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Mathematical Modelling of the Solutions of Partial Differential Equation with Physical Implications

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Abstract: *This work aimed at examining the mathematical modelling of the solutions of partial differential equation with physical implications. In particular, we looked at the consistency and well-posedness of the solution of homogeneous one-dimensional wave. The general solution of the wave equation was derived by the method of change of variable and finally the general solution derived leads us to the unique solution of the problem, called the d'Alembert's formula. We then proof that the d'Alembert's formula obtained exists, it is unique and stable. There after we used the solution obtained to analyze result and show the behaviour of our result in a table and conclude that the notion of a well-posed problem is important in applied mathematics.*

Keyword: *Partial Derivatives; Partial Differential Equation, Well-posedness, Modelling*

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I. Introduction

Generally, finding the exact solution of problems in partial differential equation may be impossible or at least poses practical challenges to obtain it. The study of partial differential equations started as a tool to analyze the models of physical science. The PDE's usually arise from the physical laws such as balancing forces (Newton's law), momentum and conservation laws etc. (Strauss, 2008).

In this work the equation of motion for the string under certain assumption has been derived which is in the form of second order partial differential equation. The governing partial differential equation represents transverse vibration of an elastic string which is known as one dimensional wave equation (King and Billingham, 2000). The analytical solution has been obtained using method of change of variable.

The solution of wave equation was one of the major mathematical problems of the mid eighteenth century. The wave equation was first derived and studied by D'Alembert in 1746. He introduced the one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

This also attracted the attention of Merwin, (2014), Marcel, (2014), Schiesser and Griffths, (2009), and Broman, (1989).

The wave equation was then generalized to two and three dimensions, i.e.

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \Delta u(x,t)$$

Where

$$\Delta = \sum_{i-1}^{3} \frac{\partial^2}{\partial x_i^{\,2}}$$

The solution of wave equation was obtained in several different forms in series of works (Benzoni-Gavage and Serre, 2007). Partial differential equations are often used to construct models of the most basic theories underlying physics and engineering (Lawrence, 2010). Unlike the theory of ordinary differential equations, which relies on the fundamental existence and uniqueness theorem, there is no single theorem which is central to the subject. Instead, there are separate theories used for each of the major types of partial differential equations that commonly arise. It is worth pointing out that the preponderance of differential equations arising in applications, in science, in engineering, and within mathematics itself, are of either first or second order, with the latter being by far the most prevalent.

Wave equation is a second-order linear hyperbolic PDE that describes the propagation of a variety of waves, such as sound or water waves. It arises in different fields such as acoustics, electromagnetics, or fluid dynamics (Sajjadi and Smith, 2008).

Partial differential equation (PDE) contains partial derivative of the independent variable, which is an unknown function in more than one variable. $x, y, t, \ldots$

$$\frac{\partial u}{\partial x} = u_x, \frac{\partial u}{\partial y} = u_y \; \& \; \frac{\partial u}{\partial t} = u_t$$

We can write the general first order PDE for $u(x,t)$ as

$$F[x,t,u(x,t),u(x,t),u(x,t)] = F(x,t,u,u_x,u_t) = 0 \tag{1.0}$$

(Strauss, 2008).

Although one can study PDEs with many independent variables as one wish, but in this research work we will be primarily concerned with PDEs in two independent variables.

So many methods have been used in solving partial differential equation for example:

I. Method of separation of variable,
II. Method of change of variable,
III. Laplace Transform method,
IV. Numerical method and
V. Method of characteristics, etc.

But in this paper, we obtain d'Alembert's general solution by the method of change of variable.

