart from other techniques.

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