Statement of the problem

In these research work, we are mainly concerned with implications of a solution to a given partial differential equation, and the good properties of that solution. We are interested in the mathematical modelling of the consistency and well-posedness of solution(s) to certain PDEs, in particular the homogeneous one-dimensional wave equation. Various physical quantities will be measured by some function $u = u(x,y,z,t)$

This could depend on all spatial variables and time, or some subset (Guo and Zhang, 2009). The partial derivatives of u will be denoted with the following condensed notation:

$$u_x = \frac{\partial u}{\partial y}, \; u_{xx} = \frac{\partial^2 u}{\partial x^2}, \; u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \; u_{xt} = \frac{\partial^2 u}{\partial x \partial t}, \; u_t = \frac{\partial u}{\partial t} \; \text{etc.}$$

Aim

The aim of this work is to present the mathematical modelling of partial differential equations with implications.

Objectives

1. To obtain a solution, examine the consistency and well-posedness of the solution of the homogeneous one-dimensional wave equation via the method of change of variable
2. To present the result in a tabular form

II. METHODS

We shall consider certain type of problem that is associated with linear hyperbolic partial differential equations. We shall consider this problem in connection with the homogeneous one-dimensional wave equation of the form

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Where *x* and *t* are the independent variables and *C* is a constant.

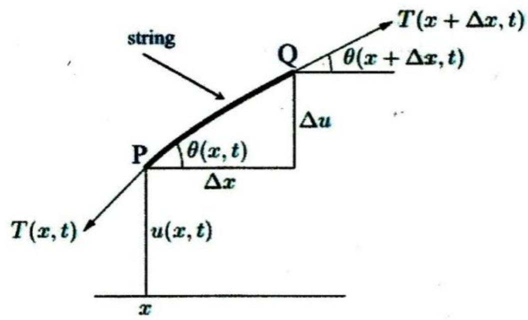
This equation, called the homogeneous one-dimensional wave equation, serve as the prototype for a class of hyperbolic differential equation. Hyperbolic equations arise in physical application as models of waves, such as acoustic, elastic, electromagnetic or gravitational waves. The qualitative properties of hyperbolic PDEs differ sharply from those of parabolic and elliptic PDEs .

The wave equation is certainly one of the most important classical equations of mathematical physics. We look at the mathematical modelling of partial differential equations as applied to the homogeneous one-dimensional wave equation. In particular, we examine the consistency and well-posedness of the solution (Guo and Zhang, 2007). The d'Alembert's general solution is to be derived by the method of change of variable which will lead us finally to the d'Alembert's formula for the solution of the wave equation.

2.1 Mathematical Formulation

The wave equation has many physical applications from sound waves in air to magnetic waves in the Sun's atmosphere. However, the simplest systems to visualize and describe are waves on a stretched elastic string.

Initially the string is horizontal with two fixed ends say a left end L and a right end R: Then from end L we shake the string and we notice a wave Propagate through the string. The aim is to try and determine the vertical Displacement from the $x-axis$ of the string, $u(x,t)$ as a function of position x And time t: That is, $u(x,t)$ is the displacement from the equilibrium at Position x and time t: A displacement of a tiny piece of the string between Points P and Q is shown in;



Where

- $\theta(x,t)$ is the angle between the string and a horizontal line at position x and time t
- $T(x,t)$ is the tension in the string at position x and time t ;
- $\rho(x)$ is the mass density of the string at position x:

To derive the wave equation we need to make some simplifying assumptions:

(1) The density of the string, ρ is constant so that the mass of the string between P and Q is simply ρ times the length of the string between P and Q where the length of the string is Δs given by

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta u)^2} = \Delta x\sqrt{1+\left(\frac{\Delta u}{\Delta x}\right)^2} \approx \Delta x\sqrt{\left(\frac{\partial u}{\partial x}\right)^2}$$

(2) The displacement, $u(x,t)$ and its derivatives are assumed small so that

$$\Delta s \approx \Delta x$$

and the mass of the portion of the string is

$$\rho\Delta x$$

(3) The only forces acting on this portion of the string are the tensions $T(x,t)$ at P and $T(x+\Delta x,t)$ at Q (In physics, tension is the magnitude of the pulling force exerted by a string). The gravitational force is neglected.

(4) Our tiny string element moves only vertically. Then the net horizontal force on it must be zero. Next, we consider the forces acting on the typical string portion shown above. These forces are:

(i) Tension pulling to the right, which has magnitude $T(x+\Delta x,t)$ and acts at an angle $\theta(x+\Delta x,t)$ above the horizontal.

(ii) Tension pulling to the left, which has magnitude $T(x,t)$, and acts at an angle $\theta(x,t)$, above the horizontal.

Now we resolve the forces into their horizontal and vertical components.-

Horizontal: The net horizontal force of the tiny string is

$$T(x+\Delta x,t)\cos\theta(x+\Delta x,t) - T(x,t)\cos\theta(x,t).$$

Since there is no horizontal motion, we must have

$$T(x,t)\cos\theta(x,t) = T(x+\Delta x,t)\cos\theta(x+\Delta x,t) = T.$$

Vertical: At P the tension force is $-T(x,t)\sin\theta(x,t)$ where as at Q the Force is $T(x+\Delta x,t)\sin\theta(x+\Delta x,t)$. Then Newton's Law of motion

Mass acceleration Applied Forces

Gives

$$\rho \Delta x \frac{\partial^2 u}{\partial t^2} = T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t).$$

$$\frac{\rho}{\rho}\Delta x \frac{\partial^2 u}{\partial t^2} = \frac{T(x+\Delta x,t)\sin\theta(x+\Delta x,t)}{T(x+\Delta x,t)\sin\theta(x+\Delta x,t)} = -\frac{T(x,t)\sin\theta(x,t)}{T(x,t)\cos\theta(x,t)}$$

$$\tan\theta(x+\Delta x,t) - \tan\theta(x,t)$$

But

$$\tan\theta(x,t) = \lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x} = u_x(x,t).$$

Likewise,

$$\tan\theta(x+\Delta x,t) = u_x(x+\Delta x,t)$$

Hence, we get

$$\frac{\rho}{T}\Delta x u_{tt}(x,t) = u_x(x+\Delta x,t) - u_x(x,t)$$

Dividing by Δx and letting $\Delta x \to 0$ we obtain

$$\frac{\rho}{T}\Delta x u_{tt}(x,t) = u_{xx}(x,t)$$

Or

$$u_{tt}(x,t) = c^2 u_{xx}(x,t)$$

Where $c^2 = \dfrac{T}{\rho}$,

We call c the wave speed. (http://hyperphysics.phyastr.gsu.edu/hbase/waves/waveq.html)
This is the partial differential equation giving the transverse vibration of the string. It is also called the one dimensional wave equation.

2.2 Homogeneous one-dimensional wave equation

We solve the wave equation on the whole real line $-\infty < x < +\infty$. Real physical situations are usually on finite intervals. We are justified in taking *x* on the whole real line for two reasons. Physically speaking, if you are sitting far away from the boundary, it will take a certain time for the boundary to have a substantial effect on you, and until that time the solutions we obtain in this chapter are valid. Mathematically speaking, the absence of a boundary is a big simplification. The most fundamental properties of the PDEs can be found most easily without the complications of boundary conditions.

$u_{tt} = c^2 u_{xx}$ for $-\infty < x < \infty, t > 0$ (1)

$u(x,0) = f(x)$ for $-\infty < x < \infty$ (2)

$u_t(x,0) = g(x)$ for $-\infty < x < \infty$ (3)

2.3 Solution via change of variable

Since the equation is hyperbolic we introduce the new variable ε, η defined by

$$\varepsilon = x + ct$$

$$\eta = x - ct$$

Then, by the chain rule, we have

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \varepsilon}.\frac{\partial \varepsilon}{\partial \eta} + \frac{\partial}{\partial \eta}.\frac{\partial \eta}{\partial t}$$

$$= C\frac{\partial}{\partial \varepsilon} + \frac{\partial}{\partial \eta}(-C)$$

Factoring out C, we have

$$C\left(\frac{\partial}{\partial \varepsilon}-\frac{\partial}{\partial \eta}\right)$$

Therefore,

$$\frac{\partial^2}{\partial t^2}=C\left(\frac{\partial}{\partial \varepsilon}-\frac{\partial}{\partial \eta}\right)C\left(\frac{\partial}{\partial \varepsilon}-\frac{\partial}{\partial \eta}\right)$$

$$\frac{\partial^2}{\partial t^2}\ =C^2\left(\frac{\partial}{\partial \varepsilon}-\frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \varepsilon}-\frac{\partial}{\partial \eta}\right)$$

$$\frac{\partial^2}{\partial t^2}\ =C^2\left(\frac{\partial^2}{\partial \varepsilon^2}-\frac{2\partial^2}{\partial \varepsilon\partial \eta}+\frac{\partial^2}{\partial \eta^2}\right)$$

Also

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial \varepsilon}.\frac{\partial \varepsilon}{\partial x}+\frac{\partial}{\partial \eta}.\frac{\partial \eta}{\partial x}$$

From the coordinate equations above, we have

$$\frac{\partial}{\partial x}\ =\frac{\partial}{\partial \varepsilon}.1+\frac{\partial}{\partial \eta}.1$$

$$\frac{\partial}{\partial x}\ =\frac{\partial}{\partial \varepsilon}+\frac{\partial}{\partial \eta}$$

Therefore,

$$\frac{\partial^2}{\partial x^2}=\left(\frac{\partial}{\partial \varepsilon}+\frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \varepsilon}+\frac{\partial}{\partial \eta}\right)$$

$$=\frac{\partial^2}{\partial \varepsilon^2}+\frac{2\partial^2}{\partial \varepsilon\partial \eta}+\frac{\partial^2}{\partial \eta^2}$$

Let transform $x,t \rightarrow \varepsilon,\eta$ such that

$$U(x,t)=W(\varepsilon,\eta)$$

Hence

$$U_{tt}-C^2U_{xx}=0$$

Becomes

$$C^2(W_{\varepsilon\varepsilon}-2W_{\varepsilon\eta}+W_{\eta\eta})-C^2(W_{\varepsilon\varepsilon}+2W_{\varepsilon\eta}+W_{\eta\eta})=0$$

$$\Rightarrow -4C^2W_{\varepsilon\eta}=0$$

Therefore, the wave equation (*) transforms into

$$\Rightarrow W_{\varepsilon\eta}=0.$$

OR

$$\frac{\partial^2 w}{\partial \varepsilon\partial \eta}=0 \qquad (4)$$

It is easy to find the general solution of the equation (4) by integrating it twice.
First suppose you integrate with respect to ε and notice that the constant of integration must depend on η to get,

$$\frac{\partial w}{\partial \eta}=G(\eta)$$

Now integrate with respect to η and notice that the constant of integration depends on ε.

$$w = \int_0^{\eta} G(\eta)d\eta + F(\varepsilon)$$

Therefore, let

$$\int_0^{\eta} G(\eta)d\eta = G(\eta)$$

So that

$$w = G(\eta) + F(\varepsilon)$$

Recall that we transformed

$$U(x,t) = w(\varepsilon, \eta)$$

Therefore

$$U(x,t) = F(x+ct) + G(x-ct) \qquad (5)$$

This is d'Alembert's general solution of the wave equation, where *f, g* are arbitrary functions.

Along with the wave equation (1), we next consider some initial conditions, to single out a particular physical solution from the general solution (5). The equation is of second order in time t, so considering *f(x) and g(x)* which are specified both for the initial displacement *u(x,0)*, and the initial velocity $u_t(x,0)$ respectively, where *f* and *g* are arbitrary functions of single variable, and together are called the initial data of the equation (1). Now we determine the function *F* and *G* so that the general equation (5) may satisfy equation (1) above.

Given;

u(x,0) = f(x) for $-\infty < x < \infty$

$u_t(x,0) = g(x)$ for $-\infty < x < \infty$

First, setting *t=0* in (5), we get

$$U(x,0) = f(x) = F(x) + G(x)$$

$$U(x,0) = F(x) + G(x) = f(x) \qquad (6)$$

Then, using the chain rule, we differentiate (5) with respect to *t* and put *t=0* to get

$$U_t(x,0) = CF'(x) + CG'(x)$$

(The primes here mean differentiation with regard to the single variable the function depends on.)

$$CF'(x) - CG'(x) = g(x)$$

$$C\big(F'(x) - G'(x)\big) = \frac{g(x)}{C} \qquad (7)$$

Integrating (ii)

$$F(x) - G(x) = \int_0^x \frac{g(s)}{C} ds + k \qquad (8)$$

Adding (i) and (ii)

$$2F(x) = \int_0^x \frac{g(s)}{C} ds + k + f(x)$$

$$F(x) = \frac{1}{2_C} \int_0^x g(s)ds + \frac{k}{2} + \frac{f(x)}{2} \qquad (9)$$

Also subtracting (i) from (ii) yield

$$2G(x) = -\frac{1}{c} \int_0^x g(s)ds - k + f(x)$$

$$G(x) = -\frac{1}{2c} \int_0^x g(s)ds - \frac{k}{2} + \frac{f(x)}{2} \qquad (10)$$

Recall that

$$U(x,t) = F(x+ct) + G(x-ct)$$

Therefore from (IV)

$$F(x+ct) = \frac{f(x+ct)}{2} + \frac{1}{2c}\int_0^{x+ct} g(s)ds + \frac{k}{2} \qquad (11)$$

And

$$G(x-ct) = \frac{f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{0} g(s)ds\frac{k}{2} \qquad (12)$$

Hence, adding (vi) and (vii) yield.

$$U(x,t) = f(x+ct) + f(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds \qquad (13)$$

This is known as d'Alembert's formula for the solution of the initial value problem above. Our derivation of d'Alembert's formula shows that any solution of (1),(2) and (3) that is twice continuously differentiable must have the representation (13), hence the solution is uniquely determined by the initial data *f* and *g*. Thus d'Alembert's formula represents the unique solution of (1), (2) and (3).

2.3 Proof of the well-Posedness of the solution

We prove the well-Posedness of the Cauchy problem (1), (2) and (3), above by proving

(a) The existence of the solution
(b) The uniqueness of the solution
(c) The stability of the solution

2.3.1 Clearly the solution exist from (13), hence proved.

3.3.2 Uniqueness of the Solution

In this section, we define an energy function *E(t)* for the wave equation, show that energy is conserved for the Cauchy problem (1), (2) and (3), and use this property to establish uniqueness of solutions of the Cauchy problem (1), (2) and (3), above.

Let's assume that

$u = u(x,t)$

To be a smooth solution of the Cauchy problem and the derivatives

$u_t(x,t)$ and $u_x(x,t)$

Are square integrable for each $t \geq 0$. Then the total energy defined by

$$E(t) = \int_{-\infty}^{\infty} (\frac{1}{2}u_t^2 + \frac{c^2}{2}u_x^2)dx$$

is finite, note that E(t) is the sum of kinetic and potential energy. The potential energy

$$PE(t) = \int_{-\infty}^{\infty} \frac{C^2}{2}u_x^2 dx$$

Is the energy stored in the string due to tension and the kinetic energy

$$KE(t) = \int_{-\infty}^{\infty} \frac{1}{2}u_t^2 dx$$

Is the form of $\frac{1}{2}mv^2$ in classical mechanics of a rigid body with mass m and velocity v .

To see how *E(t)* is connected to the one-dimensional wave equation

Let us multiply the PDE

$u_{tt} = c^2 u_{xx}$

by u_t and integrate by parts.

Multiplying $\qquad u_t u_{tt} = c^2 u_{xx} u_t$

$$\int_{-\infty}^{\infty} u_t u_{tt} dx = \int_{-\infty}^{\infty} c^2 u_{xx} u_t dx$$

Integrating by parts $\qquad \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\frac{u_t^2}{2} dx = c^2 u_x u_t \Big|_{-\infty}^{+\infty} - c^2\int_{-\infty}^{\infty} u_x u_{xt} dx$

$$= -c^2 \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\frac{u_x^2}{2} dx$$

That is $$E^{'}(t)=\frac{d}{dt}\int_{-\infty}^{\infty}(\frac{1}{2}u_t^2+\frac{c^2}{2}u_x^2)dx=0$$

Therefore, we have conservation of total energy: *E(t)* = constant, from which we deduce

$$E(t)=E(0)=\frac{1}{2}\int_{-\infty}^{\infty}(\varphi(x)^2+c^2\phi^{'}(x)^2)dx,t>o$$

This identity is an important tool for existence and regularity of solutions, but also for uniqueness of solutions, as we now discusses.

2.4 Proof of the uniqueness of Solution

To prove uniqueness, we show

$u_1=u_2$.

Let define

$u(x,t)=u_1(x.t)-u_2(x,t)$.

Then *u* satisfies the homogeneous wave equation, with zero initial data:

$$u_{tt}=c^2u_{xx}-\infty<x<\infty$$

$$u(x,0)=0,u_t(x,0)=0$$

Since *E(t) = E(0) = 0* for this problem, we have

$$E(t)=\int_{-\infty}^{\infty}(\frac{1}{2}u_t^2+\frac{c^2}{2}u_x^2)dx=0$$

Therefore

$u_t=0,$ And $u_x=0$

Thus, *u* is constant in *x* and *t*. But

u(x,0) = 0, so the constant is zero.

Hence

$u=u_1-u_2=0$

2.5 Proof of continuous dependence (stability)

Let $u=u_1$ and $u=u_2$

Be solutions of problem (1), (2) and (3) with initial data

φ_k, ψ_k, where *k = 1, 2,…,*

Those are bounded, and uniformly close in the sense of continuous functions

$$|\phi_1(x)-\phi_2(x)|<\varepsilon; \quad |\varphi_1(x)-\varphi_2(x)|<\varepsilon, \quad -\infty<x<\infty$$

Where $\varepsilon>0$ is small. From (5),

$$|u_1(x.t)-u_2(x,t)|$$

$$=\left|\frac{1}{2}(\phi_1-\phi_2)(x+ct)+\frac{1}{2}(\phi_1-\phi_2)(x-ct)+\frac{1}{2c}\int_{x-ct}^{x+ct}(\varphi_1-\varphi_2)(y)dy\right|$$

$$\le\frac{1}{2}|(\phi_2-\phi_2)(x+ct)|+\frac{1}{2}|(\phi_1-\phi_2)(x-ct)|+\frac{1}{2c}\int_{x-ct}^{x+ct}|(\varphi_1-\varphi_2)(y)|dy$$

$$\le\frac{1}{2}\varepsilon+\frac{1}{2}\varepsilon+\frac{1}{2c}2ct\varepsilon=(1+t)\varepsilon$$

It follows that if

$|\varphi_1-\varphi_2|$ and $|\psi_1-\psi_2|$ are uniformly small, then $|u_1(x,t)-u_2(x,t)|$ is small at each $x,t<\infty$.

Thus the small change in the initial data leads a small change in the solution at any positive time. One expects this, since the initial value problem is well-posed for the wave equation. Thus, the initial value problem (1), (2) and (3) is well posed.

III. Result

Considering equation (5) obtained that is,

$$U(x,t) = f(x+ct) + f(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

and the standard wave equation;

$u_{tt} = c^2 u_{xx}$ for $-\infty < x < \infty, t > 0$ (1)

$u(x,0) = f(x)$ for $-\infty < x < \infty$ (2)

$u_t(x,0) = g(x)$ for $-\infty < x < \infty$ (3)

Let solve some problems;

3.1 Problem (1)

$$U_{tt} - 25U_{xx} = 0 \qquad -\infty < x < \infty, \quad t > 0$$

$$U(x,0) = f(x) \qquad -\infty < x < \infty,$$

$$U(x,0) = \sin x \qquad -\infty < x < \infty,$$

$$U_t(x,o) = 0 \qquad -\infty < x < \infty$$

Solution

Comparing with the standard wave equation

$$U_{tt} - C^2 U_{xx} = 0 \qquad -\infty < x < \infty, \quad t > 0$$

$$U(x,0) = f(x) \qquad -\infty < x < \infty,$$

$$U_t(x,o) = g(x) \qquad -\infty < x < \infty$$

Therefore from the above question

$$C^2 = 25 \qquad \Rightarrow C = 5$$

$$f(x) = \sin x$$

$$g(x) = 0$$

Therefore, using

$$U(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(x)ds$$

We have

$$U(x,t) = \frac{1}{2}[\sin(x+ct) + \sin(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} 0ds$$

$$U(x,t) = \frac{1}{2}[\sin(x+ct) + \sin(x-ct)] + 0$$

$$U(x,t) = \frac{1}{2}[\sin(x+5t) + \sin(x-5t)] \qquad (1)$$

Recall that

$$\sin(A+B) + \sin(A-B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \qquad (2)$$

Therefore, putting (2) into (1) yield

$$U(x,t) = \frac{1}{2}\left[2\sin\left(\frac{x-5t+x-5t}{2}\right)\cos\left(\frac{x+5t-x+5t}{2}\right)\right]$$

Hence

$$U(x,t) = \sin x \cos 5t$$

Table 1 showing the values of $u(x,t)$ at varying x and t

S/N	$u(x,t)$	x	t
1	0.0872	5^0	0

2	0.1710	10^0	2
3	0.2432	15^0	4
4	0.2962	20^0	6
5	0.3237	25^0	8
6	0.3214	30^0	10
7	0.2868	35^0	12
8	0.2198	40^0	14
9	0.1228	45^0	16
10	0.000	50^0	18

Remark; At varying x and t as *x* increases *u(x,t)* also increase until maximum point is attain *u(x,t)* then start decreasing**.**

Table2 Showing result of *u(x,t)* at fixed *x* and varying *t*

S/N	$u(x,t)$	Fixed x	Varying t
1	0.0872	5	0
2	0.0859		2
3	0.0819		4
4	0.0755		6
5	0.0668		8

Remark; From initial point at fixed *x* as *t* increases *u(x,t)* decrease.

Table3 Showing result of *u(x,t)* at fixed *t* and varying *x*

S/N	$u(x,t)$	Varying x	Fixed t
1	0.0872	5	0
2	0.1736	10	
3	0.2588	15	
4	0.3420	20	
5	0.4226	25	
6	0.5000	30	
7	0.5736	35	

Remark; At fixed t as *x* increases *u(x,t)* also increases.

3.2 Problem (2)

$$U_{tt} - 9U_{xx} = 0 \qquad -\infty < x < \infty, \qquad t > 0$$

$$U(x,0) = \sin x \qquad -\infty < x < \infty,$$

$$U_t(x,o) = \cos 3x \qquad -\infty < x < \infty$$

Solution

Comparing with the standard wave equation

$$U_{tt} - C^2 U_{xx} = 0 \qquad -\infty < x < \infty, \qquad t > 0$$

$$U(x,0) = f(x) \qquad -\infty < x < \infty,$$

$$U_t(x,o) = g(x) \qquad -\infty < x < \infty$$

Where

$$C^2 = 9 \qquad \Rightarrow C = 3$$

$$f(x) = \sin x$$

$$g(x) = \cos 3x$$

Therefore, using the formula

$$U(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(x)ds$$

We have

$$U(x,t)=\frac{1}{2}[\sin(x+3t)+\sin(x-3t)]+\frac{1}{2\times 3}\int_{x-ct}^{x+ct}\cos 3sds$$

But

$$U(x,t)=\frac{1}{2}[\sin(x+c3t)+\sin(x-3t)]=\frac{2\sin x\cos 3t}{2}$$

Therefore

$$\frac{2\sin x\cos 3t}{2}+\frac{1}{6}\int_{x-ct}^{x+ct}\cos 3st$$

$$\frac{2\sin x\cos 3t}{2}+\frac{1}{6}\left[\frac{1}{3\sin 3s}\right]_{x-3t}^{x+3t}$$

$$\sin x\cos 3t+\frac{1}{18}\sin\Big|_{x-3t}^{x+3t}$$

$$=\sin x\cos 3t+\frac{1}{18}[\sin 3(x+3t)-\sin 3(x-3t)]$$

$$=\sin x\cos 3t+\frac{1}{18}[\sin 3t\cos x]$$

Hence

$$U(x,t)=\sin x\cos 3t+\frac{1}{9}\sin 3t\cos x$$

Table 4 showing the values of $u(x,t)$ at varying x and t

S/N	$u(x,t)$	x	t
1	0.6872	5^0	0
2	0.1840	10^0	2
3	0.2754	15^0	4
4	0.3576	20^0	6
5	0.4270	25^0	8
6	0.4811	30^0	10
7	0.6175	35^0	12
8	0.5346	40^0	14
9	0.5315	45^0	16
10	0.5081	50^0	18

Remark; At a point when *x* is at initial, *u(x,t)* attaint its highest point, but when x increase to 10, u(x,t) then decreases to its lowest point. Then both keep on increasing until maximum point is attain, then *u(x,t)* start decreasing again.

Table 5 Showing result of *u(x,t)* at fixed *t* and varying *x*

S/N	$u(x,t)$	Varying x	Fixed t
1	0.0872	5	0
2	0.1736	10	
3	0.2588	15	
4	0.3420	20	
5	0.4226	25	
6	0.5000	30	
7	0.5736	35	

Remark; At fixed t as *x* increases *u(x,t)* also increases

Table 6 showing result of *u(x,t)* at fixed *x* and varying *t*

S/N	$u(x,t)$	Fixed x	Varying t
1	0.0872	5	0
2	0.0983		2
3	0.1083		4
4	0.1171		6
5	0.1247		8
6	0.1347		10
7	0.0770		12

Remark; At fixed x as *t* increases *u(x,t)* also increases until, when t is 12 *u(x,t)* drastically reduce to 0.0770.

IV. Discussion

We have focused on obtaining solution, examining its consistency and its well-posedness of the wave equation via change of variable method.

Using the method, the general solution of the wave equation was derived. Then, the general solution derived led us to the unique solution of the problem, called the d'Alembert's formula, we then discussed the well-posedness of the solution and proved the uniqueness, existence and the dependency of the solution on the initial conditions which helped us to analyze and present result and showed the behaviour of our results in a tabular form.

V. Conclusion

Finally, we conclude that the notion of a well-posed problem is important in applied mathematics.
Though the classical theory of partial differential equations deals almost completely with the well-posed, ill-posed problems can be mathematically and scientifically interesting.

VI. Recommendation

It's highly recommended that, other methods for solving partial differential equation such as

a. Method of separation of variable,
b. Laplace Transform method,
c. Numerical method, etc.

Should be carried out, this will enable readers to compare any of the above method with the method used here.

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